Syntactic Confluence Criteria for Positive/Negative-Conditional Term Rewriting Systems

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SEKI Report SR–95–09
Searchable Online Edition

July 23, 1995
Revised March 6, 1996
("noetherian" replaced with "terminating". Example 14.8 added.)

Revised January and October 2005

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Abstract: We study the combination of the following already known ideas for showing confluence of unconditional or conditional term rewriting systems into practically more useful confluence criteria for conditional systems: Our syntactic separation into constructor and non-constructor symbols, Huet's introduction and Toyama's generalization of parallel closedness for non-terminating unconditional systems, the use of shallow confluence for proving confluence of terminating and non-terminating conditional systems, the idea that certain kinds of limited confluence can be assumed for checking the fulfilledness or infeasibility of the conditions of conditional critical pairs, and the idea that (when termination is given) only prime superpositions have to be considered and certain normalization restrictions can be applied for the substitutions fulfilling the conditions of conditional critical pairs. Besides combining and improving already known methods, we present the following new ideas and results: We strengthen the criterion for overlay joinable terminating systems, and, by using the expressiveness of our syntactic separation into constructor and non-constructor symbols, we are able to present criteria for level confluence that are not criteria for shallow confluence actually and also able to weaken the severe requirement of normality (stiffened with left-linearity) in the criteria for shallow confluence of terminating and non-terminating conditional systems to the easily satisfied requirement of quasi-normality. Finally, the whole paper also gives a practically useful overview of the syntactic means for showing confluence of conditional term rewriting systems.

This research was supported by the Deutsche Forschungsgemeinschaft, SFB 314 (D4-Projekt)

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1 Introduction and Overview

While[1] powerful confluence criteria for conditional term rewriting systems[2] are in great demand and while there are interesting new results for *unconditional* systems[3], hardly any new results on confluence of *conditional* term rewriting systems (besides some on modularity[4] and on the treatment of extra-variables in conditions[5]) have been published since Dershowitz &al. (1988), Toyama (1988), and Bergstra & Klop (1986), and not even a common generalization (as given by our theorems 13.6 and 15.1) of the main confluence theorems of the latter two papers (i.e. something like confluence of parallel closed conditional systems) has to our knowledge been published. We guess that this is due to the following problems:

1. A proper treatment is very tedious and technically most complicated, especially in the case of non-terminating reduction relations.[6]

2. There is a big gap between the known criteria and those criteria that are supposed to be true, even for unconditional systems.[7]

3. The usual framework for conditional term rewriting systems does not allow us to model some simple and straightforward applications naturally in such a way that the resulting reduction relation is known to be confluent, unless some sophisticated semantic or termination knowledge is postulated a priori.

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[1]Please do try not to read the footnotes for a first reading!

[2]For an introduction to the subject cf. Avenhaus & Madlener (1989) or Klop (1992).

[3]Cf. Oostrom (1994a) and Oostrom (1994b). Note that the lemmas 5.1 and 5.3 of Oostrom (1994b) do not apply for conditional systems because they are not subsumed by the notion of “patterm rewriting systems” of Oostrom (1994b).

[4]Cf. Middeldorp (1993), Gramlich (1994).

[5]Cf. Avenhaus & Loría-Sáenz (1994) for the case of decreasing systems and Suzuki &al. (1995) for the case of orthogonal systems.

[6]The technique we apply for proving our confluence criteria for non-terminating reduction relations is in essence to show strong confluence of relations whose reflexive & transitive closures are equal to that of the reduction relation.

In Bergstra & Klop (1986) another technique is used. Instead of an actual presentation of the proof there is only a pointer to Klop (1980). It would be worthwhile to reformulate this proof in modern notions (including path orderings) and notations. While we did not do this, we just try to describe here the abstract global idea of this proof:

The field of the reduction relation is changed from terms to terms with licenses in such a way that the projection to terms just yields the original reduction relation again. The transformed reduction relation becomes terminating since it consumes and inherits licenses in a wellfounded manner; thus its confluence is implied by its local confluence that is to be shown. Finally, each diverging peak of the original reduction relation is a projection of a diverging peak in the transformed reduction relation when one only provides enough licenses.

We did not apply this global proof idea since (while we were able to generalize it for allowing parallel closed critical pairs as in the corollary on page 815 in Huet (1980)) we were not able to generalize it for proving Corollary 3.2 of Toyama (1988) (which generalizes this corollary of Huet (1980)).

[7]Cf. e.g. Problem 13 of Dershowitz &al. (1991).

4. For conditional rule systems there is another big gap between the known criteria and those criteria that are required for practical purposes. This results from the difficulty to capture (with effective means) the infinite number of substitutions that must be tested for fulfilling the conditions of critical pairs.

While we are not able to contribute too much regarding the first two problems, we are able to present some progress with the latter two.

Our positive/negative-conditional rule systems including a syntactic separation between constructor and non-constructor symbols as presented in Wirth & Gramlich (1994a) offer more expressive power than the standard positive conditional rule systems and therefore allow us to model more applications naturally in such a way that their confluence is given by the new confluence criteria presented in this paper. Using the separation into constructor and non-constructor rules (generated by the syntactic separation into constructor and non-constructor function symbols) it is possible to divide the problem of showing confluence of the whole rule system into three smaller sub-problems, namely confluence of the constructor rules, confluence of the non-constructor rules, and their commutation. The important advantage of this modularization is not only the division into smaller problems, but is due to the possibility to tackle the sub-problems with different confluence criteria. E.g., when confluence of the constructor rules is not trivial then its confluence often can only be shown by sophisticated semantical considerations or by criteria that are applicable to terminating systems only. For the whole rule system, however, neither semantic confluence criteria nor confluence criteria requiring termination of the reduction relation are practically feasible in general. This is because, on the one hand, an effective application of semantic confluence criteria requires that the specification given by the whole rule system has actually been modeled before in some formalism. On the other hand, termination of the whole rule system may not be given or difficult to be shown without some confluence assumptions.[8] Fortunately, without requiring termination of the whole rule system the syntactic confluence criteria[9] presented in this paper guarantee confluence of the non-constructor rules of a class of rule systems that is sufficient for practical specification. This class of rule systems generalizes the function specification style used in the framework of classic inductive theorem proving[10] by allowing of partial functions resulting from incomplete specification as well as from non-termination. Together with the notions of inductive validity presented in Wirth & Gramlich (1994b) this extends the area of semantically clearly understood inductive specification considerably.

Regarding the last problem of the above problem list (occuring in case of conditional rule systems), by carefully including the invariants of the proofs for the confluence criteria into the conditions of the joinability tests for the conditional critical pairs we allow of more reasoning on those substitutions that fulfill the condition of a critical pair. E.g. consider the following example:

[8]When termination is assumed, there are approaches to prove confluence automatically, cf. Becker (1993) and Becker (1994).

[9]Cf. our theorems 13.3, 13.4, and 15.3.

[10]Cf. Walther (1994). Note that we can even keep the notation style similar to this function specification style, cf. Wirth & Lunde (1994).

Example 1.1 Let R:

$$\begin{array}{lcll} \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & = & \mathsf{s}(0) & \longleftarrow \ \mathsf{f}(x){=}0 \\ \mathsf{f}(\mathsf{s}(\mathsf{s}(x))) & = & 0 & \longleftarrow \ \mathsf{f}(x){=}\mathsf{s}(0) \\ \mathsf{f}(\mathsf{s}(0)) & = & \mathsf{s}(0) & \\ \mathsf{f}(0) & = & 0 & \end{array}$$

Assume 0 and $\mathsf{s}(0)$ to be irreducible.

The experts may notice that the part of R we are given in this example is rather well-behaved: It is left-linear and normal; it may be decreasing; and the only critical pair is an overlay. Now, for showing the critical pair between the first two rules to be joinable, one has to show that it is impossible that both conditions hold simultaneously for a substitution $\{x \mapsto t\}$. One could argue the following way: If both conditions were fulfilled, then $\mathsf{f}(t)$ would reduce to 0 as well as to $\mathsf{s}(0)$, which contradicts confluence below $\mathsf{f}(t)$. But, as our aim is to establish confluence, it is not all clear that we are allowed to assume confluence for the joinability test here. None of the theorems in Dershowitz &al. (1988) or Bergstra & Klop (1986) provides us with such a confluence assumption, even if their proofs could do so with little additional effort. For practical purposes, however, it is important that the joinability test allows us to assume a sufficient kind of confluence for the condition terms. Therefore, all our joinability notions provide us with sufficient assumptions that allow us to easily establish the infeasibility of the condition of a critical pair, without knowing the proofs for the confluence criteria by heart. This applies for example, when two rules with same left-hand side are meant to express a case distinction that is established by the condition of the one containing a condition literal "$p{=}\mathsf{true}$" or "$u{=}v$" and the condition of the other containing the condition literal "$p{=}\mathsf{false}$" or "$u{\neq}v$".[11]

For terminating reduction relations we carefully investigate whether the joinability test can be restricted by certain irreducibility requirements, e.g. whether the substitutions which must be tested for fulfilling the conditions of critical pairs can be required to be normalized, cf. § 14, esp. Example 14.3. The restrictions on the infinite number of substitutions for which the condition of a critical pair must be tested for fulfilledness may be a great help in practice. However, they do not solve the principle problem that the number of substitutions is still infinite.

Another important point is that we weaken the severe restriction imposed on terminating systems by Theorem 2 of Dershowitz &al. (1988) and on non-terminating systems by Theorem 3.5 of Bergstra & Klop (1986), namely normality, which in our framework can be considerably weakened to the so-called *quasi-normality*, cf. our theorems 13.6 and 14.5.

Moreover, besides these two criteria for shallow confluence, we present to our knowledge the first criteria for level confluence that are not criteria for shallow confluence actually[12], cf. our theorems 13.9 and 14.6.

Finally, we considerably improve the notion of "quasi overlay joinability" of Wirth & Gramlich (1994a), generalizing the notion of "overlay joinability" of Dershowitz &al. (1988). This results in a stronger criterion with a simpler proof, cf. § 9 and Theorem 14.7.

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[11]In Definition 4.4 of Avenhaus & Loría-Sáenz (1994) the critical pair resulting from such two rules is called "infeasible" (in the case with "$p{=}\mathsf{true}$" and "$p{=}\mathsf{false}$"). We will call it "complementary" instead (in both cases), cf. Theorem 13.3.

[12]as is the case with Suzuki &al. (1995).

Since our main interest is in positive/negative-conditional rule systems with two kinds of variables and two kinds of function symbols as presented in Wirth &al. (1993) and Wirth & Gramlich (1994a), the whole paper is based on this framework. We know that this is problematic because the paper may also be of interest for readers interested in positive conditional rule systems with one kind of variables and function symbols only: With the exception of our generalization of normality to quasi-normality and our criteria for level confluence, our results also have interesting implications for this special case (which is subsumed by our approach). Nevertheless we prefer our more expressive framework for this presentation because it provides us with more power for most of our confluence criteria which is lost when restricting them to the standard framework. Therefore in the following section we are going to repeat those results of Wirth & Gramlich (1994a) which are essential for this paper. Those readers who are only interested in the implications of this paper for standard positive conditional rule systems with one kind of variables and function symbols should try to read only the theorems presented or pointed at in § 15, which have been supplied with independent proofs for allowing a direct understanding. The contents of the other sections are explained by their titles. For a first reading sections 7 and 8 should only be skimmed and its definitions looked up by need. Due to their enormous length, most of the proofs have been put into D.

We conclude this section with a list on where in this paper to find generalizations of known theorems:

Parallel Closed + Left-Linear + Unconditional:
The corollary on page 815 in Huet (1980) as well as Corollary 3.2 in Toyama (1988) are generalized by our theorems 13.6(I), 13.6(III), 13.6(IV), 13.9(I), 13.9(III), 13.9(IV), and 15.1(I).

No Critical Pairs + Left-Linear + Normal:
Theorem 3.5 in Bergstra & Klop (1986) as well as Theorem 1 in Dershowitz &al. (1988) are generalized by our theorems 13.3, 13.4, 13.6, 15.1, and 15.3.

Strongly Joinable + Strong Variable Restriction:
Lemma 3.2 of Huet (1980) as well as the translation of Theorem 5.2 in Avenhaus & Becker (1994) into our framework is generalized by our theorems 13.6(II) and 13.9(II).

Shallow Joinable + Left-Linear + Normal + Terminating:
Theorem 2 in Dershowitz &al. (1988) is generalized by our theorems 14.5 and 15.4.

Overlay Joinable + Terminating:
Theorem 4 in Dershowitz &al. (1988) as well as Theorem 6.3 in Wirth & Gramlich (1994a) are generalized by our theorem 14.7.

Joinable + Variable Restriction + Terminating:
Theorem 7.18 in Wirth & Gramlich (1994a) is generalized by our theorem 14.4.

Joinable + Decreasing:
Theorem 3.3 in Kaplan (1987), Theorem 4.2 in Kaplan (1988), Theorem 3 in Dershowitz &al. (1988), as well as Theorem 7.17 in Wirth & Gramlich (1994a) are generalized by our theorems 14.2 and 14.4.

2 Positive/Negative-Conditional Rule Systems

We use '$\uplus$' for the union of disjoint classes and 'id' for the identity function. '$\mathbf{N}$' denotes the set of natural numbers and we define $\mathbf{N}_+ := \{\, n{\in}\mathbf{N} \mid 0{\neq}n \,\}$. For classes A,B we define: $\mathrm{dom}(A) := \{\, a \mid \exists b.\ (a,b){\in}A \,\}$; $\mathrm{ran}(A) := \{\, b \mid \exists a.\ (a,b){\in}A \,\}$; $B[A] := \{\, b \mid \exists a{\in}A.\ (a,b){\in}B \,\}$. This use of "[...]" should not be confused with our habit of stating two definitions, lemmas, or theorems (and their proofs &c.) in one, where the parts between '[' and ']' are optional and are meant to be all included or all omitted. Furthermore, we use '$\emptyset$' to denote the empty set as well as the empty function or empty word.

2.1 Terms and Substitutions

Since our approach is based on the consequent syntactic distinction of constructors, we have to be quite explicit about terms and substitutions.

We will consider terms of fixed arity over many-sorted signatures. A *signature* $\mathrm{sig} = (\mathbb{F},\mathbb{S},\alpha)$ consists of an enumerable set of function symbols $\mathbb{F}$, a finite set of sorts $\mathbb{S}$ (disjoint from $\mathbb{F}$), and a computable arity-function $\alpha : \mathbb{F} \to \mathbb{S}^+$. For $f \in \mathbb{F}$. $\alpha(f)$ is the list of argument sorts augmented by the sort of the result of f; to ease reading we will sometimes insert a '$\to$' between a nonempty list of argument sorts and the result sort. A *constructor sub-signature of the signature* $(\mathbb{F},\mathbb{S},\alpha)$ is a signature $\mathrm{cons} = (\mathbb{C},\mathbb{S},{}_{\mathbb{C}}\!\upharpoonright\!\alpha)$ such that the set $\mathbb{C}$ is a decidable subset of $\mathbb{F}$. $\mathbb{C}$ is called the set of *constructor symbols*; the complement $\mathbb{N} = \mathbb{F} \setminus \mathbb{C}$ is called the set of *non-constructor symbols*.

Example 2.1 (Signature with Constructor Sub-Signature)

$$\begin{array}{rcl} \mathbb{C} &=& \{0,\mathsf{s},\mathsf{false},\mathsf{true},\mathsf{nil},\mathsf{cons}\} \\ \mathbb{N} &=& \{-,\mathsf{mbp}\} \\ \mathbb{S} &=& \{\mathsf{nat},\mathsf{bool},\mathsf{list}\} \\ \\ \alpha(0) &=& \mathsf{nat} \\ \alpha(\mathsf{s}) &=& \mathsf{nat} \to \mathsf{nat} \end{array} \qquad \begin{array}{rcl} \alpha(\mathsf{false}) &=& \mathsf{bool} \\ \alpha(\mathsf{true}) &=& \mathsf{bool} \\ \alpha(\mathsf{nil}) &=& \mathsf{list} \\ \alpha(\mathsf{cons}) &=& \mathsf{nat\ list} \to \mathsf{list} \\ \alpha(-) &=& \mathsf{nat\ nat} \to \mathsf{nat} \\ \alpha(\mathsf{mbp}) &=& \mathsf{nat\ list} \to \mathsf{bool} \end{array}$$

To reduce declaration effort, in all examples (unless stated otherwise) in this and the following sections we will have only one sort; 'a', 'b', 'c', 'd', 'e', and '0' will always be constants; 's', 'p', 'f', 'g', and 'h' will always denote functions with one argument; '+' and '$-$' take two arguments in infix notation; 'W', 'X', 'Y', 'Z' are variables from $\mathrm{V}_{\mathrm{SIG}}$ (cf. below).

A *variable-system for a signature* $(\mathbb{F},\mathbb{S},\alpha)$ is an $\mathbb{S}$-sorted family of decidable sets of variable symbols which are mutually disjoint and disjoint from $\mathbb{F}$. By abuse of notation we will use the symbol 'X' for an $\mathbb{S}$-sorted family to denote not only the family $X = (X_s)_{s\in\mathbb{S}}$ itself, but also the union of its *ranges*: $\bigcup_{s\in\mathbb{S}} X_s$. As the basis for our terms throughout the whole paper we assume two fixed disjoint variable-systems $\mathrm{V}_{\mathrm{SIG}}$ of *general variables* and $\mathrm{V}_{\mathcal{C}}$ of *constructor variables* such that $\mathrm{V}_{\mathrm{SIG},s}$ as well as $\mathrm{V}_{\mathcal{C},s}$ contain infinitely many elements for each $s \in \mathbb{S}$.

$\mathcal{T}(\mathrm{sig}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ denotes the $\mathbb{S}$-sorted family of all well-sorted *(variable-mixed) terms* over '$\mathrm{sig}/\mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}}$', while $\mathcal{GT}(\mathrm{sig})$ denotes the $\mathbb{S}$-sorted family of all well-sorted *ground terms* over 'sig'. Similarly, $\mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ denotes the $\mathbb{S}$-sorted family of all *(variable-mixed) constructor terms*, $\mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ denotes the $\mathbb{S}$-sorted family of all *pure constructor terms*, while $\mathcal{GT}(\mathrm{cons})$ denotes the $\mathbb{S}$-sorted family of all *constructor ground terms*. To avoid problems with empty sorts, we assume $\mathcal{GT}(\mathrm{cons})$ to have nonempty ranges only.

We define $\mathrm{V} := (\mathrm{V}_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathcal{C}\}\times\mathbb{S}}$ and call it a *variable-system for a signature* $(\mathbb{F}, \mathbb{S}, \alpha)$ *with constructor sub-signature*. We use $\mathcal{V}(A)$ to denote the $\{\mathrm{SIG}, \mathcal{C}\}\times\mathbb{S}$-sorted family of variables occurring in a structure A (e.g. a term or a set or list of terms). Let $\mathrm{X} \subseteq \mathrm{V}$ be a variable-system. We define $\mathcal{T}(\mathrm{X}) = (\mathcal{T}(\mathrm{X})_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathcal{C}\}\times\mathbb{S}}$ by ($s \in \mathbb{S}$): $\mathcal{T}(\mathrm{X})_{\mathrm{SIG},s} := \mathcal{T}(\mathrm{sig}, \mathrm{X})_s$ and $\mathcal{T}(\mathrm{X})_{\mathcal{C},s} := \mathcal{T}(\mathrm{cons}, \mathrm{X}_{\mathcal{C}})_s$. To avoid confusion: Note that $\mathcal{T}(\mathrm{X})_{\mathcal{C},s} \subseteq \mathcal{T}(\mathrm{X})_{\mathrm{SIG},s}$ for $s \in \mathbb{S}$, whereas $\mathrm{V}_{\mathcal{C},s} \cap \mathrm{V}_{\mathrm{SIG},s} = \emptyset$. Furthermore we write $\mathcal{GT}$ for $\mathcal{T}(\emptyset)$ as well as $\mathcal{T}$ for $\mathcal{T}(\mathrm{V})$. Our custom of reusing the symbol of a family for the union of its ranges now allows us to write $\mathcal{T}$ as a shorthand for $\mathcal{T}(\mathrm{sig}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$.

For a term $t \in \mathcal{T}$ we denote by $\mathcal{POS}(t)$ the *set of its positions* (which are lists of positive natural numbers), by t/p the subterm of t at position p, and by $t[p \leftarrow t']$ the result of replacing t/p with t' at position p in t. We write $p \parallel q$ to express that neither p is a prefix of q, nor q a prefix of p. For $\Pi \subseteq \mathcal{POS}(t)$ with $\forall p,q \in \Pi.\ (p=q \vee p \parallel q)$ we denote by $t[p \leftarrow t'_p \mid p \in \Pi]$ the result of replacing, for each $p \in \Pi$, the subterm at position p in the term t with the term t'_p. t is *linear* if $\forall p,q \in \mathcal{POS}(t).\ (t/p = t/q \in \mathrm{V} \Rightarrow p = q)$.

The set of *substitutions* from V to a $\{\mathrm{SIG}, \mathcal{C}\}\times\mathbb{S}$-sorted family of sets $T = (T_{\varsigma,s})_{(\varsigma,s)\in\{\mathrm{SIG},\mathcal{C}\}\times\mathbb{S}}$ is defined to be

$$\mathcal{SUB}(\mathrm{V}, T) := \{\ \sigma : \mathrm{V} \to T \mid \forall(\varsigma,s) \in \{\mathrm{SIG}, \mathcal{C}\}\times\mathbb{S}.\ \forall x \in \mathrm{V}_{\varsigma,s}.\ \sigma(x) \in T_{\varsigma,s}\ \}.$$

Note that $\forall\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}).\ \forall(\varsigma,s) \in \{\mathrm{SIG}, \mathcal{C}\}\times\mathbb{S}.\ \forall t \in \mathcal{T}_{\varsigma,s}.\ t\sigma \in \mathcal{T}_{\varsigma,s}$.

Let E be a finite set of equations and X a finite subset of V. A substitution $\sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T})$ is called *a unifier for* E if $E\sigma \subseteq \mathrm{id}$. Such a unifier is called *most general on* X if for each unifier μ for E there is some $\tau \in \mathcal{SUB}(\mathrm{V}, \mathcal{T})$ such that ${}_{\mathrm{X}}\!\upharpoonleft\!(\sigma\tau) = {}_{\mathrm{X}}\!\upharpoonleft\!\mu$. If E has a unifier, then it also has a most general unifier[13] on X, denoted by $\mathrm{mgu}(E, \mathrm{X})$.

[13]For this most general unifier σ we could, as usual, even require $\sigma\sigma = \sigma$ but *not* $\mathcal{V}(\sigma[\mathcal{V}(E)]) \subseteq \mathcal{V}(E)$.

2.2 Relations

Let X⊆V. Let T ⊆ $\mathcal{T}$. A relation R on $\mathcal{T}$ is called:

sort-invariant if $\forall (t,t')\in R.\ \exists s\in\mathbb{S}.\ t,t'\in\mathcal{T}_{\mathrm{SIG},s}$

X-*stable (w.r.t. substitution)* if $\forall (t_0,\ldots,t_{n-1})\in R.\ \forall\sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$
$(t_0\sigma,\ldots,t_{n-1}\sigma)\in R$

T-*monotonic* if $\forall (t',t'')\in R.\ \forall t\in\mathcal{T}.\ \forall p\in\mathcal{POS}(t).$

$$\left(\left(\begin{array}{ll} & \exists s\in\mathbb{S}.\ t/p,t',t''\in\mathcal{T}_{\mathrm{SIG},s} \\ \wedge & t[p\leftarrow t']\in\mathrm{T} \end{array} \right) \Rightarrow \left(\begin{array}{ll} & (t[p\leftarrow t'],t[p\leftarrow t''])\in R \\ \wedge & t[p\leftarrow t'']\in\mathrm{T} \end{array} \right) \right)$$

The *subterm ordering* $\lhd_{\mathrm{ST}}$ *on* $\mathcal{T}$ is the V-stable and wellfounded ordering defined by: $t\unlhd_{\mathrm{ST}}t'$ if $\exists p\in\mathcal{POS}(t').\ t=t'/p.$ A *termination-pair* over sig/V is a pair $(>,\rhd)$ of V-stable, wellfounded orderings on $\mathcal{T}$ such that $>$ is $\mathcal{T}$-monotonic, $>\subseteq\rhd$, and $\rhd_{\mathrm{ST}}\subseteq\rhd$. Cf. Wirth & Gramlich (1994a) for further theoretical aspects of termination-pairs, and Geser (1994) for interesting practical examples. For further details on orderings cf. Dershowitz (1987).

The reflexive, symmetric, transitive, and reflexive & transitive closure of a relation $\longrightarrow$ will be denoted by $\stackrel{=}{\longrightarrow}$, $\longleftrightarrow$, $\stackrel{+}{\longrightarrow}$, and $\stackrel{*}{\longrightarrow}$, resp..[14] Two terms v, w are called *joinable w.r.t.* $\longrightarrow$ if $v{\downarrow}w$, i.e. if $v\stackrel{*}{\longrightarrow}\circ\stackrel{*}{\longleftarrow}w$. They are *strongly joinable w.r.t.* $\longrightarrow$ if $v{\downarrow\downarrow}w$, i.e. if $v\stackrel{=}{\longrightarrow}\circ\stackrel{*}{\longleftarrow}w\stackrel{=}{\longrightarrow}\circ\stackrel{*}{\longleftarrow}v$. $\longrightarrow$ is called *terminating below u* if there is no $s:\mathbf{N}\rightarrow\mathrm{dom}(\longrightarrow)$ such that $u=s_0\wedge\forall i\in\mathbf{N}.\ s_i\longrightarrow s_{i+1}$.

[14]Note that this is actually an abuse of notation since A^+ now denotes the transitive closure of A as well as the set of nonempty words over A and since A^* now denotes the reflexive & transitive closure of A as well as the set of words over A. In our former papers we prefered to denote different things different but now we have found back to this standard abuse of notion for the sake of convenient readability, because the reader will easily find out what is meant with any application with the exception of those in the proof of Lemma B.7.

2.3 The Reduction Relation

In the definition below we restrict our constructor rules to contain no non-constructor function symbols, to be extra-variable free, and to contain no negative literals. This is important for our approach (cf. Lemma 2.10, Lemma 2.11, and Lemma 2.12) and should always be kept in mind when reading the following sections.

Definition 2.2 (Syntax of CRS)
$\mathcal{CONDLIT}(\mathrm{sig},\mathrm{V})$ is the set of *condition literals* over the following predicate symbols on terms from $\mathcal{T}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$: '$=$', '$\neq$' (binary, symmetric, sort-invariant), and 'Def' (singulary). The terms[15] of a list C of condition literals are called *condition terms* and their set is denoted by $\mathcal{TERMS}(C)$. A (positive/negative-) *conditional rule system* (*CRS*) R over sig/cons/V is a finite subset of the *set of rules* over sig/cons/V, which is defined by

$$\left\{\ \big(\ (l,r),\ C\ \big)\ \middle|\ \left(\begin{array}{ll} & \exists s\in\mathbb{S}.\ l,r\in\mathcal{T}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})_s\\ \wedge & C\in(\mathcal{CONDLIT}(\mathrm{sig},\mathrm{V}))^*\\ \wedge & \left(\begin{array}{l} l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\ \Rightarrow\\ \left(\begin{array}{ll} & \{r\}\cup\mathcal{TERMS}(C)\ \subseteq\ \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ \wedge & \mathcal{V}(\{r\}\cup\mathcal{TERMS}(C))\ \subseteq\ \mathcal{V}(l)\\ \wedge & \forall L \text{ in } C.\ \forall u,v.\ \ L\neq(u{\neq}v)\end{array}\right)\end{array}\right)\end{array}\right)\right\}.$$

A rule $((l,r),\emptyset)$ with an empty condition will be written $l{=}r$. Note that $l{=}r$ differs from $r{=}l$ whenever the equation is used as a reduction rule. A rule $((l,r),C)$ with condition C will be written $l{=}r{\longleftarrow}C$. We call l the *left-hand side* and r the *right-hand side* of the rule $l{=}r{\longleftarrow}C$. A rule is said to be *left-linear* (or else *right-linear*) if its left-hand (or else right-hand) side is a linear term. A rule $l{=}r{\longleftarrow}C$ is said to be *extra-variable free* if $\mathcal{V}(\{r\}\cup\mathcal{TERMS}(C))\subseteq\mathcal{V}(l)$. The whole CRS R is said to have one of these properties if each of its rules has it. A rule $l{=}r{\longleftarrow}C$ is called a *constructor rule* if its left-hand side is a constructor term, i.e. $l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$.

In the following example we define the subtraction operation '$-$' partially (due to a non-complete defining case distinction), whereas we define a member-predicate 'mbp' totally on the constructor ground terms.

Example 2.3 *(continuing Example 2.1)*
Let $x,y\in\mathrm{V}_{\mathcal{C},\mathsf{nat}}$ and $l\in\mathrm{V}_{\mathcal{C},\mathsf{list}}$.

$$\begin{array}{lrcl|rclcl} \mathrm{R}_{2.3}: & x-0 & = & x & \mathsf{mbp}(x,\mathsf{nil}) & = & \mathsf{false} & & \\ & \mathsf{s}(x)-\mathsf{s}(y) & = & x-y & \mathsf{mbp}(x,\mathsf{cons}(y,l)) & = & \mathsf{true} & \longleftarrow & x{=}y\\ & & & & \mathsf{mbp}(x,\mathsf{cons}(y,l)) & = & \mathsf{mbp}(x,l) & \longleftarrow & x{\neq}y \end{array}$$

[15] To avoid misunderstanding: For a condition list, say " $s{=}t,\ u{\neq}v,\ \mathrm{Def}\,w$ ", we mean the top level terms $s,t,u,v,w\in\mathcal{T}(\mathrm{sig},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$, but neither their proper subterms nor the literals "$s{=}t$", "$u{\neq}v$", "$\mathrm{Def}\,w$" themselves.

Definition 2.4 (Fulfilledness)
A list $D \in \mathcal{CONDLIT}(\mathrm{sig},\mathrm{X})^*$ of condition literals is said to be *fulfilled w.r.t. some relation* $\longrightarrow$ if

$$\forall u,v \in \mathcal{T}.\left(\begin{array}{lllr} & \mathbf{(}\,((\,u{=}v\,)\text{ in }D) & \Rightarrow & u{\downarrow}v\ \mathbf{)} \\ \wedge & \mathbf{(}\,((\mathrm{Def}\,u)\text{ in }D) & \Rightarrow\ \exists\hat{u}\in\mathcal{GT}(\mathrm{cons}). & u\overset{*}{\longrightarrow}\hat{u}\,\mathbf{)} \\ \wedge & \mathbf{(}\,((\,u{\neq}v\,)\text{ in }D) & \Rightarrow\ \exists\hat{u},\hat{v}\in\mathcal{GT}(\mathrm{cons}). & u\overset{*}{\longrightarrow}\hat{u}{\not\updownarrow}\hat{v}\overset{*}{\longleftarrow}v\,\mathbf{)}\end{array}\right).$$

To avoid a non-monotonic behaviour of our negative conditions, we define our reduction relation $\longrightarrow_{\mathrm{R,X}}$ via a double closure: First we define $\longrightarrow_{\mathrm{R,X},\omega}$ by using the constructor rules only. Then we define $\longrightarrow_{\mathrm{R,X},\omega+\omega}$ via a second closure including all rules.

Definition 2.5 ($\longrightarrow_{\mathrm{R,X}}$)
Let R be a CRS over sig/cons/V. Let X⊆V. Let $\prec$ denote the ordering on the ordinal numbers. For $\beta \preceq \omega{+}\omega$ and $p \in \mathbf{N}_+^*$ the reduction relations $\longrightarrow_{\mathrm{R,X},\beta}$ and $\longrightarrow_{\mathrm{R,X},\beta,p}$ on $\mathcal{T}(\mathrm{sig},\mathrm{X})$ are inductively defined as follows: For $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})$:
$s\longrightarrow_{\mathrm{R,X},\beta}t$ if $\exists p{\in}\mathcal{POS}(s).\ s\longrightarrow_{\mathrm{R,X},\beta,p}t.$
For $p \in \mathbf{N}_+^*$: $\longrightarrow_{\mathrm{R,X},0,p} := \emptyset$. For $i \in \mathbf{N}$; $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})$:

$$s\longrightarrow_{\mathrm{R,X},i+1,p}t \text{ if } \exists\left\langle\begin{array}{l}((l,r),C)\in\mathrm{R}\\ \sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))\end{array}\right\rangle.\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ \wedge & s/p=l\sigma\\ \wedge & t=s[p\leftarrow r\sigma]\\ \wedge & C\sigma\text{ is fulfilled w.r.t. }\longrightarrow_{\mathrm{R,X},i}\end{array}\right).$$

$\longrightarrow_{\mathrm{R,X},\omega,p} := \bigcup_{i\in\mathbf{N}}\longrightarrow_{\mathrm{R,X},i,p}$. For $i \in \mathbf{N}$; $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})$: $s\longrightarrow_{\mathrm{R,X},\omega+i+1,p}t$ if

$$s\longrightarrow_{\mathrm{R,X},\omega,p}t\ \vee\ \exists\left\langle\begin{array}{l}((l,r),C)\in\mathrm{R}\\ \sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))\end{array}\right\rangle.\left(\begin{array}{ll} & s/p=l\sigma\\ \wedge & t=s[p\leftarrow r\sigma]\\ \wedge & C\sigma\text{ is fulfilled w.r.t. }\longrightarrow_{\mathrm{R,X},\omega+i}\end{array}\right).$$

$\longrightarrow_{\mathrm{R,X},\omega+\omega,p} := \bigcup_{i\in\mathbf{N}}\longrightarrow_{\mathrm{R,X},\omega+i,p}$; $\longrightarrow_{\mathrm{R,X}} := \longrightarrow_{\mathrm{R,X},\omega+\omega}$.

We will drop "R, X" in $\longrightarrow_{\mathrm{R,X}}$ and $\longrightarrow_{\mathrm{R,X},\beta}$ &c. when referring to some fixed R, X.

Corollary 2.6
$\longrightarrow_{\mathrm{R,X},\omega}$ is the minimum (w.r.t. set-inclusion) of all relations $\leadsto$ on $\mathcal{T}$ satisfying for all $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})$: $s \leadsto t$ if

$$\exists\left\langle\begin{array}{l}p\in\mathcal{POS}(s)\\((l,r),C)\in\mathrm{R}\\ \sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))\end{array}\right\rangle.\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ \wedge & s/p=l\sigma\\ \wedge & t=s[p\leftarrow r\sigma]\\ \wedge & C\sigma\textit{ is fulfilled w.r.t. }\leadsto\end{array}\right).$$

Lemma 2.7 *Let $S_{\mathrm{R,X}}$ be the set of all relations $\leadsto$ on $\mathcal{T}$ satisfying*

1. $(\leadsto \cap (\mathcal{GT}(\mathrm{cons})\times\mathcal{T})) \subseteq \longrightarrow_{\mathrm{R,X},\omega}$ *as well as*
2. *for all $s,t \in \mathcal{T}(\mathrm{sig},\mathrm{X})$:*

$$s \leadsto t\ \textit{if}\ \exists\left\langle\begin{array}{l}p\in\mathcal{POS}(s)\\((l,r),C)\in\mathrm{R}\\ \sigma\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))\end{array}\right\rangle.\left(\begin{array}{ll} & s/p=l\sigma\\ \wedge & t=s[p\leftarrow r\sigma]\\ \wedge & C\sigma\textit{ is fulfilled w.r.t. }\leadsto\end{array}\right).$$

Now $\longrightarrow_{\mathrm{R,X}}$ is the minimum (w.r.t. set-inclusion) in $S_{\mathrm{R,X}}$, and $S_{\mathrm{R,X}}$ is closed under nonempty intersection.

Corollary 2.8 (Monotonicity of $\longrightarrow$ w.r.t. Replacement)
$\longrightarrow_{\mathrm{R,X},\beta}$ *(for* $\beta \preceq \omega{+}\omega$*) and* $\longrightarrow_{\mathrm{R,X}}$ *are* $\mathcal{T}(\mathrm{sig},\mathrm{X})$*-monotonic as well as* $\overset{*}{\longrightarrow}_{\mathrm{R,X}}[\mathrm{T}]$*-monotonic for each* $\mathrm{T} \subseteq \mathcal{T}(\mathrm{sig},\mathrm{X})$.

Corollary 2.9 (Stability of $\longrightarrow$)
$\longrightarrow_{\mathrm{R,X},\beta}$ *(for* $\beta \preceq \omega{+}\omega$*),* $\longrightarrow_{\mathrm{R,X}}$*, and their respective fulfilledness-predicates are* X*-stable.*

Lemma 2.10 *For* $\mathrm{X} \subseteq \mathrm{Y} \subseteq \mathrm{V}$*:*

$$\forall n\in\mathbf{N}.\ \forall s\in\mathcal{T}(\mathrm{cons},\mathrm{X}).\ \forall t.\ \left(s\overset{n}{\longrightarrow}_{\mathrm{R,Y}}t\ \Rightarrow\ (s\overset{n}{\longrightarrow}_{\mathrm{R,Y},\omega}t \in \mathcal{T}(\mathrm{cons},\mathrm{X}))\right)$$

Lemma 2.11 $\downarrow \cap \left(\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}}) \times \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\right) \ \subseteq\ \downarrow_{\omega}$

Lemma 2.12 (Monotonicity of $\longrightarrow_{\beta}$ and of Fulfilledness w.r.t. $\longrightarrow_{\beta}$ in β)
For $\beta \preceq \gamma \preceq \omega{+}\omega$*:* $\longrightarrow_{\beta}\ \subseteq\ \longrightarrow_{\gamma}\ \subseteq\ \longrightarrow$ *; and if C is fulfilled w.r.t.* $\longrightarrow_{\beta}$ *and* $\omega \preceq \beta\ \vee\ \forall u,v.\ ((u{\neq}v)$ *is not in C*$)$ *, then C is fulfilled w.r.t.* $\longrightarrow_{\gamma}$ *and w.r.t.* $\longrightarrow$*.*

Note that monotonicity of fulfilledness is not given in general for $\beta \prec \omega$ and a negative literal which may become invalid during the growth of the reduction relation on constructor terms.

For the proofs cf. Wirth & Gramlich (1994a).

2.4 The Parallel Reduction Relation

The following relation is essential for sophisticated joinability notions as well as for most of our proofs:

Definition 2.13 (Parallel Reduction)
For $\beta \preceq \omega{+}\omega$ we define the *parallel reduction relation* $\twoheadrightarrow_{\mathrm{R,X},\beta}$ on $\mathcal{T}(\mathrm{sig},\mathrm{X})$:

$s \twoheadrightarrow_{\mathrm{R,X},\beta} t$ if $\exists \Pi \subseteq \mathcal{POS}(s).\ s\twoheadrightarrow_{\mathrm{R,X},\beta,\Pi}t$, where

$$s\twoheadrightarrow_{\mathrm{R,X},\beta,\Pi}t \quad\text{if}\quad \left(\begin{array}{ll} & \forall p,q\in\Pi.\ (\ p{=}q\ \vee\ p\parallel q\ ) \\ \wedge & t{=}s[p\leftarrow t/p \mid p\in\Pi] \\ \wedge & \forall p\in\Pi.\ s/p\longrightarrow_{\mathrm{R,X},\beta}t/p \end{array}\right).$$

Corollary 2.14 $\forall\beta\preceq\omega{+}\omega.\ \longrightarrow_{\mathrm{R,X},\beta} \subseteq \twoheadrightarrow_{\mathrm{R,X},\beta} \subseteq \overset{*}{\longrightarrow}_{\mathrm{R,X},\beta}$.

3 Confluence

The following notions and lemmas have become folklore, cf. e.g. Klop (1980) or Huet (1980) for more information.

Definition 3.1 (Commutation and Confluence)

Two relations $\longrightarrow_0$ and $\longrightarrow_1$ are *commuting* if

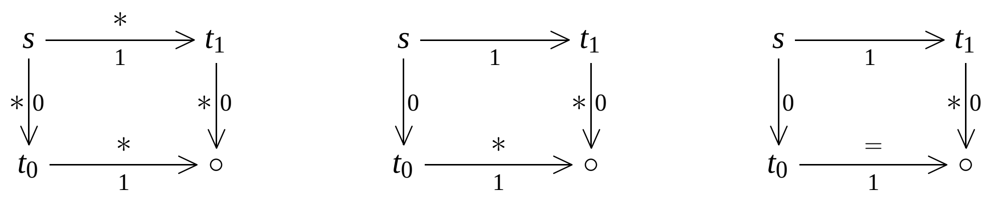
$$\forall s,t_0,t_1.\ \left(t_0 \overset{*}{\longleftarrow}_0 s \overset{*}{\longrightarrow}_1 t_1 \;\Rightarrow\; t_0 \overset{*}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t_1 \right).$$

$\longrightarrow_0$ and $\longrightarrow_1$ are *locally commuting* if

$$\forall s,t_0,t_1.\ \left(t_0 \longleftarrow_0 s \longrightarrow_1 t_1 \;\Rightarrow\; t_0 \overset{*}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t_1 \right).$$

$\longrightarrow_1$ *strongly commutes over* $\longrightarrow_0$ if

$$\forall s,t_0,t_1.\ \left(t_0 \longleftarrow_0 s \longrightarrow_1 t_1 \;\Rightarrow\; t_0 \overset{=}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t_1 \right).$$



$\longrightarrow_0$ and $\longrightarrow_1$ are commuting

$\longrightarrow_0$ and $\longrightarrow_1$ are locally commuting

$\longrightarrow_1$ strongly commutes over $\longrightarrow_0$

A single relation $\longrightarrow$ is called *[locally] confluent* if $\longrightarrow$ and $\longrightarrow$ are [locally] commuting. It is called *strongly confluent* if $\longrightarrow$ strongly commutes over $\longrightarrow$. It is called *confluent below u* if $\forall v,w.\ (v \overset{*}{\longleftarrow} u \overset{*}{\longrightarrow} w \;\Rightarrow\; v{\downarrow}w)$.

Lemma 3.2 (Generalized Newman Lemma)
If $\longrightarrow_0$ and $\longrightarrow_1$ are commuting, then they are locally commuting, too.
Furthermore, if $\longrightarrow_0 \cup \longrightarrow_1$ is terminating or if $\longrightarrow_0$ or $\longrightarrow_1$ is transitive, then also the converse is true, i.e. $\longrightarrow_0$ and $\longrightarrow_1$ are commuting iff they are locally commuting.

Lemma 3.3
The following three properties are logically equivalent:

1. *$\longrightarrow_1$ strongly commutes over $\longrightarrow_0$.*
2. *$\longrightarrow_1$ strongly commutes over $\overset{+}{\longrightarrow}_0$.*
3. *$\longrightarrow_1$ strongly commutes over $\overset{*}{\longrightarrow}_0$.*

Moreover, each of them implies that $\longrightarrow_0$ and $\longrightarrow_1$ are commuting.

Lemma 3.4 (Church-Rosser)
Assume that $\longrightarrow$ is confluent. Now: $\overset{*}{\longleftrightarrow} \subseteq \downarrow$.

Besides strong confluence there are two other important versions of strengthened confluence for conditional rule systems. They are based on the depth of the reduction steps, i.e. on the β of $\longrightarrow_{\mathrm{R,X},\beta}$. Therefore they actually are properties of R,X instead of $\longrightarrow_{\mathrm{R,X}}$, unless one considers $\longrightarrow_{\mathrm{R,X}}$ to be the family $(\longrightarrow_{\mathrm{R,X},\beta})_{\beta\preceq\omega+\omega}$. These two strengthened versions of confluence are *shallow confluence* and *level confluence*. Their generalizations to our generalized framework here are called 0-*shallow confluence* for the closure w.r.t. our constructor rules, as well as ω-*shallow confluence* and ω-*level confluence* for our second closure. Shallow and level confluence are interesting: On the on hand, they provide us with stronger induction hypotheses for the proofs of our confluence criteria. On the other hand, the stronger confluence properties may be essential for certain kinds of reasoning with the specification of a rule system; for level joinability cf. Middeldorp & Hamoen (1994).

Before we define our notions of shallow and level confluence we present some operations on ordinal numbers:

Definition 3.5 ($+_0$, $+_\omega$, $\dot{-}$)
Let $\alpha \in \{0,\omega\}$. Let '$+$' be the addition of ordinal numbers.
Define '$+_0$', '$+_\omega$', and '$\dot{-}$' for $n_0, n_1 \prec \omega$:

$$
\begin{array}{lcl}
0{+_\alpha}n_1 & := & n_1 \\
n_0{+_\alpha}0 & := & n_0 \\
(n_0{+}1){+_\alpha}(n_1{+}1) & := & \alpha + n_0{+}1 + n_1{+}1 \\
 & & \\
(n_0{+}n_1)\dot{-}n_1 & := & n_0 \\
n_0\dot{-}(n_0{+}n_1) & := & 0
\end{array}
$$

Note that the subscript of the operator '$+_\omega$' is chosen to remind that it adds an extra ω to the left if both arguments are different from 0. Moreover, note that $\mathbf{N}{\times}\mathbf{N}{\restriction}{+_0} = \mathbf{N}{\times}\mathbf{N}{\restriction}{+}$. '$\dot{-}$' is sometimes called *monus*.

Since we want to use shallow and level confluence also for terminating reduction relations we have to parameterize them w.r.t. wellfounded orderings. Let '$\succ$' as before be the wellordering of the ordinal numbers. Let '$\rhd$' be some wellfounded ordering on $\mathcal{T}$. We denote the lexicographic combination of $\succ$ and $\rhd$ by '$\succ\!\rhd$', its reverse by '$\prec\!\lhd$', and the reflexive closure of the latter by '$\underline{\prec\!\lhd}$'.

Definition 3.6 (0-Shallow Confluent / ω-Shallow Confluent)
Let $\alpha \in \{0,\omega\}$. Let $\beta \preceq \omega{+}\omega$. Let $s \in \mathcal{T}$.
R, X is said to be α-*shallow confluent up to* β *and s in* ⊲ if

$$\forall n_0, n_1 \prec \omega.\ \forall u,v,w.\ \left(\begin{array}{l} \left(\begin{array}{l} (\ n_0 +_\alpha n_1,\ u\ ) \underline{\prec\lhd} (\ \beta,\ s\ ) \\ \wedge\ \ v \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} u \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} w \end{array} \right) \\ \Rightarrow\ v \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} w \end{array} \right).$$

R, X is said to be α-*shallow confluent up to* β if
R, X is α-shallow confluent up to β and[16] s for all $s \in \mathcal{T}$.
R, X is said to be α-*shallow confluent* if R, X is α-shallow confluent up to $\omega{+}\alpha$.

Definition 3.7 (ω-Level Confluent)
Let $\beta \preceq \omega$. Let $s \in \mathcal{T}$. R, X is said to be ω-*level confluent up to* β *and s in* ⊲ if

$$\forall n_0, n_1 \prec \omega.\ \forall u,v,w.\ \left(\begin{array}{l} \left(\begin{array}{l} (\ \max\{n_0,n_1\},\ u\ ) \underline{\prec\lhd} (\ \beta,\ s\ ) \\ \wedge\ \ v \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_0} u \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} w \end{array} \right) \\ \Rightarrow\ v \downarrow_{\mathrm{R,X},\omega+\max\{n_0,n_1\}} w \end{array} \right).$$

R, X is said to be ω-*level confluent up to* β if
R, X is ω-level confluent up to β and[16] s for all $s \in \mathcal{T}$.
R, X is said to be ω-*level confluent* if R, X is ω-level confluent up to ω.

Note that ω-level and ω-shallow confluence specialize to the standard definitions of level and shallow confluence, resp., for the case that all symbols are considered to be non-constructor symbols (where n becomes the standard depth of $\longrightarrow_{\mathrm{R,X},\omega+n}$); and that 0-shallow confluence specializes to the standard definition of shallow confluence for the case that all symbols are considered to be constructor symbols.

Corollary 3.8 (ω-Shallow Confluent ⇒ ω-Level Confluent ⇒ Confluent)
If R, X *is* ω-*shallow confluent, then* R, X *is* ω-*level confluent.*
If R, X *is* ω-*level confluent, then* $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

Corollary 3.9
R, X *is* ω-*shallow confluent up to* 0 *iff*
R, X *is* ω-*level confluent up to* 0 *iff*
$\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent.*

[16] Note that reference to a special ⊲ becomes irrelevant here

4 Critical Peaks

Critical peaks describe those possible sources of non-confluence that directly arise from the syntax of the given rule system. While the so-called *variable overlaps* can hardly be approached via syntactic means, the critical peaks describe the non-variable overlaps resulting from an instantiated left-hand side being subterm of an instantiated left-hand side at a non-variable position. Our critical peaks capture more information than the standard *critical pairs*: Besides the pair, they contain the peak term and its overlap position. Furthermore, each element of the pair is augmented with the condition that must be fulfilled for enabling the reduction step down from the peak term, and with a bit indicating whether the rule applied was a non-constructor rule or not.

Definition 4.1 (Critical Peak)
If the left-hand side of a rule $l_0{=}r_0{\longleftarrow}C_0$ and
the subterm at non-variable (i.e. $l_1/p \notin \mathrm{V}$) position $p \in \mathcal{POS}(l_1)$
of the left-hand side of a rule $l_1{=}r_1{\longleftarrow}C_1$
(assuming $\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0) \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$ w.l.o.g.[17]) are unifiable by

$$\sigma = \mathrm{mgu}(\ \{(l_0, l_1/p)\},\ \mathcal{V}(l_0{=}r_0{\longleftarrow}C_0, l_1{=}r_1{\longleftarrow}C_1)),$$

if (for $i \prec 2$) $\quad \Lambda_i = \left\{ \begin{array}{ll} 0 & \text{if } l_i \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}}) \\ 1 & \text{otherwise} \end{array} \right\}$,
and if the resulting critical pair is non-trivial (i.e. $l_1[p \leftarrow r_0]\sigma \neq r_1\sigma$), then

$$\big(\ (l_1[p \leftarrow r_0],\ C_0,\ \Lambda_0),\ \ (r_1,\ C_1,\ \Lambda_1),\ \ l_1,\ \ \sigma,\ \ p\ \big)$$

is a (non-trivial) *critical peak (of the form* (Λ_0, Λ_1)*)* consisting of the conditional critical pair, its peak term l_1, the most general unifier σ, and the overlap position p.

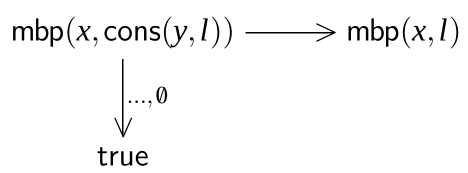
For convenience we usually identify this critical peak with its instantiated version

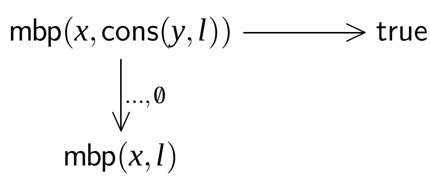
$$\big(\ (l_1[p \leftarrow r_0]\sigma,\ C_0\sigma,\ \Lambda_0),\ \ (r_1\sigma,\ C_1\sigma,\ \Lambda_1),\ \ l_1\sigma,\ \ p\ \big)$$

which should not lead to confusion because the tuple is shorter.

The set of all critical peaks of a CRS R is denoted by CP(R).

Example 4.2 *(continuing Example 2.3)*
$\mathrm{CP}(\mathrm{R}_{2.3})$ contains two critical peaks, namely (in the instantiated version)
$\big(\ (\mathsf{true}, (x{=}y), 1),\ (\mathsf{mbp}(x,l), (x{\neq}y), 1),\ \mathsf{mbp}(x, \mathsf{cons}(y,l)),\ \emptyset\ \big)$ and
$\big(\ (\mathsf{mbp}(x,l), (x{\neq}y), 1),\ (\mathsf{true}, (x{=}y), 1),\ \mathsf{mbp}(x, \mathsf{cons}(y,l)),\ \emptyset\ \big)$
which we would (partially) display as





Note that we omit the position at the arrow to the right because it is always $\emptyset$. Furthermore, note that the two critical peaks are different although they look similar. Namely, the one is the symmetric overlay (cf. below) of the other.

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[17]To achieve this, let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)] \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$ and then replace $l_0{=}r_0{\longleftarrow}C_0$ with $(l_0{=}r_0{\longleftarrow}C_0)\xi$.

5 Basic Forms of Joinability of Critical Peaks

A critical peak

$$((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)$$

is *joinable w.r.t.* R,X (for $\mathrm{X}\subseteq\mathrm{V}$) if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$
$$\left(\ ((D_0D_1)\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}) \Rightarrow t_0\sigma\varphi\downarrow_{\mathrm{R,X}}t_1\sigma\varphi\ \right).$$

It is an *overlay* if $p=\emptyset$. It is a *non-overlay* if $p\neq\emptyset$.
It is *overlay joinable w.r.t.* R,X if it is joinable w.r.t. R,X and is an overlay.

In the following two definitions ‘true’ and ‘false’ denote two arbitrary irreducible ground terms. Their special names have only been chosen to make clear the intuition behind.

The above critical peak is *complementary w.r.t.* R,X if

$$\left(\begin{array}{ll} & \exists u,v\in\mathcal{T}.\ \exists i\prec 2.\ \left(\begin{array}{ll} & (u{=}v) \text{ occurs in } D_i\sigma \\ \wedge & (u{\neq}v) \text{ occurs in } D_{1-i}\sigma \end{array}\right) \\ \vee & \exists p\in\mathcal{T}.\ \exists \mathsf{true},\mathsf{false}\in\mathcal{GT}\setminus\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}).\ \exists i\prec 2.\ \left(\begin{array}{ll} & (p{=}\mathsf{true}) \text{ occurs in } D_i\sigma \\ \wedge & (p{=}\mathsf{false}) \text{ occurs in } D_{1-i}\sigma \\ \wedge & \mathsf{true}\neq\mathsf{false} \end{array}\right) \end{array}\right).$$

It is *weakly complementary w.r.t.* R,X if

$$\left(\begin{array}{ll} & \exists u,v\in\mathcal{T}.\ \left(\begin{array}{l} (u{=}v) \text{ and} \\ (u{\neq}v) \text{ occur in } (D_0D_1)\sigma \end{array}\right) \\ \vee & \exists p\in\mathcal{T}.\ \exists \mathsf{true},\mathsf{false}\in\mathcal{GT}\setminus\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}).\ \left(\begin{array}{ll} & (p{=}\mathsf{true}) \text{ and} \\ & (p{=}\mathsf{false}) \text{ occur in } (D_0D_1)\sigma \\ \wedge & \mathsf{true}\neq\mathsf{false} \end{array}\right) \end{array}\right).$$

It is *strongly joinable w.r.t.* R,X if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$
$$\left(\ ((D_0D_1)\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}) \Rightarrow t_0\sigma\varphi\downarrow\!\downarrow_{\mathrm{R,X}}t_1\sigma\varphi\ \right).$$

In the following definition ‘A’ is an arbitrary function from positions to sets of terms.

The above critical peak is $\rhd$*-weakly joinable w.r.t.* R,X *[besides A]* if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\left(\begin{array}{ll} & (D_0D_1)\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}} \\ \wedge & \forall u.\ (\ u\lhd\hat{t}\sigma\varphi\ \Rightarrow\ \longrightarrow_{\mathrm{R,X}} \text{ is confluent below } u\ ) \\ \wedge & \forall x\in\mathrm{V}.\ x\varphi\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \\ \wedge & (\ p\neq\emptyset\ \Rightarrow\ \forall x\in\mathcal{V}(\hat{t}).\ x\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})\ ) \\ {[}\wedge & \hat{t}\sigma\varphi\notin A(p)\quad{]} \end{array}\right) \Rightarrow\ t_0\sigma\varphi\downarrow_{\mathrm{R,X}}t_1\sigma\varphi\ \right).$$

Note that $\rhd$-weak joinability adds several useful features to the single condition of joinability, forming a conjunctive condition list. The first new feature allows to assume confluence below all terms that are strictly smaller than the peak term. The following features allow us to assume some irreducibilities for the joinability test, where the optional one is an interface that is to be specified by the confluence criteria using it, cf. our theorems 14.2 and 14.4. For a demonstration of the usefulness of these additional features cf. Example 14.3.

Lemma 5.1 (Joinability of Critical Peaks is Necessary for Confluence)
If $\longrightarrow_{\mathrm{R,X}}$ *is confluent, then all critical peaks in* $\mathrm{CP}(\mathrm{R})$ *are joinable w.r.t.* R,X.

6 Basic Forms of Shallow and Level Joinability

Just like confluence and strong confluence, also level and shallow confluence have their corresponding joinability notion. Sorry to say, they are pretty complicated, however.

Definition 6.1 (0-Shallow Joinable / ω-Shallow Joinable)
Let $\alpha \in \{0,\omega\}$. Let $\beta \preceq \omega{+}\alpha$. Let $s \in \mathcal{T}$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ \sigma,\ p)$ is α-*shallow joinable up to* β *and* s *w.r.t.* R, X *and* $\lhd$ *[besides A]* if
$\forall \varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})).\ \forall n_0, n_1 \prec \omega.$

$$\left(\begin{array}{l} \left(\begin{array}{l} (\, n_0 +_\alpha n_1,\ \hat{t}\sigma\varphi\,) \,\underline{\prec\lhd}\, (\,\beta,\ s\,) \\ \wedge\ \ \forall i \prec 2.\ \left(\begin{array}{l} (\,\alpha = 0 \ \Rightarrow\ \Lambda_i = 0 \prec n_i\,) \\ \wedge\ \ (\,\alpha=\omega \ \Rightarrow\ \Lambda_i \preceq n_i\,) \\ \wedge\ \ D_i\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\alpha+(n_i \dot{-} 1)} \end{array}\right) \\ \wedge\ \ \forall(\,\delta,\ s'\,) \prec\lhd (\, n_0 +_\alpha n_1,\ \hat{t}\sigma\varphi\,).\ \left(\begin{array}{l}\mathrm{R,X} \text{ is } \alpha\text{-shallow confluent} \\ \text{up to } \delta \text{ and } s' \text{ in } \lhd\end{array}\right) \\ \wedge\ \ \forall x \in \mathrm{V}.\ x\varphi \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\alpha+\min\{n_0,n_1\}}) \\ \wedge\ \ \left(\, p \neq \emptyset \ \Rightarrow\ \forall x \in \mathcal{V}(\hat{t}).\ x\sigma\varphi \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\alpha+\min\{n_0,n_1\}})\,\right) \\ [\wedge\ \ \hat{t}\sigma\varphi \notin A(p, \min\{n_0,n_1\})\quad] \end{array}\right) \\ \Rightarrow\ \ \left(\, t_0\sigma\varphi \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_1\sigma\varphi\,\right)\end{array}\right).$$

It is called α-*shallow joinable up to* β *w.r.t.* R, X *and* $\lhd$ *[besides A]* if
it is α-shallow joinable up to β and s w.r.t. R, X and $\lhd$ [besides A] for all $s \in \mathcal{T}$.
It is called α-*shallow joinable w.r.t.* R, X *and* $\lhd$ *[besides A]* if
it is α-shallow joinable up to $\omega{+}\alpha$ w.r.t. R, X and $\lhd$ [besides A].
When $\lhd$ is not specified, we tacitly assume it to be $\lhd_{\mathrm{ST}}$.

Definition 6.2 (ω-Level Joinable)
Let $\beta \preceq \omega$. Let $s \in \mathcal{T}$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ \sigma,\ p)$ is ω-*level joinable up to* β *and* s *w.r.t.* R, X *and* $\lhd$ *[besides A]* if
$\forall \varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})).\ \forall n_0, n_1 \prec \omega.$

$$\left(\begin{array}{l} \left(\begin{array}{l} (\, \max\{n_0,n_1\},\ \hat{t}\sigma\varphi\,) \,\underline{\prec\lhd}\, (\,\beta,\ s\,) \\ \wedge\ \ \forall i \prec 2.\ \left(\begin{array}{l} \Lambda_i \preceq n_i \\ \wedge\ \ D_i\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n_i \dot{-} 1)} \end{array}\right) \\ \wedge\ \ \forall(\,\delta,\ s'\,) \prec\lhd (\, \max\{n_0,n_1\},\ \hat{t}\sigma\varphi\,).\ \left(\begin{array}{l}\mathrm{R,X} \text{ is } \omega\text{-level confluent} \\ \text{up to } \delta \text{ and } s' \text{ in } \lhd\end{array}\right) \\ \wedge\ \ \forall x \in \mathrm{V}.\ x\varphi \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\max\{n_0,n_1\}}) \\ \wedge\ \ \left(\, p \neq \emptyset \ \Rightarrow\ \forall x \in \mathcal{V}(\hat{t}).\ x\sigma\varphi \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\max\{n_0,n_1\}})\,\right) \\ [\wedge\ \ \hat{t}\sigma\varphi \notin A(p, \max\{n_0,n_1\})\quad] \end{array}\right) \\ \Rightarrow\ \ \left(\, t_0\sigma\varphi \downarrow_{\mathrm{R,X},\omega+\max\{n_0,n_1\}} t_1\sigma\varphi\,\right)\end{array}\right).$$

It is called ω-*level joinable up to* β *w.r.t.* R, X *and* $\lhd$ *[besides A]* if
it is ω-level joinable up to β and s w.r.t. R, X and $\lhd$ [besides A] for all $s \in \mathcal{T}$.
It is called ω-*level joinable w.r.t.* R, X *and* $\lhd$ *[besides A]* if
it is ω-level joinable up to ω w.r.t. R, X and $\lhd$ [besides A].
When $\lhd$ is not specified, we tacitly assume it to be $\lhd_{\mathrm{ST}}$.

Please notice the generic structure of these and the following definitions that makes them actually less complicated than they look like. While the conclusions of their implications should be clear, the elements of their conjunctive condition lists have the following purposes: The first just parameterizes the notion in β and s. The second requires the appropriate fulfilledness of the conditions of the critical peak, where $\Lambda_i \preceq n_i$ allows us to assume $1 \preceq n_i$ when the term t_i is generated by a non-constructor rule which is important since otherwise the conclusion is very unlikely to be fulfilled, cf. also below. The third allows us to assume a certain confluence property which can be applied when checking the fulfilledness of the conditions. E.g., this condition sometimes implies that the fulfilledness assumptions of the second element for "$i = 0$" and "$i = 1$" are contradictory. An example for this are the critical peaks of Example 4.2 which are both ω-level and ω-shallow confluent since the condition list can never be fulfilled. But how do we know that? Suppose that $(x{=}y)\varphi$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+(n_0\dot{-}1)}$ and that $(x{\neq}y)\varphi$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+(n_1\dot{-}1)}$. Then there are $\hat{u},\hat{v} \in \mathcal{GT}(\mathrm{cons})$ such that $x\varphi\downarrow_{\mathrm{R,X},\omega+(n_0\dot{-}1)}y\varphi$ and $x\varphi\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+(n_1\dot{-}1)}\hat{u}\not\downarrow_{\mathrm{R,X},\omega+(n_1\dot{-}1)}\hat{v}\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+(n_1\dot{-}1)}y\varphi$. By $x,y \in \mathrm{V}_{\mathcal{C}}$ we get $x\varphi, y\varphi \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and thus by Lemma 2.10 we get $x\varphi\downarrow_{\mathrm{R,X},\omega}y\varphi$ and $x\varphi\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}\hat{u}\not\downarrow_{\mathrm{R,X},\omega}\hat{v}\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega}y\varphi$. This contradicts confluence of $\longrightarrow_{\mathrm{R,X},\omega}$ and then by Corollary 3.9 it also contradicts ω-level and ω-shallow confluence up to 0. However, we are allowed to assume this since we know $0 \prec \max\{n_0, n_1\}$ and $0 \prec n_0 +_\omega n_1$ due to $\Lambda_0 = \Lambda_1 = 1$ (and $\Lambda_i \preceq n_i$). A more general argumentation of this kind proves theorems 13.3, 13.4, and 15.3, which are confluence criteria for rule systems with complementary critical peaks. Finally, the following items in the conjunctive condition lists allow us to assume some irreducibilities similar to those for $\rhd$-weak joinability but less powerful.

Lemma 6.3 (α-Shallow Joinability is Necessary for α-Shallow Confluence)
Let $\alpha \in \{0, \omega\}$. *If* $\mathrm{R,X}$ *is* α*-shallow confluent [up to* β *[and s in* $\lhd$*]], then all critical peaks in* $\mathrm{CP(R)}$ *are* α*-shallow joinable [up to* β *[and s]] w.r.t.* $\mathrm{R,X}$ *[[and* $\lhd$*]].*

Lemma 6.4 (ω-Level Joinability is Necessary for ω-Level Confluence)
If $\mathrm{R,X}$ *is* ω*-level confluent [up to* β *[and s in* $\lhd$*]], then all critical peaks in* $\mathrm{CP(R)}$ *are* ω*-level joinable [up to* β *[and s]] w.r.t.* $\mathrm{R,X}$ *[[and* $\lhd$*]].*

7 Sophisticated Forms of Shallow Joinability

For a first reading this section should only be skimmed and its definitions looked up by need. At least § 12 should be read before.[18]

The ω-shallow joinability notions of this section are only necessary for understanding the sophisticated Theorem 13.6 and its interrelation with the examples in the following sections, but not for the important practical consequence of this theorem, namely Theorem 13.3, which is easy to understand and sufficient for many practical applications. The 0-shallow joinability notions are needed for Theorem 15.1 only.

The following notion will be applied for non-overlays of the forms $(1,0)$ and $(1,1)$ for "$\alpha = \omega$" and of the form $(0,0)$ for "$\alpha = 0$":

Definition 7.1 (0-Shallow Parallel Closed / ω-Shallow Parallel Closed)
Let $\alpha \in \{0,\omega\}$. Let $\beta \preceq \omega{+}\alpha$. A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$ is α-*shallow parallel closed up to* β *w.r.t.* R,X if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n_0,n_1 \prec \omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & 0 \prec n_0 \succeq n_1 \\ \wedge & n_0 +_\alpha n_1 \preceq \beta \\ \wedge & \forall i \prec 2.\ \left(\begin{array}{ll} & (\ \alpha = 0\ \Rightarrow\ \Lambda_i = 0 \prec n_i\ ) \\ \wedge & (\ \alpha = \omega\ \Rightarrow\ \Lambda_i \preceq n_i\ ) \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R},\mathrm{X},\alpha+(n_i \dot{-} 1)} \end{array}\right) \\ \wedge & \forall\delta \prec n_0 +_\alpha n_1.\ \mathrm{R},\mathrm{X} \text{ is } \alpha\text{-shallow confluent up to } \delta \end{array}\right) \\ \Rightarrow & \left(\begin{array}{ll} & (\ n_1 = 0\ \Rightarrow\ t_0\varphi \mathrel{-\!\!\!\parallel\!\!\!\rightarrow}_{\mathrm{R},\mathrm{X},\alpha} t_1\varphi\ ) \\ \wedge & t_0\varphi \mathrel{-\!\!\!\parallel\!\!\!\rightarrow}_{\mathrm{R},\mathrm{X},\alpha+n_1} \circ \overset{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\alpha+(n_1 \dot{-} 1)} \circ \overset{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\alpha} t_1\varphi \end{array}\right) \end{array}\right).$$

It is called α-*shallow parallel closed w.r.t.* R,X if
it is α-shallow parallel closed up to $\omega{+}\alpha$ w.r.t. R,X.

The following notion will be applied for critical peaks of the forms $(0,1)$ and $(1,1)$ for "$\alpha = \omega$" and of the form $(0,0)$ for "$\alpha = 0$":

Definition 7.2 (0-Shallow / ω-Shallow [Noisy] Parallel Joinable)
Let $\alpha \in \{0,\omega\}$. Let $\beta \preceq \omega{+}\alpha$. A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$ is α-*shallow* [*noisy*] *parallel joinable up to* β *w.r.t.* R,X if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n_0,n_1 \prec \omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & n_0 \preceq n_1 \succ 0 \\ \wedge & n_0 +_\alpha n_1 \preceq \beta \\ \wedge & \forall i \prec 2.\ \left(\begin{array}{ll} & (\ \alpha = 0\ \Rightarrow\ \Lambda_i = 0 \prec n_i\ ) \\ \wedge & (\ \alpha = \omega\ \Rightarrow\ \Lambda_i \preceq n_i\ ) \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R},\mathrm{X},\alpha+(n_i \dot{-} 1)} \end{array}\right) \\ \wedge & \forall\delta \prec n_0 +_\alpha n_1.\ \mathrm{R},\mathrm{X} \text{ is } \alpha\text{-shallow confluent up to } \delta \end{array}\right) \\ \Rightarrow & t_0\varphi \mathrel{-\!\!\!\parallel\!\!\!\rightarrow}_{\mathrm{R},\mathrm{X},\alpha+n_1} \circ \overset{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\alpha[+(n_1 \dot{-} 1)]} \circ \overset{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\alpha+n_0} t_1\varphi \end{array}\right).$$

It is called α-*shallow* [*noisy*] *parallel joinable w.r.t.* R,X if
it is α-shallow [noisy] parallel joinable up to $\omega{+}\alpha$ w.r.t. R,X.

[18] We put this section here because we do not want to scatter our later discussion with a big definition section and because we do not want to use the (for a first reading not essential) joinability labels in the boxes of the examples in the following sections before defining them.

Note that α-shallow parallel closedness specializes to the standard definition of parallel closedness of Huet (1980) for the case that all symbols are considered to be non-constructor symbols in case of $\alpha = \omega$ (or else constructor symbols in case of $\alpha = 0$) and the rule system is unconditional (since then $\longrightarrow_{\mathrm{R,X},\alpha} = \emptyset$ and $\longrightarrow_{\mathrm{R,X},\alpha+1} = \longrightarrow_{\mathrm{R,X}}$). Similarly, α-shallow parallel joinability specializes for these cases to the joinability required for overlays in Toyama (1988). Moreover, note that the notions whose names end with "closed" are always restricted to "$0 \prec n_0 \succeq n_1$", whereas those whose names end with "joinable" are always restricted to "$n_0 \preceq n_1 \succ 0$". Finally, note that some notions have "noisy" variants which are weaker since they allow some "noise", i.e. some reduction on a smaller depth than the preceding reduction step.[19]

The following notion will be applied for non-overlays of the forms $(1,0)$ and $(1,1)$ for "$\alpha = \omega$" and of the form $(0,0)$ for "$\alpha = 0$":

Definition 7.3 (0-Shallow / ω-Shallow [Noisy] Anti-Closed)
Let $\alpha \in \{0, \omega\}$. Let $\beta \preceq \omega{+}\alpha$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$ is α-*shallow* [*noisy*] *anti-closed up to* β *w.r.t.* R, X if $\forall \varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})).\ \forall n_0, n_1 \prec \omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & 0 \prec n_0 \succeq n_1 \\ \wedge & n_0 +_\alpha n_1 \preceq \beta \\ \wedge & \forall i \prec 2.\ \left(\begin{array}{ll} & (\ \alpha = 0 \ \Rightarrow\ \Lambda_i = 0 \prec n_i\ ) \\ \wedge & (\ \alpha = \omega \ \Rightarrow\ \Lambda_i \preceq n_i\ ) \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\alpha+(n_i \dot{-} 1)} \end{array}\right) \\ \wedge & \forall \delta \prec n_0 +_\alpha n_1.\ \mathrm{R,X} \text{ is } \alpha\text{-shallow confluent up to } \delta \end{array}\right) \\ \Rightarrow & \left(\begin{array}{ll} & \left(\ n_1 = 0 \ \Rightarrow\ t_0\varphi \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha[+(n_0 \dot{-} 1)]} \circ \overset{=}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_1\varphi\ \right) \\ \wedge & t_0\varphi \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha[+(n_0 \dot{-} 1)]} \circ \overset{=}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha} t_1\varphi \end{array}\right) \end{array}\right).$$

It is called α-*shallow* [*noisy*] *anti-closed w.r.t.* R, X if
it is α-shallow [noisy] anti-closed up to $\omega{+}\alpha$ w.r.t. R, X.

The following notion will be applied for critical peaks of the form $(0,1)$ and $(1,1)$ for "$\alpha = \omega$" and of the form $(0,0)$ for "$\alpha = 0$":

Definition 7.4 (0-Shallow / ω-Shallow [Noisy] Strongly Joinable)
Let $\alpha \in \{0, \omega\}$. Let $\beta \preceq \omega{+}\alpha$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$ is α-*shallow* [*noisy*] *strongly joinable up to* β *w.r.t.* R, X if $\forall \varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})).\ \forall n_0, n_1 \prec \omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & n_0 \preceq n_1 \succ 0 \\ \wedge & n_0 +_\alpha n_1 \preceq \beta \\ \wedge & \forall i \prec 2.\ \left(\begin{array}{ll} & (\ \alpha = 0 \ \Rightarrow\ \Lambda_i = 0 \prec n_i\ ) \\ \wedge & (\ \alpha = \omega \ \Rightarrow\ \Lambda_i \preceq n_i\ ) \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\alpha+(n_i \dot{-} 1)} \end{array}\right) \\ \wedge & \forall \delta \prec n_0 +_\alpha n_1.\ \mathrm{R,X} \text{ is } \alpha\text{-shallow confluent up to } \delta \end{array}\right) \\ \Rightarrow & \left(\begin{array}{ll} & \left(\ n_0 = 0 \ \Rightarrow\ t_0\varphi \overset{=}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha[+(n_1 \dot{-} 1)]} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha} t_1\varphi\ \right) \\ \wedge & t_0\varphi \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha} \circ \overset{=}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha[+(n_1 \dot{-} 1)]} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_1\varphi \end{array}\right) \end{array}\right).$$

It is called α-*shallow* [*noisy*] *strongly joinable w.r.t.* R, X if
it is α-shallow [noisy] strongly joinable up to $\omega{+}\alpha$ w.r.t. R, X.

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[19] The name for the notion was inspired by Oostrom (1994a).

The following notion will be applied for non-overlays of the forms $(1,0)$ and $(1,1)$:

Definition 7.5 (ω-Shallow Closed)
Let $\beta \preceq \omega{+}\omega$. A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$ is ω-*shallow closed up to* β *w.r.t.* R, X if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n_0,n_1\prec\omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & 0\prec n_0\succeq n_1 \\ \wedge & n_0{+_\omega}n_1\preceq\beta \\ \wedge & \forall i\prec 2.\left(\begin{array}{ll} & \Lambda_i\preceq n_i \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n_i\dot{-}1)}\end{array}\right) \\ \wedge & \forall\delta\prec n_0{+_\omega}n_1.\ \text{R, X is }\omega\text{-shallow confluent up to }\delta\end{array}\right) \\ \Rightarrow & \left(\begin{array}{ll} & \left(\ n_1=0\ \Rightarrow\ t_0\varphi\overset{=}{\longrightarrow}_{\mathrm{R,X},\omega}\circ\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega}t_1\varphi\ \right) \\ \wedge & t_0\varphi\overset{=}{\longrightarrow}_{\mathrm{R,X},\omega+n_1}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+(n_1\dot{-}1)}\circ\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega}t_1\varphi\end{array}\right)\end{array}\right).$$

It is called ω-*shallow closed w.r.t.* R, X if
it is ω-shallow closed up to $\omega{+}\omega$ w.r.t. R, X.

The following notion will be applied for critical peaks of the forms $(0,1)$ and $(1,1)$:

Definition 7.6 (ω-Shallow [Noisy] Weak Parallel Joinable)
Let $\beta \preceq \omega{+}\omega$. A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$ is ω-*shallow* [*noisy*] *weak parallel joinable up to* β *w.r.t.* R, X if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n_0,n_1\prec\omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & n_0\preceq n_1\succ 0 \\ \wedge & n_0{+_\omega}n_1\preceq\beta \\ \wedge & \forall i\prec 2.\left(\begin{array}{ll} & \Lambda_i\preceq n_i \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n_i\dot{-}1)}\end{array}\right) \\ \wedge & \forall\delta\prec n_0{+_\omega}n_1.\ \text{R, X is }\omega\text{-shallow confluent up to }\delta\end{array}\right) \\ \Rightarrow & t_0\varphi\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}\circ\ \longrightarrow\!\!\!\!\!\!\!+\!\!\!+_{\mathrm{R,X},\omega+n_1}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega[+(n_1\dot{-}1)]}\circ\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_0}t_1\varphi\end{array}\right).$$

It is called ω-*shallow* [*noisy*] *weak parallel joinable w.r.t.* R, X if
it is ω-shallow [noisy] weak parallel joinable up to $\omega{+}\omega$ w.r.t. R, X.

The following are corollaries of Corollary 2.14:

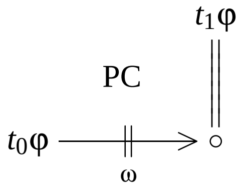
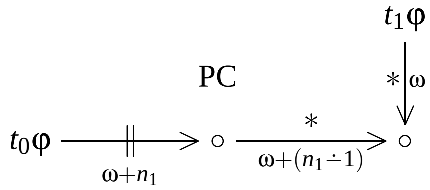
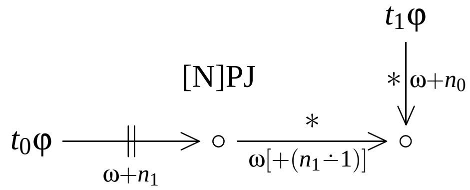
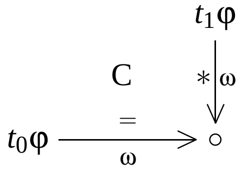
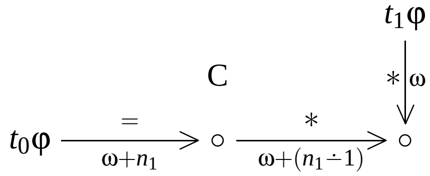
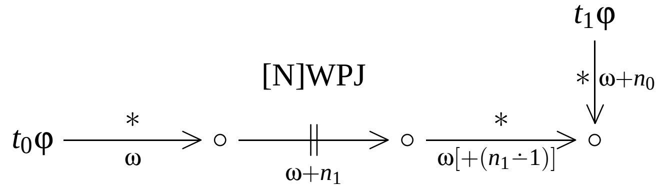
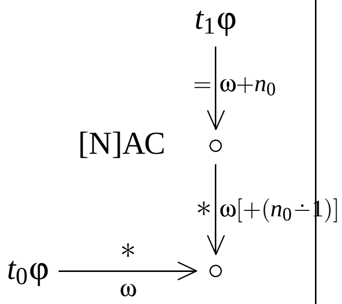
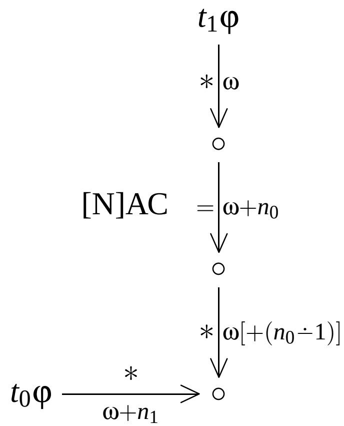
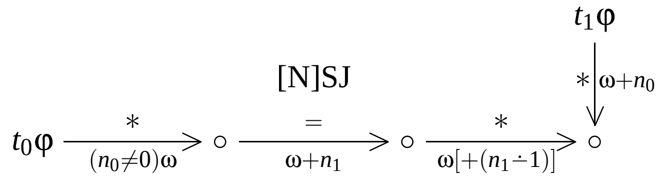
Corollary 7.7 *Let* $\alpha\in\{0,\omega\}$. *Now w.r.t.* R, X *the following holds:*
If a critical peak is ω-*shallow* [*noisy*] *parallel joinable up to* $\beta\preceq\omega{+}\omega$,
then it is ω-*shallow* [*noisy*] *weak parallel joinable up to* β.
If a critical peak is ω-*shallow* [*noisy*] *strongly joinable up to* $\beta\preceq\omega{+}\omega$,
then it is ω-*shallow* [*noisy*] *weak parallel joinable up to* β.
If a critical peak is α-*shallow* [*noisy*] *strongly joinable up to* $\beta\preceq\omega$,
then it is α-*shallow* [*noisy*] *parallel joinable up to* β.

Corollary 7.8 *Let* $\alpha\in\{0,\omega\}$. *Let* $\beta\preceq\omega{+}\alpha$. *Now w.r.t.* R, X *the following holds:*
If a critical peak is α-*shallow parallel closed or (for* $\alpha=\omega$*)* α-*shallow closed up to* β, *then it is* α-*shallow* [*noisy*] *anti-closed up to* β.

Overview over sophisticated forms of ω-Shallow … of $\big((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p\big)$

Generally assumed condition for $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; $n_0, n_1 \prec \omega$:
$$\left(\begin{array}{l} \quad\text{“Property 1”} \ \wedge\ n_0 {+_\omega} n_1 \preceq \beta \\ \wedge\ \ \forall i \prec 2.\ \big(\ \Lambda_i \preceq n_i\ \wedge\ D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n_i \dot{-} 1)}\ \big) \\ \wedge\ \ \forall \delta \prec n_0 {+_\omega} n_1.\ \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \delta \end{array}\right)$$

Required conclusion (P := Parallel; C := Closed; N := Noisy; J := Joinable; W := Weak; A := Anti-; S := Strongly):

Property 1 := …	$0 \prec n_0 \succeq n_1$		$n_0 \preceq n_1 \succ 0$
In case of …	$n_1 = 0$	$n_1 \succ 0$	
	PC 	PC 	[N]PJ 
	C 	C 	[N]WPJ 
	[N]AC 	[N]AC 	[N]SJ 

8 Sophisticated Forms of Level Joinability

For a first reading this section should only be skimmed and its definitions looked up by need. At least § 7 should be read before.

This section is only necessary for understanding the sophisticated Theorem 13.9 and its interrelation with the examples in the following sections, but not for the easy to understand consequence of this theorem, namely Theorem 13.4.

Having completed our special notions for shallow confluence, we now present some for level confluence.

The following notion will be applied for non-overlays of the form $(1,1)$:

Definition 8.1 (ω-Level Parallel Closed)
Let $\beta \preceq \omega$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$ is
ω-*level parallel closed up to* β *w.r.t.* R, X if
$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n\prec\omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & 0\prec n \\ \wedge & n\preceq\beta \\ \wedge & \forall i\prec 2.\left(\begin{array}{ll} & \Lambda_i\preceq n \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n\dot{-}1)}\end{array}\right) \\ \wedge & \forall\delta\prec n.\ \mathrm{R,X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \omega\end{array}\right) \\ \Rightarrow & t_0\varphi \mathrel{-\!\!+\!\!\!+\!\!\!\rightarrow}_{\mathrm{R,X},\omega+n}\circ\stackrel{*}{\longrightarrow}_{\mathrm{R,X},\omega}\circ\stackrel{*}{\longleftarrow}_{\mathrm{R,X},\omega} t_1\varphi\end{array}\right).$$

It is called ω-*level parallel closed w.r.t.* R, X if
it is ω-level parallel closed up to ω w.r.t. R, X.

The following notion will be applied for critical peaks of the form $(1,1)$:

Definition 8.2 (ω-Level Parallel Joinable)
Let $\beta \preceq \omega$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$ is
ω-*level parallel joinable up to* β *w.r.t.* R, X if
$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n\prec\omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & n\succ 0 \\ \wedge & n\preceq\beta \\ \wedge & \forall i\prec 2.\left(\begin{array}{ll} & \Lambda_i\preceq n \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n\dot{-}1)}\end{array}\right) \\ \wedge & \forall\delta\prec n.\ \mathrm{R,X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \omega\end{array}\right) \\ \Rightarrow & t_0\varphi \mathrel{-\!\!+\!\!\!+\!\!\!\rightarrow}_{\mathrm{R,X},\omega+n}\circ\stackrel{*}{\longrightarrow}_{\mathrm{R,X},\omega}\circ\stackrel{*}{\longleftarrow}_{\mathrm{R,X},\omega+n} t_1\varphi\end{array}\right).$$

It is called ω-*level parallel joinable w.r.t.* R, X if
it is ω-level parallel joinable up to ω w.r.t. R, X.

The following notion will be applied for non-overlays of the form $(1,1)$:

Definition 8.3 (ω-Level Anti-Closed)
Let $\beta \preceq \omega$. A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$
is ω-*level anti-closed up to* β *w.r.t.* R,X if
$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n \prec \omega.$

$$\left(\left(\begin{array}{ll} & 0 \prec n \\ \wedge & n \preceq \beta \\ \wedge & \forall i \prec 2. \left(\begin{array}{ll} & \Lambda_i \preceq n \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R},\mathrm{X},\omega+(n\dot{-}1)} \end{array}\right) \\ \wedge & \forall\delta\prec n.\ \mathrm{R},\mathrm{X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R},\mathrm{X} \text{ is } \omega\text{-shallow confluent up to } \omega \end{array}\right) \Rightarrow\ t_0\varphi \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \stackrel{=}{\longleftarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\omega} t_1\varphi\right).$$

It is called ω-*level anti-closed w.r.t.* R,X if
it is ω-level anti-closed up to ω w.r.t. R,X.

The following notion will be applied for critical peaks of the form $(1,1)$:

Definition 8.4 (ω-Level Strongly Joinable)
Let $\beta \preceq \omega$. A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$
is ω-*level strongly joinable up to* β *w.r.t.* R,X if
$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall n \prec \omega.$

$$\left(\left(\begin{array}{ll} & n \succ 0 \\ \wedge & n \preceq \beta \\ \wedge & \forall i \prec 2. \left(\begin{array}{ll} & \Lambda_i \preceq n \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R},\mathrm{X},\omega+(n\dot{-}1)} \end{array}\right) \\ \wedge & \forall\delta\prec n.\ \mathrm{R},\mathrm{X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R},\mathrm{X} \text{ is } \omega\text{-shallow confluent up to } \omega \end{array}\right) \Rightarrow\ t_0\varphi \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \stackrel{=}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \stackrel{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\omega+n} t_1\varphi\right).$$

It is called ω-*level strongly joinable w.r.t.* R,X if
it is ω-level strongly joinable up to ω w.r.t. R,X.

The following notion will be applied for non-overlays of the form $(1,1)$:

Definition 8.5 (ω-Level Closed)
Let $\beta \preceq \omega$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$
is ω-*level closed up to* β *w.r.t.* R, X if
$\forall \varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})).\ \forall n \prec \omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & 0 \prec n \\ \wedge & n \preceq \beta \\ \wedge & \forall i \prec 2. \left(\begin{array}{ll} & \Lambda_i \preceq n \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n\dot{-}1)} \end{array}\right) \\ \wedge & \forall \delta \prec n.\ \mathrm{R,X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \omega \end{array}\right) \\ \Rightarrow & t_0\varphi \overset{=}{\longrightarrow}_{\mathrm{R,X},\omega+n} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega} t_1\varphi \end{array}\right).$$

It is called ω-*level closed w.r.t.* R, X if
it is ω-level closed up to ω w.r.t. R, X.

The following notion will be applied for critical peaks of the form $(1,1)$:

Definition 8.6 (ω-Level Weak Parallel Joinable)
Let $\beta \preceq \omega$. A critical peak $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$
is ω-*level weak parallel joinable up to* β *w.r.t.* R, X if
$\forall \varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X})).\ \forall n \prec \omega.$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & n \succ 0 \\ \wedge & n \preceq \beta \\ \wedge & \forall i \prec 2. \left(\begin{array}{ll} & \Lambda_i \preceq n \\ \wedge & D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n\dot{-}1)} \end{array}\right) \\ \wedge & \forall \delta \prec n.\ \mathrm{R,X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \omega \end{array}\right) \\ \Rightarrow & t_0\varphi \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega} \circ \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\mathrm{R,X},\omega+n} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n} t_1\varphi \end{array}\right).$$

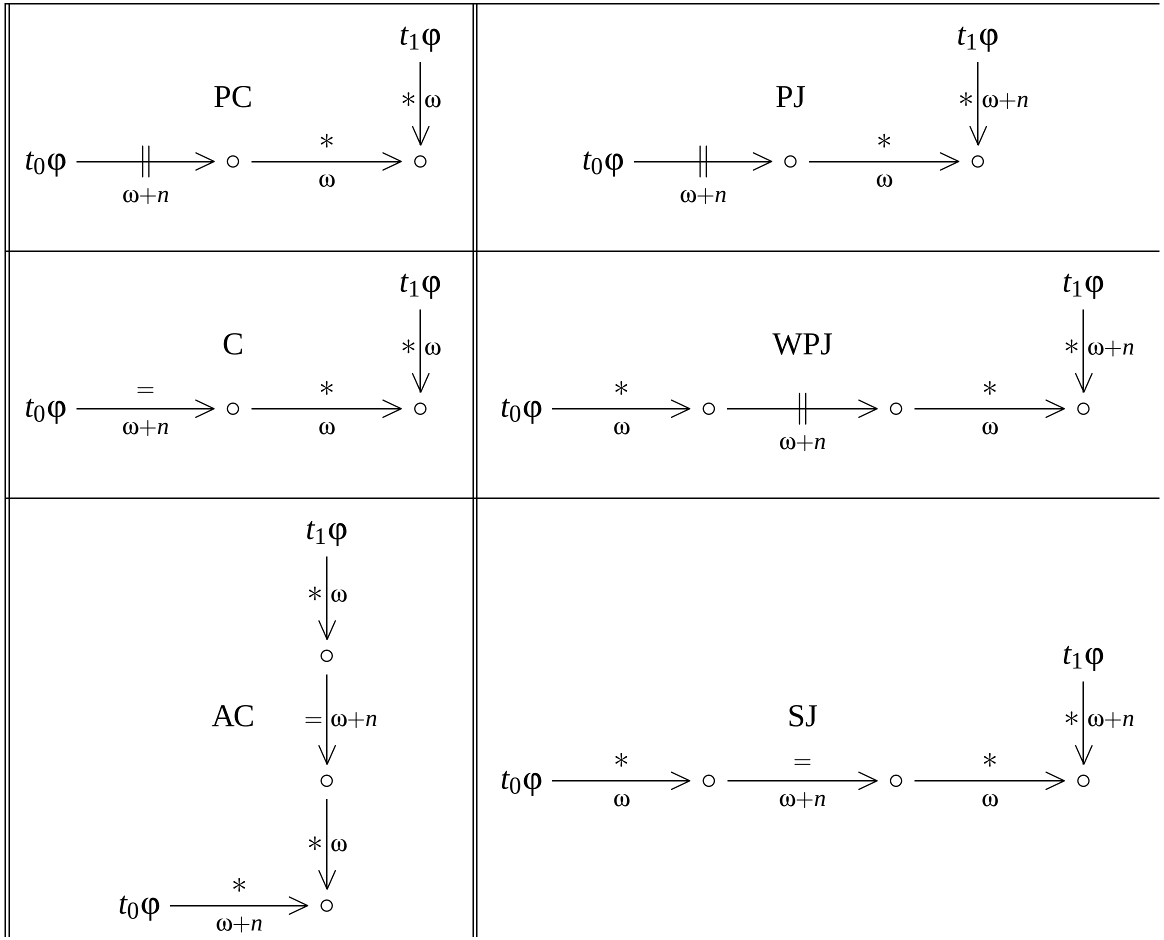
It is called ω-*level weak parallel joinable w.r.t.* R, X if
it is ω-level weak parallel joinable up to ω w.r.t. R, X.

Overview over sophisticated forms of ω-Level … of $((t_0, D_0, \Lambda_0),\ (t_1, D_1, \Lambda_1),\ \hat{t},\ p)$

Generally assumed condition for $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; $n \prec \omega$:

$$\left(\begin{array}{ll} & 0 \prec n \ \wedge\ n \preceq \beta \\ \wedge & \forall i \prec 2.\ \big(\ \Lambda_i \preceq n\ \wedge\ D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+(n\dot{-}1)}\ \big) \\ \wedge & \forall \delta \prec n.\ \mathrm{R,X} \text{ is } \omega\text{-level confluent up to } \delta \\ \wedge & \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \omega \end{array}\right)$$

Required conclusion (P := Parallel; C := Closed; J := Joinable; W := Weak; A := Anti-; S := Strongly):



9 Quasi Overlay Joinability

According to Theorem 4 of Dershowitz &al. (1988), a terminating positive conditional rule system is confluent if it is overlay joinable. The remainder of this section is only relevant for Theorem 14.7 and even this can be applied without knowing about $\rhd$-quasi overlay joinability when one just knows:

Lemma 9.1 (Overlay Joinable $\Rightarrow$ $\rhd$-Quasi Overlay Joinable)
W.r.t. R, X *the following holds for each critical peak:*
If it is overlay joinable, then it is $\rhd$-quasi overlay joinable.

In Wirth & Gramlich (1994a) we introduced the following definition:
A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)$ is *quasi overlay joinable w.r.t.* R, X if

$$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\left(\begin{array}{c}\left(\ (D_0D_1)\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ \right)\\ \Rightarrow\ \left(\begin{array}{c} t_1\sigma\varphi = t_0\sigma\varphi[p\leftarrow t_1\sigma\varphi/p] \\ \wedge\ \ (t_0/p)\sigma\varphi\downarrow_{\mathrm{R,X}} t_1\sigma\varphi/p\ (\longleftarrow_{\mathrm{R,X}}\cup\lhd_{\mathrm{ST}})^+\ (\hat{t}/p)\sigma\varphi \end{array}\right)\end{array}\right).$$

This notion of quasi overlay joinability, however, has turned out to produce a wondrous effect in case that for some critical peak, w.l.o.g. say

$$((l_1[p\leftarrow r_0],C_0,\Lambda_0),\ (r_1,C_1,\Lambda_1),\ l_1,\ \sigma,\ p\ )$$

generated by two rules $((l_0,r_0),C_0)$, $((l_1,r_1),C_1)$ (with w.l.o.g. no variables in common) due to $\sigma=\mathrm{mgu}(\{(l_0,l_1/p)\},\mathrm{Y})$ for $\mathrm{Y}:=\mathcal{V}(((l_0,r_0),C_0),((l_1,r_1),C_1))$, and for some $\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $(C_0C_1)\sigma\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$, there is some $p'\in\mathcal{POS}(l_1)\setminus\{p\}$ with $l_1/p'\notin\mathrm{V}$ and $l_0\sigma\varphi=(l_1/p')\sigma\varphi$; i.e. the left-hand side of the rule $((l_0,r_0),C_0)$ occurs a second time in the instantiated peak term (or superposition term) at a non-variable position p'. In this case due to $l_0\sigma\varphi=(l_1/p')\sigma\varphi$ there are $\sigma'=\mathrm{mgu}(\{(l_0,l_1/p')\},\mathrm{Y})$ and $\varphi'\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma'\varphi')={}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi)$ and then (unless $l_1[p'\leftarrow r_0]\sigma'=r_1\sigma'$) we get another critical peak

$$((l_1[p'\leftarrow r_0],C_0,\Lambda_0),\ (r_1,C_1,\Lambda_1),\ l_1,\ \sigma',\ p'\ ).$$

Now (since $(C_0C_1)\sigma'\varphi'=(C_0C_1)\sigma\varphi$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$), if both critical peaks are quasi overlay joinable, then we get by the first conclusion in the above definition:

$$\begin{array}{rcl} r_1\sigma\varphi &=& l_1[p\leftarrow r_0]\,\sigma\varphi\ [p\leftarrow r_1\sigma\varphi/p]\quad ;\\ r_1\sigma'\varphi' &=& l_1[p'\leftarrow r_0]\sigma'\varphi'[p'\leftarrow r_1\sigma'\varphi'/p'] \end{array}$$

(unless $l_1[p'\leftarrow r_0]\sigma'\varphi'=r_1\sigma'\varphi'$). Simplified, this means:

$$\begin{array}{rcl} r_1\sigma\varphi &=& l_1\sigma\varphi[p\leftarrow r_1\sigma\varphi/p]\quad ;\\ r_1\sigma\varphi &=& l_1\sigma\varphi[p'\leftarrow r_1\sigma\varphi/p'] \end{array}$$

(unless $r_1\sigma\varphi=l_1\sigma\varphi[p'\leftarrow r_0\sigma\varphi]$). Thus, in any case, we get

$$l_1\sigma\varphi[p\leftarrow\ldots]=r_1\sigma\varphi=l_1\sigma\varphi[p'\leftarrow\ldots].$$

Since (due to $p\neq p'$ and $(l_1/p')\sigma\varphi=l_0\sigma\varphi=(l_1/p)\sigma\varphi$) we have $p'\parallel p$, this has the wondrous result

$$l_1\sigma\varphi=r_1\sigma\varphi. \tag{!}$$

Using the second conclusion of the quasi overlay joinability we get
$l_1\sigma\varphi/p=r_1\sigma\varphi/p\ (\longleftarrow\cup\lhd_{\mathrm{ST}})^+\ (l_1/p)\sigma\varphi$ which implies

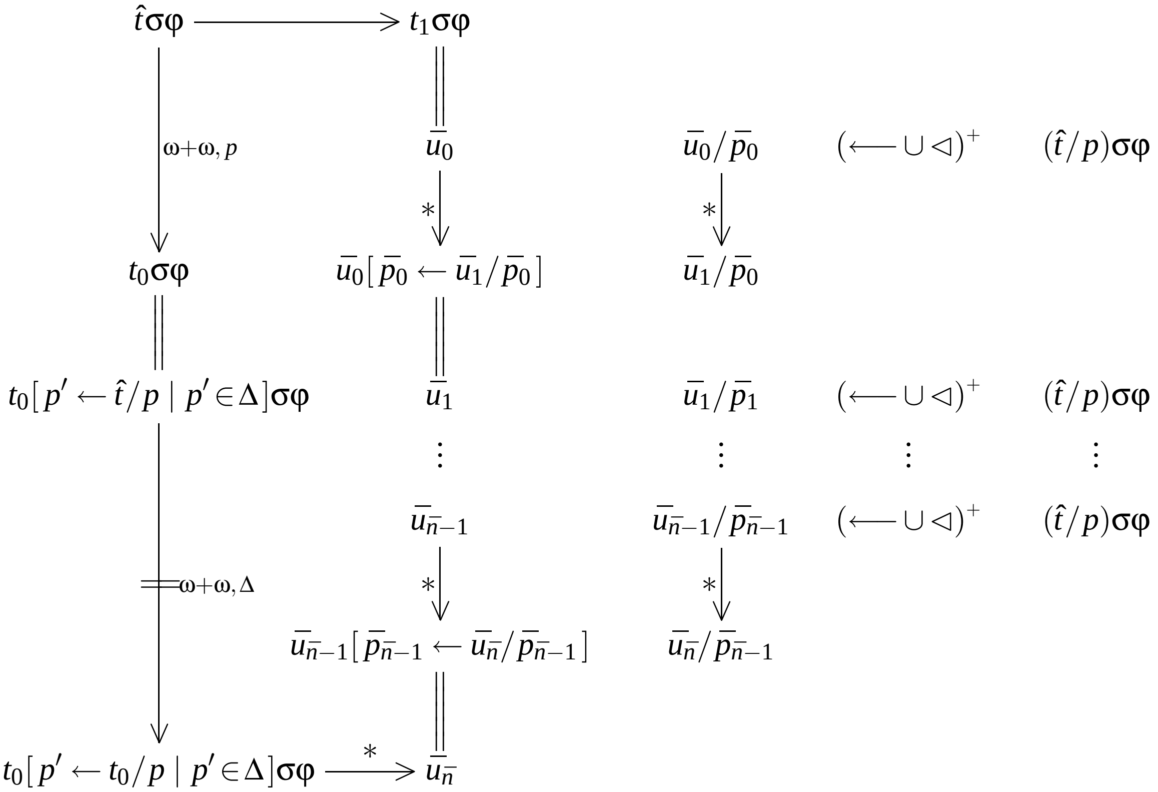
$$l_0\sigma\varphi\ (\longleftarrow\cup\lhd_{\mathrm{ST}})^+\ l_0\sigma\varphi. \tag{!!}$$

Since both results (!) and (!!) are absurd for a property which is only to be used for a noetherian reduction relation $\longrightarrow_{\mathrm{R,X}}$, we now generalize our notion of quasi overlay joinability.

Definition 9.2 ($\rhd$-Quasi Overlay Joinability)
A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)$ is $\rhd$-*quasi overlay joinable w.r.t.* R,X if $\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \forall\Delta.$

$$\left(\begin{array}{l}\left(\begin{array}{ll} & (D_0D_1)\sigma\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}} \\ \wedge & \Delta=\{\ p'\in\mathcal{POS}(\hat{t})\setminus\{p\}\mid \hat{t}/p'\not\in\mathrm{V}\wedge(\hat{t}/p')\sigma\varphi=(\hat{t}/p)\sigma\varphi\ \} \\ \wedge & \forall w\,(\longleftarrow_{\mathrm{R,X}}\cup\lhd)^{+}\ (\hat{t}/p)\sigma\varphi.\ \ \longrightarrow_{\mathrm{R,X}} \text{ is confluent below } w \\ \wedge & \forall p''\in\mathcal{POS}((\hat{t}/p)\sigma\varphi)\setminus\{\emptyset\}.\ (\hat{t}/p)\sigma\varphi/p''\not\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})\end{array}\right) \\ \Rightarrow\quad \exists\bar{n}\in\mathbf{N}.\ \exists\bar{p}.\ \exists\bar{u}.\left(\begin{array}{ll} & \bar{p}:\{0,\ldots,\bar{n}-1\}\to\mathbf{N}^{*} \\ \wedge & \bar{u}:\{0,\ldots,\bar{n}\}\to\mathcal{T} \\ \wedge & t_0[p'\leftarrow t_0/p\mid p'\in\Delta]\sigma\varphi\overset{*}{\longrightarrow}_{\mathrm{R,X}}\bar{u}_{\bar{n}} \\ \wedge & \forall i\prec\bar{n}.\left(\begin{array}{ll} & \bar{u}_{i+1}=\bar{u}_i[\bar{p}_i\leftarrow\bar{u}_{i+1}/\bar{p}_i] \\ \wedge & \bar{u}_{i+1}/\bar{p}_i\overset{*}{\longleftarrow}_{\mathrm{R,X}}\bar{u}_i/\bar{p}_i\ (\longleftarrow_{\mathrm{R,X}}\cup\lhd)^{+}\ (\hat{t}/p)\sigma\varphi\end{array}\right) \\ \wedge & \bar{u}_0=t_1\sigma\varphi\end{array}\right)\end{array}\right).$$

For $\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ and $\Delta\subseteq\mathcal{POS}(\hat{t})\setminus\{\emptyset\}$ with $(D_0D_1)\sigma\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$ and $\forall p'\in\Delta.\ (\ \hat{t}/p'\not\in\mathrm{V}\ \wedge\ (\hat{t}/p')\sigma\varphi=(\hat{t}/p)\sigma\varphi\ )$ the critical peak, the further reduction of its left part, and the required joinability after this reduction can be depicted as follows:[20]



[20]It should be noted that the fact that the parallel reduction can be restricted not only to non-variable positions of $\hat{t}$ but also to the same identical redex $(\hat{t}/p)\sigma\varphi$ (and the necessity of the analogous restriction in the proof) was especially brought to our attention by Bernhard Gramlich (cf. Gramlich (1995b)) who already had similar but less general ideas on the weakening of overlay joinability.

It is rather easy to see that $\rhd_{\mathrm{ST}}$-quasi overlay joinability of a critical peak generalizes the old notion of quasi overlay joinability:

In case that $\Delta=\emptyset$: For quasi overlay joinability of $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)$, i.e. for $t_1\sigma\varphi=t_0\sigma\varphi[p\leftarrow t_1\sigma\varphi/p]$; $(t_0/p)\sigma\varphi\overset{*}{\longrightarrow}_{\mathrm{R,X}}w\overset{*}{\longleftarrow}_{\mathrm{R,X}}t_1\sigma\varphi/p\,(\longleftarrow_{\mathrm{R,X}}\cup\lhd_{\mathrm{ST}})^+\,(\hat{t}/p)\sigma\varphi$; we simply choose $\bar{n}:=1$; $\bar{u}_0:=t_1\sigma\varphi$; $\bar{u}_1:=\bar{u}_0[p\leftarrow w]$; and get

$$t_0\sigma\varphi=t_1\sigma\varphi[p\leftarrow t_0\sigma\varphi/p]\overset{*}{\longrightarrow}_{\mathrm{R,X}}t_1\sigma\varphi[p\leftarrow w]=\bar{u}_1$$

and $$\bar{u}_1/p=w\overset{*}{\longleftarrow}_{\mathrm{R,X}}t_1\sigma\varphi/p=\bar{u}_0/p=t_1\sigma\varphi/p\,(\longleftarrow_{\mathrm{R,X}}\cup\lhd_{\mathrm{ST}})^+\,(\hat{t}/p)\sigma\varphi.$$

In case that $\Delta\neq\emptyset$: $\rhd_{\mathrm{ST}}$-quasi overlay joinability of some critical peak, w.l.o.g. say $((l_1[p\leftarrow r_0],C_0,\Lambda_0),\ (r_1,C_1,$ generated by two rules $((l_0,r_0),C_0)$, $((l_1,r_1),C_1)$ (with w.l.o.g. no variables in common) due to $\sigma=\mathrm{mgu}(\{(l_0,l_1/p)\},\mathrm{Y})$ for $\mathrm{Y}:=\mathcal{V}(((l_0,r_0),C_0),((l_1,r_1),C_1))$, generalizes quasi overlay joinability of the critical peaks resulting from overlapping $((l_0,r_0),C_0)$ into $((l_1,r_1),C_1)$. While we are not going to discuss the (then obvious) general case in detail here, the case of $\Delta=\{p'\}$ was just discussed before the definition above and we complete this discussion now as follows: Defining $\hat{t}:=l_1$; $t_0:=l_1[p\leftarrow r_0]$; $t_1:=r_1$; $\bar{n}:=2$; $\bar{u}_0:=t_1\sigma\varphi$; $\bar{u}_1:=\bar{u}_0[p\leftarrow r_0\sigma\varphi]$; $\bar{u}_2:=\bar{u}_1[p'\leftarrow r_0\sigma\varphi]$; due to (!) we have

$$t_0[p'\leftarrow t_0/p\mid p'\in\Delta]\sigma\varphi=l_1[p\leftarrow r_0][p'\leftarrow r_0]\sigma\varphi=r_1\sigma\varphi[p\leftarrow r_0\sigma\varphi][p'\leftarrow r_0\sigma\varphi]=\bar{u}_2$$

and due to (!!) we have $\bar{u}_2/p'=\bar{u}_1/p=r_0\sigma\varphi\longleftarrow_{\mathrm{R,X}}l_0\sigma\varphi\,(\longleftarrow\cup\lhd_{\mathrm{ST}})^+\,l_0\sigma\varphi=(\hat{t}/p)\sigma\varphi$ where

by (!) $$l_0\sigma\varphi=(l_1/p)\sigma\varphi=l_1\sigma\varphi/p=t_1\sigma\varphi/p=\bar{u}_0/p$$

and $$l_0\sigma\varphi=(l_1/p)\sigma\varphi=(l_1/p')\sigma\varphi=l_1\sigma\varphi/p'=t_1\sigma\varphi/p'=\bar{u}_0/p'=\bar{u}_1/p'.$$

In the case of an arbitrary $\Delta\neq\emptyset$, quasi overlay joinability of any two of the critical peaks involved implies that the diagram from above then looks the following way (where $\bar{n}:=|\{p\}\cup\Delta|$):

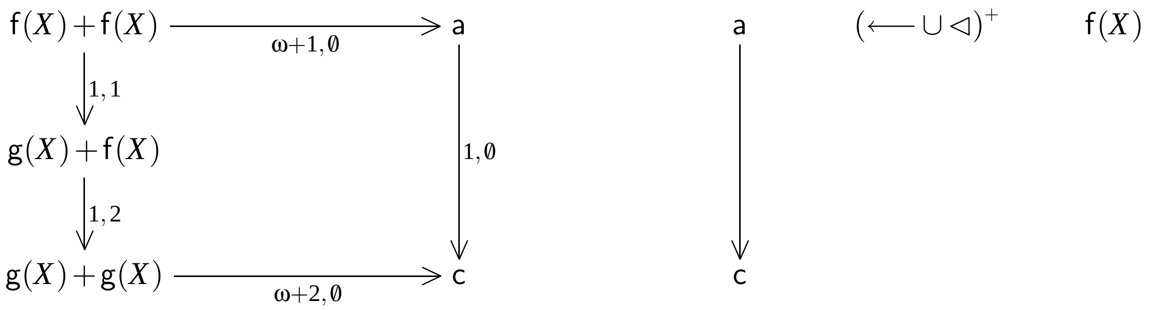
$$\begin{array}{ccc}
\hat{t}\sigma\varphi & \text{=========} & t_1\sigma\varphi \\
\Big\downarrow{\scriptstyle \omega+\omega,\{p\}\cup\Delta} & & \Big\downarrow{\scriptstyle \omega+\omega,\{p\}\cup\Delta} \\
t_0[p'\leftarrow t_0/p\mid p'\in\Delta]\sigma\varphi & \text{====} & \bar{u}_{\bar{n}}
\end{array}$$

That the wondrous results of quasi overlay joinability in the above reported case can be overcome with the new notion of $\rhd$-quasi overlay joinability can be seen from the following example:

Example 9.3 $\mathbb{C} := \{\mathsf{f},\mathsf{g},\mathsf{a},\mathsf{c}\}$
$\mathbb{N} := \{+\}$
$\mathrm{R}_{9.3}$:

$$\begin{array}{rcl} \mathsf{f}(X) & = & \mathsf{g}(X) \\ \mathsf{a} & = & \mathsf{c} \\ \mathsf{f}(X)+\mathsf{f}(X) & = & \mathsf{a} \\ \mathsf{g}(X)+\mathsf{g}(X) & = & \mathsf{c} \qquad \longleftarrow \ \mathsf{f}(X)+\mathsf{f}(X) = \mathsf{c} \end{array}$$

Now the unconditional version of $\mathrm{R}_{9.3}$ is compatible with the lexicographic path ordering $\rhd$ resulting from the following precedence on function symbols (in decreasing order): f, g, a, c. The critical peak $((\mathsf{g}(X)+\mathsf{f}(X),\emptyset,0),\ (\mathsf{a},\emptyset,1),\ \mathsf{f}(X)+\mathsf{f}(X),\ \emptyset,\ 1\ )$ cannot be quasi overlay joinable because $\mathsf{a}/1$ is undefined. It is, however, $\rhd$-quasi overlay joinable:



That the Δ in the notion of $\rhd_{\mathrm{ST}}$-quasi overlay joinability cannot be restricted to be empty can be seen from Example 12.2.

10 Some Unconditional Examples

Our main goal in this and the following sections is to find confluence criteria that do not depend on termination arguments but on the structure of the joinability of critical peaks only. Finally in § 14 we will investigate how termination can strengthen our criteria. Up to then, however, we are not going to use termination arguments. Instead, we are looking for confluence criteria of the form "If all critical peaks of a ... (e.g. normal, left-linear, &c.) rule system are joinable according to the pattern ... (e.g. shallow joinable, parallel closed, &c.) then the reduction relation is confluent."

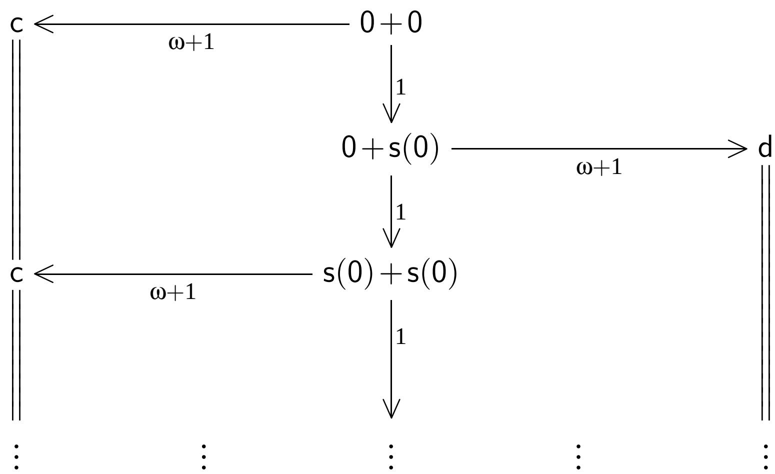
First we want to make clear that this approach has its limits. We do this by giving some examples. To distinguish confluent from non-confluent examples the rule systems of the latter ones are displayed in a box at the right margin while in a connected box to the left we list the example's crucial properties, concerning joinability structure of their critical peaks, variable occurrence, condition properties, &c.. The reader should not try to understand the sophisticated joinability labels in the boxes at a first reading. This is not necessary for understanding the examples. The sophisticated joinability labels are only needed for § 13.

In this section we start with some unconditional examples. The first one shows that left-linearity is essential[21]:

Example 10.1 (Huet (1980))

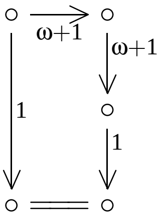
No Critical Peaks	$\mathbb{C} := \{0, \mathsf{s}, \mathsf{c}, \mathsf{d}\}$	$\mathrm{R}_{10.1}$: $0 = \mathsf{s}(0)$
Not Left-Linear	$\mathbb{N} := \{+\}$	$X + X = \mathsf{c}$
Unconditional		$X + \mathsf{s}(X) = \mathsf{d}$
Not Terminating		

There are no critical peaks. Nevertheless, $\longrightarrow_{\mathrm{R}_{10.1},\emptyset}$ is not confluent:



[21] Since this counterexample for confluence is unconditional it must be non-terminating of course. For conditional systems, however, left-linearity is essential also for terminating systems for joinability of critical peaks to imply confluence, cf. the transformation described in § 11 applied to Example 11.3 as described in § 11.

Example 10.2



ω-Level [[Weak] Parallel] Joinable
ω-Level Strongly Joinable
Not ω-Shallow [[Noisy] Parallel] Joinable up to ω

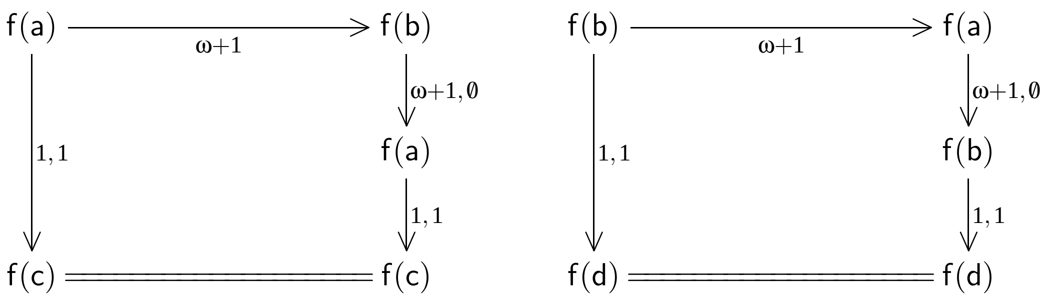
Ground

Unconditional

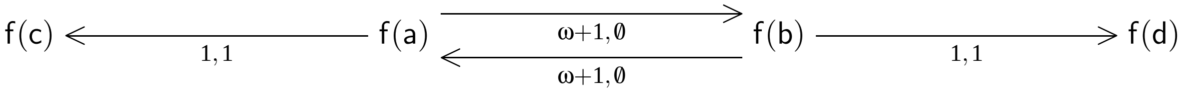
Not Terminating

$\mathbb{C} := \{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\}$
$\mathbb{N} := \{\mathsf{f}\}$
$\mathrm{R}_{10.2}$: $\mathsf{a} = \mathsf{c}$
$\mathsf{b} = \mathsf{d}$
$\mathsf{f}(\mathsf{a}) = \mathsf{f}(\mathsf{b})$
$\mathsf{f}(\mathsf{b}) = \mathsf{f}(\mathsf{a})$

The critical peaks are all of the form $(0,1)$ and can be closed as follows:



However, $\longrightarrow_{\mathrm{R}_{10.2},\emptyset}$ is not confluent:



Example 10.3
$$\begin{array}{lcl} \mathbb{C} & := & \{0,\mathsf{s},\mathsf{p}\} \\ \mathbb{N} & := & \{+\} \end{array} \quad \Bigg| \quad \mathrm{R}_{10.3} : \quad \begin{array}{c} \mathsf{s}(\mathsf{p}(X)) = X \\ \mathsf{p}(\mathsf{s}(X)) = X \\ 0 + Y = Y \\ \mathsf{s}(X) + Y = \mathsf{s}(X+Y) \\ \mathsf{p}(X) + Y = \mathsf{p}(X+Y) \end{array}$$

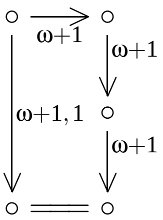
The critical peaks are all of the form $(0,1)$ and can be closed as follows:

$$\begin{array}{ccc} \mathsf{s}(\mathsf{p}(X))+Y & \xrightarrow[\omega+1]{} & \mathsf{s}(\mathsf{p}(X)+Y) \\ & & \downarrow{\scriptstyle \omega+1,1} \\ \downarrow{\scriptstyle 1,1} & & \mathsf{s}(\mathsf{p}(X+Y)) \\ & & \downarrow{\scriptstyle 1,\emptyset} \\ X+Y & = & X+Y \end{array} \qquad \begin{array}{ccc} \mathsf{p}(\mathsf{s}(X))+Y & \xrightarrow[\omega+1]{} & \mathsf{p}(\mathsf{s}(X)+Y) \\ & & \downarrow{\scriptstyle \omega+1,1} \\ \downarrow{\scriptstyle 1,1} & & \mathsf{p}(\mathsf{s}(X+Y)) \\ & & \downarrow{\scriptstyle 1,\emptyset} \\ X+Y & = & X+Y \end{array}$$

Since the reduction relation is terminating, we have confluence here. However, note that the structure of the joinability of the critical peaks is identical to that of Example 10.2 (with the exception of the positions). Thus, argumentation on the joinability structure of critical peaks must fail to infer confluence for this example (at least if we do not take positions into account).

The following example results from Example 10.2 just by changing ‘a’ and ‘b’ into non-constructors. While Example 10.2 was able to discourage generalizations of Theorem 13.9, by the slight change the following example is able to discourage generalizations of Theorem 13.6 regarding the required ω-shallow parallel closedness (for part (I) of Theorem 13.6), ω-shallow noisy anti-closedness (for part (II)), or ω-shallow closedness (for parts (III) and (IV)) of the non-overlays of the form $(1,1)$.

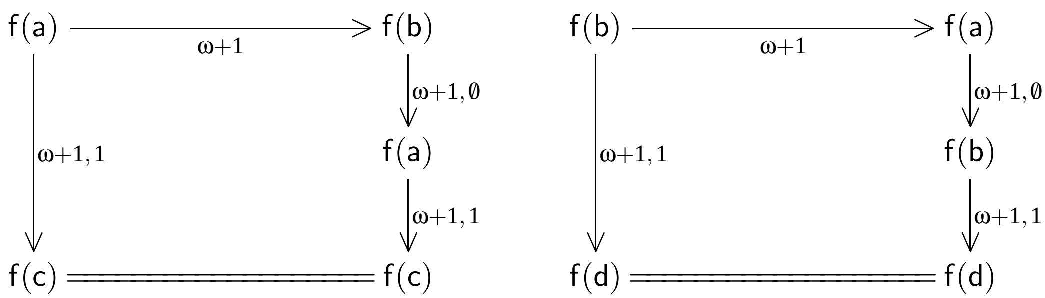
Example 10.4



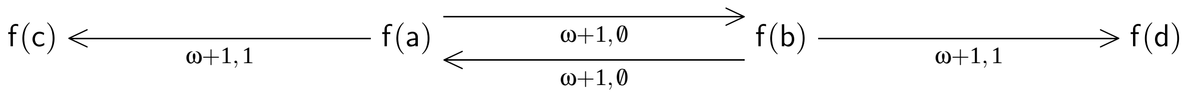
$\mathbb{C} := \{\mathsf{c},\mathsf{d}\}$
$\mathbb{N} := \{\mathsf{a},\mathsf{b},\mathsf{f}\}$
$R_{10.4}$: $\mathsf{a}=\mathsf{c}$
$\mathsf{b}=\mathsf{d}$
$\mathsf{f}(\mathsf{a})=\mathsf{f}(\mathsf{b})$
$\mathsf{f}(\mathsf{b})=\mathsf{f}(\mathsf{a})$

ω-Shallow [[Noisy] [Weak] Parallel] Joinable
ω-Shallow [Noisy] Strongly Joinable
Non-Overlay is Not ω-Shallow [Parallel] Closed
Non-Overlay is Not ω-Shallow [Noisy] Anti-Closed
Ground
Unconditional
Not Terminating

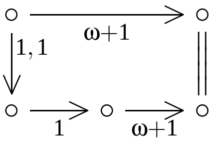
The critical peaks are all of the form $(1,1)$ now and can be closed as follows:



However, $\longrightarrow_{R_{10.4},\emptyset}$ is not confluent:



Example 10.5



$\rhd_{ST}$-Quasi Overlay Joinable
ω-Shallow [[Noisy] Weak Parallel] Joinable
Not ω-Shallow [Noisy] Parallel Joinable up to ω
Not ω-Shallow [Noisy] Strongly Joinable up to ω

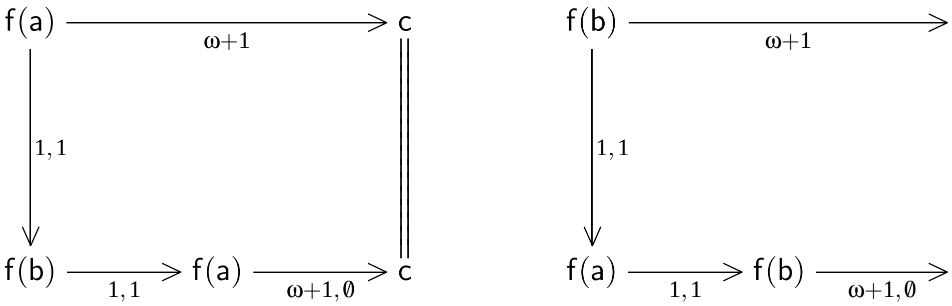
Ground

Unconditional

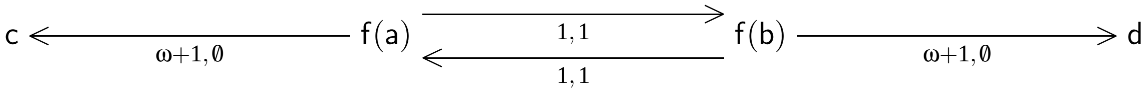
Not Terminating

$$\begin{array}{lll} \mathbb{C} & := & \{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\} \\ \mathbb{N} & := & \{\mathsf{f}\} \\ \mathrm{R}_{10.5} & : & \mathsf{a}=\mathsf{b} \\ & & \mathsf{b}=\mathsf{a} \\ & & \mathsf{f}(\mathsf{a})=\mathsf{c} \\ & & \mathsf{f}(\mathsf{b})=\mathsf{d} \end{array}$$

The critical peaks are all of the form $(0,1)$ and can be closed as follows:

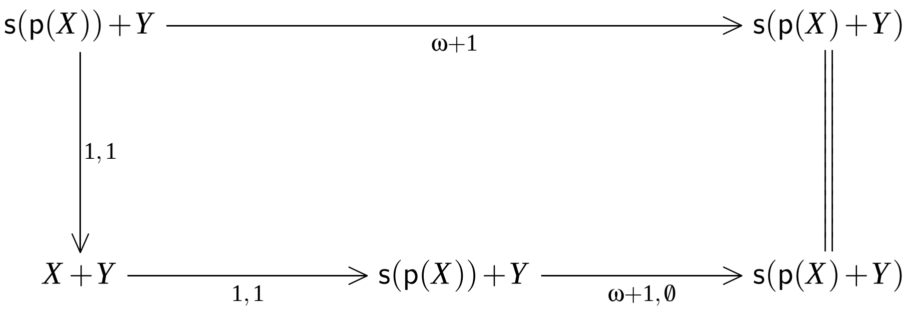


However, $\longrightarrow_{\mathrm{R}_{10.5},\emptyset}$ is not confluent:



Example 10.6 $\begin{array}{lll} \mathbb{C} & := & \{0,\mathsf{s},\mathsf{p}\} \\ \mathbb{N} & := & \{+\} \end{array}$ | $\mathrm{R}_{10.6} \ : \ \mathrm{R}_{10.3} \ + \ \begin{array}{l} X=\mathsf{s}(\mathsf{p}(X)) \\ X=\mathsf{p}(\mathsf{s}(X)) \end{array}$

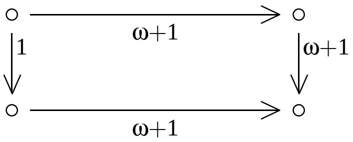
Note that we have added two rules to the system from Example 10.3: The critical peaks of the form $(0,1)$ of Example 10.3 still exist but can now be closed in different way; e.g., the first one can be closed as follows:



Since $\longrightarrow_{\mathrm{R}_{10.3},\emptyset}$ is confluent and $\longrightarrow_{\mathrm{R}_{10.3},\emptyset} \subseteq \longrightarrow_{\mathrm{R}_{10.6},\emptyset} \subseteq \overset{*}{\longleftrightarrow}_{\mathrm{R}_{10.3},\emptyset}$, $\longrightarrow_{\mathrm{R}_{10.6},\emptyset}$ is confluent, too (cf. Lemma 3.4). However, note that the structure of the joinability of the critical peaks is identical to that of Example 10.5. Thus, argumentation on the joinability structure of critical peaks must fail to infer confluence for this example.

According to Lemma 3.2 of Huet (1980), unconditional left- and right-linear rule systems with strongly joinable critical peaks are [strongly] confluent. That the severe restriction of right-linearity is essential here can be seen from the following example:

Example 10.7 (Jean-Jacques Lévy as cited in Huet (1980))



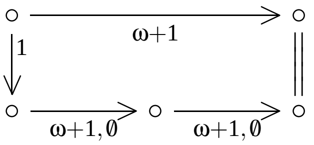
ω-Level [Parallel] Joinable
[ω-Level] [Strongly] Joinable
Not ω-Shallow [Noisy] Parallel Joinable up to ω
Not ω-Shallow [Noisy] Strongly Joinable up to ω

Left-Linear
Right-Linear Constructor Rules
Not Right-Linear

Unconditional

Not Terminating

$$
\begin{array}{lll}
\mathbb{C} & := & \{\mathsf{a},\mathsf{b},\mathsf{c},\mathsf{d}\} \\
\mathbb{N} & := & \{+,-\} \\
\mathrm{R}_{10.7} & : & \mathsf{a}=\mathsf{c} \\
 & & \mathsf{b}=\mathsf{d} \\
 & & \mathsf{a}+\mathsf{a}=\mathsf{b}-\mathsf{b} \\
 & & \mathsf{c}+X=X+X \\
 & & X+\mathsf{c}=X+X \\
 & & \mathsf{b}-\mathsf{b}=\mathsf{a}+\mathsf{a} \\
 & & \mathsf{d}-X=X-X \\
 & & X-\mathsf{d}=X-X
\end{array}
$$



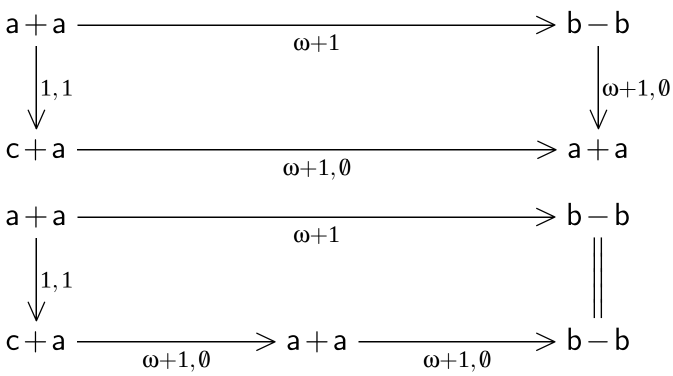
[ω-Shallow] Joinable
Not ω-Shallow [Noisy] Parallel Joinable up to ω
Not ω-Shallow [Noisy] Strongly Joinable up to ω

Left-Linear
Right-Linear Constructor Rules
Not Right-Linear

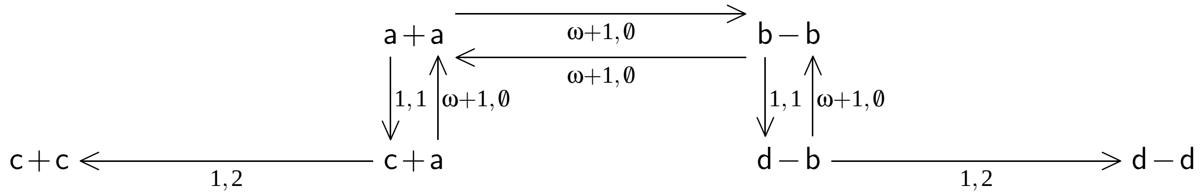
Unconditional

Not Terminating

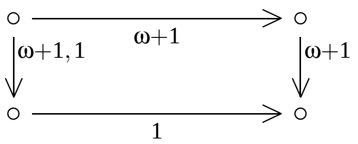
There are only four critical peaks and they are all of the form $(0,1)$. Using the symmetry of $+$ in its arguments as well the symmetry of a, c, $+$ with b, d, $-$, all other critical peaks are symmetric to the following one, which can be closed in the following two different ways:



Nevertheless, $\longrightarrow_{\mathrm{R}_{10.7},\emptyset}$ is not confluent:



We now use the same $\mathrm{R}_{10.7}$ to show that even another structure of joinability is insufficient for confluence. We do this by changing the separation into constructors and non-constructors:



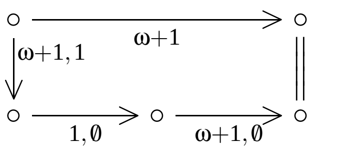
ω-Shallow [[Noisy] [Weak] Parallel] Joinable
Non-Overlay is Not ω-Shallow [Parallel] Closed
[ω-Shallow] Strongly Joinable
ω-Shallow Anti-Closed

Left-Linear
[Constructor Rules] Not Right-Linear

Unconditional

Not Terminating

$\mathbb{C}$:= $\{\mathsf{c}, \mathsf{d}, +, -\}$
$\mathbb{N}$:= $\{\mathsf{a}, \mathsf{b}\}$
$\mathrm{R}_{10.7}$:
$\mathsf{c} + X = X + X$
$X + \mathsf{c} = X + X$
$\mathsf{d} - X = X - X$
$X - \mathsf{d} = X - X$
$\mathsf{a} = \mathsf{c}$
$\mathsf{b} = \mathsf{d}$
$\mathsf{a} + \mathsf{a} = \mathsf{b} - \mathsf{b}$
$\mathsf{b} - \mathsf{b} = \mathsf{a} + \mathsf{a}$



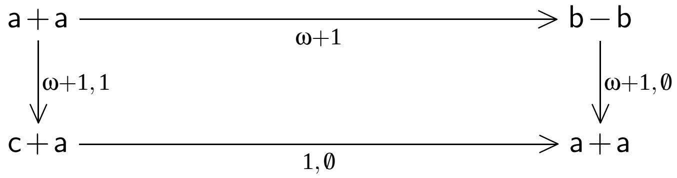
ω-Shallow [[Noisy] Weak Parallel] Joinable
Non-Overlay is Not ω-Shallow [Parallel] Closed
ω-Shallow Strongly Joinable
ω-Shallow Anti-Closed

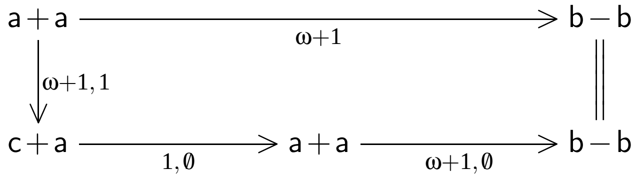
Left-Linear
[Constructor Rules] Not Right-Linear

Unconditional

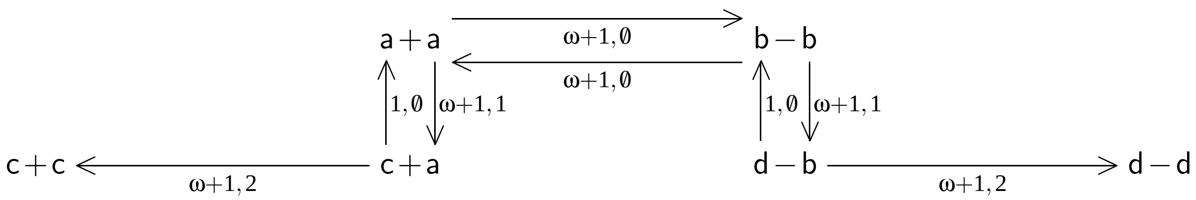
Not Terminating

Note that the rule system is not changed, but only reordered to have the constructor rules precede the non-constructor rules. The rewrite relation $\longrightarrow_{\mathrm{R}_{10.7},\emptyset}$ is not changed by this constructor re-declaration. (Note $X \in \mathrm{V}_{\mathrm{SIG}}$.) The critical peaks only have changed their form from $(0,1)$ to $(1,1)$ and are still all symmetric to the following one that closes in the two following ways:



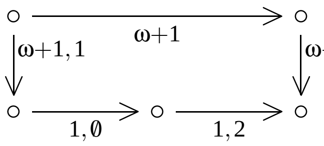


Finally, the divergence looks the following way now (Please note that now $\longrightarrow_{\mathrm{R}_{10.7},\emptyset,\omega}$ and $\longrightarrow_{\mathrm{R}_{10.7},\emptyset}$ are commuting, which was not the case before.):



The following example is a slight variation of Example 10.7 which is interesting w.r.t. Example 10.9.

Example 10.8



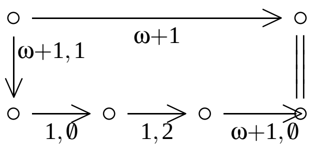
ω-Shallow [[Noisy] Parallel] Joinable
Non-Overlay is Not ω-Shallow [Parallel] Closed
ω-Shallow Strongly Joinable
ω-Shallow Anti-Closed

Left-Linear
[Constructor Rules] Not Right-Linear

Unconditional

Not Terminating

$$
\begin{array}{lll}
\mathbb{C} & := & \{\mathsf{c},\mathsf{d},+,-,\mathsf{f},\mathsf{g}\} \\
\mathbb{N} & := & \{\mathsf{a},\mathsf{b}\} \\
\mathrm{R}_{10.8} & : & \mathsf{c}+X=X+\mathsf{f}(X) \\
 & & X+\mathsf{c}=X+\mathsf{f}(X) \\
 & & \mathsf{f}(X)=X \\
 & & \mathsf{d}-X=X-\mathsf{g}(X) \\
 & & X-\mathsf{d}=X-\mathsf{g}(X) \\
 & & \mathsf{g}(X)=X \\
 & & \mathsf{a}=\mathsf{c} \\
 & & \mathsf{b}=\mathsf{d} \\
 & & \mathsf{a}+\mathsf{a}=\mathsf{b}-\mathsf{b} \\
 & & \mathsf{b}-\mathsf{b}=\mathsf{a}+\mathsf{a}
\end{array}
$$



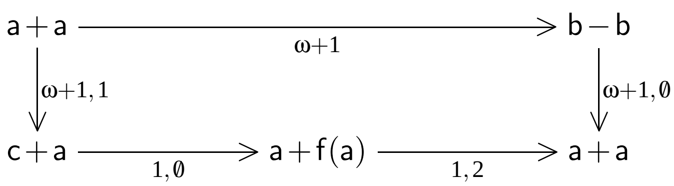
ω-Shallow [[Noisy] Weak Parallel] Joinable
Non-Overlay is Not ω-Shallow [Parallel] Closed
ω-Shallow Strongly Joinable
ω-Shallow Anti-Closed

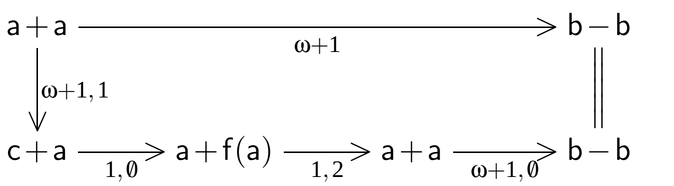
Left-Linear
[Constructor Rules] Not Right-Linear

Unconditional

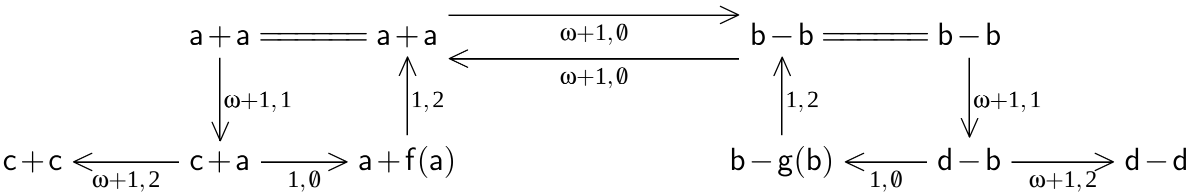
Not Terminating

There are only four critical peaks and they are all of the form $(1,1)$. Using the symmetry of $+$ in its relevant arguments as well the symmetry of a, c, $+$, f with b, d, $-$, g, all other critical peaks are symmetric to the following one, which can be closed in the following two different ways:



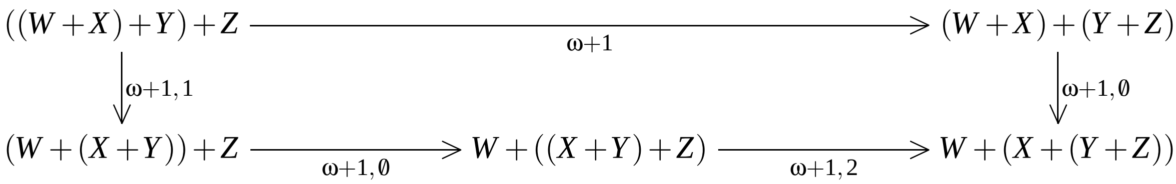


Finally, the divergence looks the following way now:



Example 10.9 $\mathbb{C} := \{0\}$
$\mathbb{N} := \{+\}$
$\mathrm{R}_{10.9}$: $(X+Y)+Z = X+(Y+Z)$

There is only one critical peak. It is of the form $(1,1)$ and can be closed as follows:



However, note that the structure of the joinability of the critical peak is weaker than the first alternative of Example 10.8. Thus, argumentation on the joinability structure of critical peaks must fail to infer confluence for this example.

11 Normality

When we now start to consider conditional besides unconditional rule systems, the first to notice is that we have to impose some normality restriction, as can be seen from Example 11.2 below.

A rule system is called *normal* if for all equations “$u_0{=}u_1$” in the condition lists of the rules, at least one of u_0, u_1 is an irreducible ground term.

Normality is no serious restriction unless left-linearity is required, too. This is because each non-normal system can be transformed into a normal but then not left-linear system without changing the reduction relation on the old sorts:

One just adds for each old sort s a new constructor function symbol eq_s with arity $s\,s \rightarrow s_{\rm new}$ (where $s_{\rm new}$ is a new sort) and a new constructor constant symbol $\perp$ of the sort $s_{\rm new}$. Then in each condition of each rule one transforms each equation of the form “$u{=}v$” with $u,v \in \mathcal{T}(\mathrm{sig}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})_s$ into “$\mathsf{eq}_s(u,v){=}\perp$” and adds for each old sort s the rule $\mathsf{eq}_s(X_s,X_s)=\perp$ (where $X_s \in \mathrm{V}_{\mathrm{SIG},s}$). Furthermore one adds the condition “$\mathsf{eq}_s(\mathsf{a},\mathsf{a}){=}\perp$” to each unconditional rule for some arbitrary constant a of an arbitrary old sort s.

The only change this transformation brings for the old sorts is that exactly those reductions which were possible with $\longrightarrow_{\mathrm{R,X},n}$ (for $n \prec \omega$) become exactly those reductions which are possible with $\longrightarrow_{\mathrm{R,X},n+1}$ after the transformation. $\longrightarrow_{\mathrm{R,X},\omega+n}$, however, is not changed by the transformation. E.g. for the rule system of Example 11.3 the transformation yields a ω-shallow [parallel] joinable, terminating system that is normal now but not left-linear anymore.

Now we return to the question whether joinability implies confluence. While Lemma 5.1 states the converse, actually little is known about the other direction unless the rule system is decreasing. Theorems 1 (which is taken from Bergstra & Klop (1986)) and 2 of Dershowitz &al. (1988) state that left-linear and normal rule systems are confluent if they have no critical pairs or are both shallow joinable and terminating. That normality is essential to imply confluence of systems with no critical pairs can be seen from Example 11.2. That normality is also essential to imply confluence of shallow joinable and terminating systems can be seen from Example 11.3. That left-linearity too is essential in both cases follows from the transformation described above.

In our framework, normality can be generalized and weakened to quasi-normality, which is a major result of this paper.

Definition 11.1 (Quasi-Normal)

Let $\alpha \in \{0, \omega\}$.

A rule $l{=}r{\longleftarrow}C$ is said to be α-*quasi-normal w.r.t.* R,X if
$\forall\tau{\in}\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\begin{array}{l}\left(\;C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R},\mathrm{X},\omega+\alpha}\;\right)\\ \Rightarrow\quad \forall(u_0{=}u_1) \text{ in } C.\left(\begin{array}{ll} & \left(\begin{array}{ll} & \alpha=\omega\\ \wedge & \mathcal{V}(u_0,u_1)\subseteq \mathrm{V}_C\end{array}\right)\\ \vee & \mathcal{V}(u_0,u_1)\subseteq\emptyset\\ \vee & \exists i{\prec}2.\left(\begin{array}{ll} & u_i\tau\notin\mathrm{dom}(\longrightarrow_{\mathrm{R},\mathrm{X},\omega+\alpha})\\ \vee & \left(\begin{array}{ll} & \alpha=\omega\\ \wedge & (\mathrm{Def}\,u_i\tau) \text{ occurs in } C\tau\end{array}\right)\end{array}\right)\end{array}\right)\end{array}\right).$$

R,X is said to be ω-*quasi-normal* if
all rules in R are ω-quasi-normal w.r.t. R,X.
R,X is said to be 0-*quasi-normal* if
all constructor rules in R are 0-quasi-normal w.r.t. R,X.

Since the case of “$\alpha=\omega$” is more important than the case of “$\alpha=0$”, we use “*quasi-normal*” as an abbreviation for “ω-quasi-normal”.

First note that we have added a condition that may reduce the instantiations of a rule we have to consider. While this may be useless in practice most of the time, it may allow of further theoretical treatment.

Also the fact that we have given up the requirement that the irreducible term has to be ground may be of minor importance: In practice this usually allows only for constructor variables or variables of sorts having only irreducible terms.

Important, however, is the fact that equations containing only constructor variables are not restricted by quasi-normality anymore. E.g., the rule system of Example 2.3 is quasi-normal but not normal.

Besides this, it is important that quasi-normality also allows to make any system quasi-normal simply by replacing any equation “$u{=}v$” in a condition with “$u{=}v,\;\mathrm{Def}\,v$”.

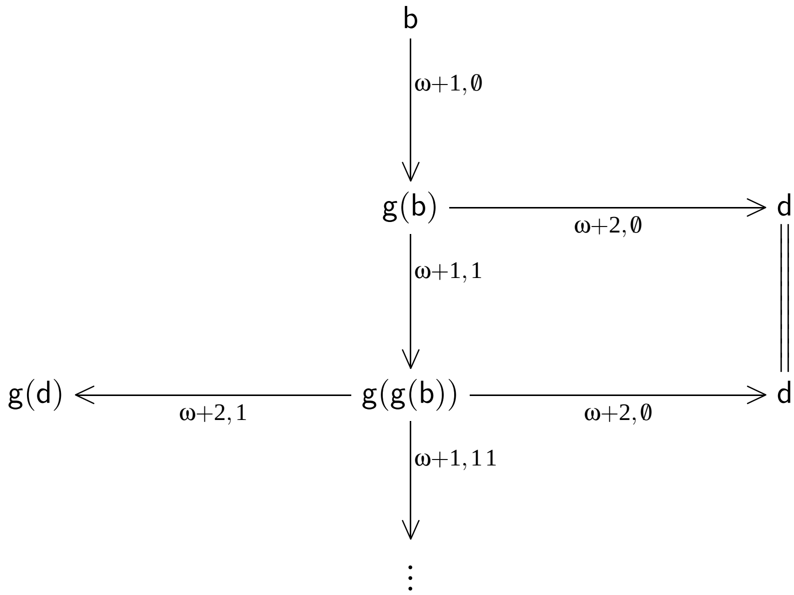
Furthermore, note that no restrictions are imposed on Def- and $\neq$-literals.

Example 11.2 (Bergstra & Klop (1986))

No Critical Peaks
Left- & Right-Linear
Not [Quasi-] Normal
Not Terminating

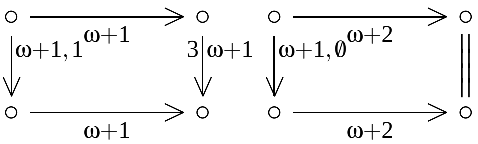
$$\begin{array}{lll} \mathbb{C} & := & \{\mathsf{d}\} \\ \mathbb{N} & := & \{\mathsf{b},\mathsf{g}\} \\ \mathrm{R}_{11.2} & : & \mathsf{b}=\mathsf{g}(\mathsf{b}) \\ & & \mathsf{g}(X)=\mathsf{d} \quad \longleftarrow \mathsf{g}(X)=X \end{array}$$

There are no critical peaks. Nevertheless, $\longrightarrow_{\mathrm{R}_{11.2},\emptyset}$ is not confluent:



The following example shows that normality is also required for terminating systems. Note that this was already shown by Example C of Dershowitz &al. (1988) which, however, is more complicated because it has there additional critical peaks.

Example 11.3



ω-Shallow [[Noisy] Parallel] Joinable
ω-Shallow [Noisy] Strongly Joinable
Non-Overlay is
Neither ω-Shallow [Parallel] Closed
Nor ω-Shallow [Noisy] Anti-Closed
Not [$\rhd_{\mathrm{ST}}$-Quasi] Overlay Joinable

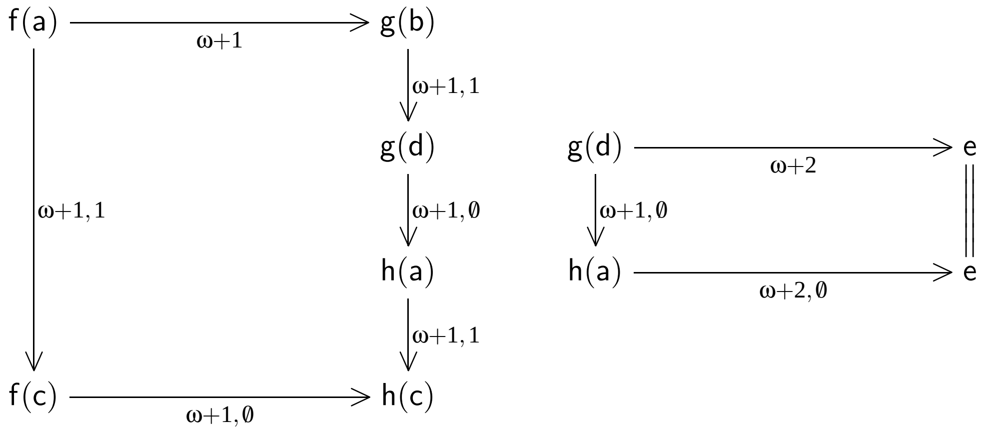
Left- & Right-Linear

Not [Quasi-] Normal

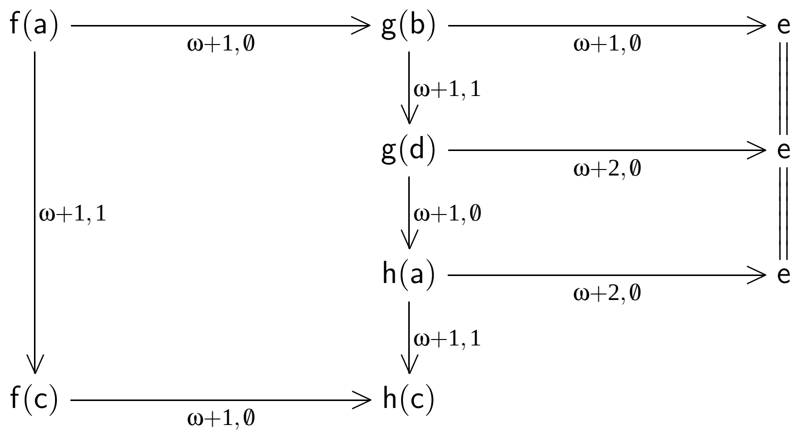
Terminating

$$\begin{array}{lll} \mathbb{C} & := & \{\mathsf{c},\mathsf{d},\mathsf{e}\} \\ \mathbb{N} & := & \{\mathsf{a},\mathsf{b},\mathsf{f},\mathsf{g},\mathsf{h}\} \\ \mathrm{R}_{11.3} & : & \mathsf{a}=\mathsf{c} \\ & & \mathsf{b}=\mathsf{d} \\ & & \mathsf{f}(\mathsf{a})=\mathsf{g}(\mathsf{b}) \\ & & \mathsf{f}(\mathsf{c})=\mathsf{h}(\mathsf{c}) \\ & & \mathsf{g}(\mathsf{d})=\mathsf{h}(\mathsf{a}) \\ & & \mathsf{g}(X)=\mathsf{e} \quad \longleftarrow X=\mathsf{b} \\ & & \mathsf{h}(X)=\mathsf{e} \quad \longleftarrow \mathsf{f}(X)=\mathsf{e} \end{array}$$

There are three critical peaks and they are all of the form $(1,1)$. Since the third is the symmetric overlay of the second, we do not depict it. The first and the second are joinable as follows:



Nevertheless, $\longrightarrow_{R_{11.3},\emptyset}$ is not confluent:



Note that the overlay would lose its shallow joinability if we made the system normal (or else quasi-normal) by writing the condition of the one but last rule in the form "$X{=}\mathsf{d}$" (or else in the form "$X{=}\mathsf{b},\ \mathrm{Def}\,\mathsf{b}$" and declaring b to be a constructor), since then we would have $\mathsf{g}(\mathsf{d})\longrightarrow_{\omega+1}\mathsf{e}$. Similarly, the overlay would lose its shallow joinability if we made the system quasi-normal by writing the condition of the one but last rule in the form "$X{=}\mathsf{b},\ \mathrm{Def}\,\mathsf{b}$" or by substituting X with a variable from V_C, since then we would have $\mathsf{h}(\mathsf{a})\longrightarrow_{\omega+3}\mathsf{e}$ only (since $\mathsf{g}(\mathsf{b})\not\longrightarrow_{\omega+1}\mathsf{e}$).

12 Counterexamples for Closed Systems

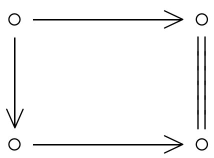
From the examples of the previous sections we can draw the following conclusions:

1. For being able to apply syntactic confluence criteria to non-terminating conditional rule systems, some kind of [quasi-] normality must be required.
2. Syntactic confluence criteria based solely on the joinability structure of the critical peaks must fail on some rather simple and common joinability structures.

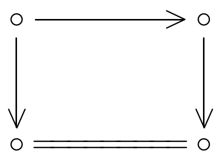
Therefore, it is now the time to have a look at the two most simple non-trivial joinability structures under the requirement of normality.

These two most simple joinability structures of critical peaks are *closedness* and *anti-closedness*, cf. below. Regarding the names of notions below, "parallel closed" is taken from Huet (1980), "closed" and "anti-closed" have been derived from "parallel closed" in an obvious manner, and "parallel joinable" was the simplest name[22] we found for the last important variant.

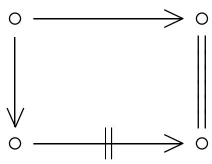
Closed:



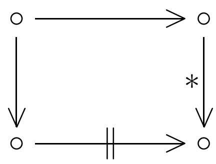
Anti-Closed:



Parallel Closed:



Parallel Joinable:



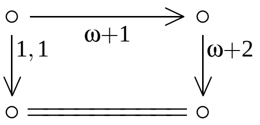
It may seem to be surprising that the question whether anti-closedness of critical peaks implies confluence for left-linear, non-right-linear, unconditional systems was listed as Problem 13 in the list of open problems of Dershowitz &al. (1991) and still seems to be open.

For the question whether closedness of critical peaks, a positive answer follows from the corollary on page 815 in Huet (1980) which says that a left-linear and unconditional system is confluent if all its critical pairs are parallel closed. The condition of parallel closedness was weakened in Corollary 3.2 of Toyama (1988) for the overlays which are required to be only parallel joinable instead of parallel closed.

For conditional systems, however, neither closedness nor anti-closedness implies confluence. And this situation does not change when we additionally require the rule systems to be terminating and normal, as can be seen from the following examples:

[22] The only obvious wrong intuitions it could rise are either meaningless (since the transitive closures of reduction and parallel reduction are always identical) or an unnecessary sharpening of our notion.

Example 12.1 (Aart Middeldorp, modified by Bernhard Gramlich)



Anti-Closed
Strongly Joinable
Not* ω*-Level Joinable
Not* ω*-Shallow Joinable
Not [*$\rhd_{ST}$*-Quasi] Overlay Joinable

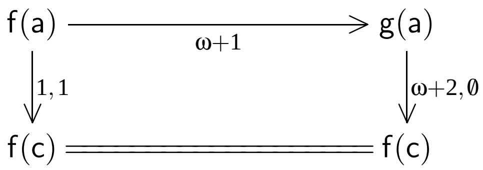
Left-Linear & Right-Linear

[Quasi-] Normal

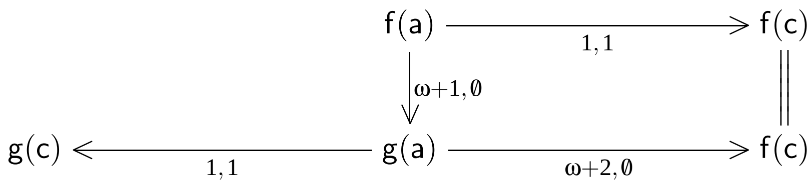
Terminating
Not Decreasing

$$
\begin{array}{lll}
\mathbb{C} & := & \{\mathsf{a},\mathsf{c}\} \\
\mathbb{N} & := & \{\mathsf{f},\mathsf{g}\} \\
\mathrm{R}_{12.1} & : & \mathsf{a} = \mathsf{c} \\
 & & \mathsf{f}(\mathsf{a}) = \mathsf{g}(\mathsf{a}) \\
 & & \mathsf{g}(X) = \mathsf{f}(\mathsf{c}) \longleftarrow \mathsf{f}(X) = \mathsf{g}(\mathsf{c})
\end{array}
$$

There is only the following critical peak and is of the form $(0,1)$:



Nevertheless, $\longrightarrow_{\mathrm{R}_{12.1},\emptyset}$ is not confluent:



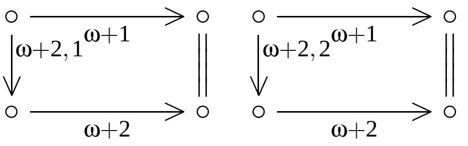
Since all critical peaks are joinable, $\mathrm{R}_{12.1}$ is necessarily non-decreasing and not compatible with a termination-pair.[23] Nevertheless, it is obviously terminating, since $\{X \mapsto \mathsf{a}\}$ is the only solution for the condition of the last equation. Furthermore, $\mathrm{R}_{12.1}$ is left-linear, right-linear, and normal[24]. Thus (since it is not confluent), it can be neither overlay joinable nor ω-shallow joinable.[25] It is, however, not ω-level joinable and we did not find a ω-level anti-closed but non-confluent system, though we spent some time searching for such an example.

[23] Cf. Definition 14.1 and Theorem 14.2

[24] even if some authors would not call it "normal" since the left-hand side of the last rule matches the right-hand side of the equation of its condition

[25] Cf. theorems 14.7 and 14.5

Example 12.2



[ω-Level [Parallel]] Closed
ω-Level Anti-Closed
[ω-Level] [Strongly] Joinable
ω-Level [Weak] Parallel Joinable
Not ω-Shallow Joinable
Not [$\triangleright_{\mathrm{ST}}$-Quasi] Overlay Joinable

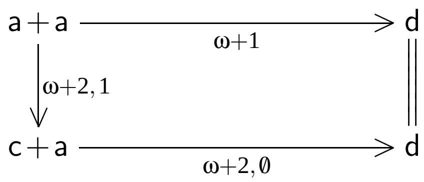
Left-Linear & Right-Linear
Conditions contain General variables

[Quasi-] Normal

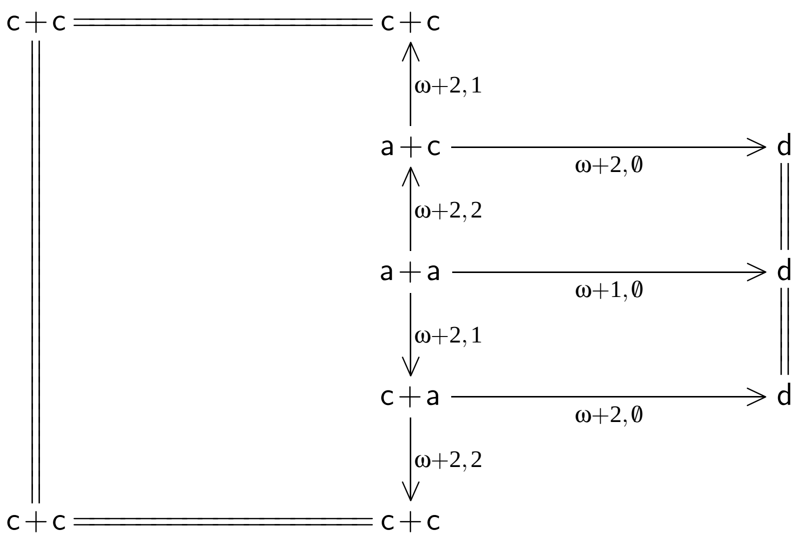
Terminating
Not Decreasing

$$
\begin{array}{lll}
\mathbb{C} & := & \{\mathsf{c},\mathsf{d}\} \\
\mathbb{N} & := & \{\mathsf{a},\mathsf{b},+\} \\
\mathrm{R}_{12.2} & : & \mathsf{a}=\mathsf{c} \longleftarrow \mathsf{b}=\mathsf{d} \\
& & \mathsf{b}=\mathsf{d} \\
& & \mathsf{a}+\mathsf{a}=\mathsf{d} \\
& & \mathsf{c}+X=\mathsf{d} \longleftarrow X+X=\mathsf{d} \\
& & X+\mathsf{c}=\mathsf{d} \longleftarrow X+X=\mathsf{d}
\end{array}
$$

There are only two critical peaks and they are of the form $(1,1)$. Using the symmetry of $+$ in its arguments, the other critical peak is symmetric to the following one.



Nevertheless, $\longrightarrow_{\mathrm{R}_{12.2},\emptyset}$ is not confluent:



Since all critical peaks are joinable, our system is necessarily non-decreasing, cf. Theorem 14.2. Nevertheless, it is obviously terminating, left-linear, right-linear, and normal. Thus (since it is not confluent), it can be neither overlay joinable nor ω-shallow joinable, cf. theorems 14.7 and 14.5. Due to the given forms of ω-level joinability, the occurrence of general variables in the conditions is essential for this example, cf. theorems 13.9 and 14.6.

13 Criteria for Confluence

Most of the theorems we present in this and the following section assume the constructor subsystem $\longrightarrow_{\mathrm{R,X},\omega}$ to be confluent and then suggest how to find out that the whole system $\longrightarrow_{\mathrm{R,X}}$ is confluent, too. How to find out that $\longrightarrow_{\mathrm{R,X},\omega}$ is confluent will be discussed in § 15.

In this section we present confluence criteria that do not rely on termination. They are, of course, also applicable to terminating systems, which might be very attractive if one does not know how to show termination or if the correctness of the technique for proving termination requires confluence.

Before we state our main theorems it is convenient to introduce some further syntactic restriction. By disallowing non-constructor variables in conditions of constructor equations we disentangle the fulfilledness of conditions of constructor equations from the influence of non-constructor rules.

Definition 13.1 (Conservative Constructors)
R is said to have *conservative constructors* if

$$\forall((l,r),C)\in\mathrm{R}.\ \big(\ l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\ \Rightarrow\ \mathcal{V}(C)\subseteq\mathrm{V}_{\mathcal{C}}\ \big).$$

Let us consider a rule system with conservative constructors. Together with our global restrictions on constructor rules (cf. Definition 2.2) this means that the condition terms of constructor rules are *pure* constructor terms. This has the advantage that (contrary to the general case) the condition terms of constructor rules still are constructor terms after they have been instantiated with some substitution. By Lemma 2.10 this means that the reducibility with constructor rules does not depend on the new possibilities which could be added by the non-constructor rules later on, i.e. that the constructor rules are conservative w.r.t. their decision not to reduce a given term because non-constructor rules cannot generate additional solutions for their conditions.[26]

The condition of conservative constructors is very natural and not very restrictive. (Note that even now constructor rules may have general variables in their left- and right-hand sides.) That conservative constructors make the construction of confluence criteria much easier can be seen from the following lemma which can treat a special case of possible divergence, namely a subcase of the "variable overlap case". In this case it is important that a reduction with a certain rule can still be done after the instantiating substitution has been reduced.

[26] Since "conservative constructors" is actually a property not of the constructors (i.e. constructor function symbols) but of the constructor *rules*, the notion should actually be called "conservative constructor rules". But the commonplace notion of "free constructors" is just the same.

Lemma 13.2
Let $\mu,\nu \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$. *Let* $((l,r),C) \in \mathrm{R}$ *with* $l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}} \uplus \mathrm{V}_C)$.
Assume that $\left(\begin{array}{ll} & \mathrm{R} \textit{ has conservative constructors} \\ \vee & \mathcal{V}(C) \subseteq \mathrm{V}_C \\ \vee & \mathcal{TERMS}(C\mu) \subseteq \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}} \uplus \mathrm{V}_C) \end{array}\right)$.
Assume $\overset{*}{\longrightarrow}_{\mathrm{R,X,\omega}}$ *to be confluent.*
Now, if $C\mu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R,X}}$ *and* $\forall x \in \mathrm{V}.\ x\mu \overset{*}{\longrightarrow}_{\mathrm{R,X}} x\nu$,
then $C\nu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R,X,\omega}}$ *and* $l\nu \longrightarrow_{\mathrm{R,X,\omega}} r\nu$.

While the conditions of our main theorems of this section, Theorem 13.6 and Theorem 13.9, are rather complicated and difficult to check, they are always satisfied for a certain class of rule systems captured by Theorem 13.3 (being a consequence of Theorem 13.6) and Theorem 13.4 (being a consequence of Theorem 13.9) below.

This class consists of left-linear rule systems with conservative constructors that achieve quasi-normality just by requiring the presence of a Def-literal for each equation not containing an irreducible ground term in a condition of a rule, and satisfy the joinability requirements due to the critical peaks being complementary, i.e. having complementary literals in their condition lists, cf. § 5. Furthermore, rule systems of this class are quite useful in practice. It generalizes the function specification style that is usually required in the framework of classic inductive theorem proving (cf. e.g. Walther (1994)) by allowing for partial functions resulting from non-complete defining case distinctions as well as resulting from non-termination.

Theorem 13.3 (Syntactic Confluence Criterion)
Let R *be a left-linear CRS over* sig/cons/V *with conservative constructors.*
Assume $\forall((l,r),C) \in \mathrm{R}.\ \forall(u_0{=}u_1)$ *in* C. $\exists i \prec 2.\ \left(\begin{array}{ll} & (\mathrm{Def}\, u_i) \textit{ occurs in } C \\ \vee & u_i \in \mathcal{GT} \setminus \mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \end{array}\right)$.
Assume that $\longrightarrow_{\mathrm{R,X,\omega}}$ *is confluent. Now:*

If each critical peak in $\mathrm{CP(R)}$ *of the form* $(0,1)$, $(1,0)$, *or* $(1,1)$ *is complementary, then* $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

Theorem 13.4 (Syntactic Confluence Criterion)
Let R *be a left-linear CRS over* sig/cons/V *with* $\forall((l,r),C) \in \mathrm{R}.\ \mathcal{V}(C) \subseteq \mathrm{V}_C$.
Assume $\forall((l,r),C) \in \mathrm{R}.\ \forall(u_0{=}u_1)$ *in* C. $\exists i \prec 2.\ \left(\begin{array}{ll} & (\mathrm{Def}\, u_i) \textit{ occurs in } C \\ \vee & u_i \in \mathcal{GT} \setminus \mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \end{array}\right)$.
Assume that $\longrightarrow_{\mathrm{R,X,\omega}}$ *is confluent. Now:*

If each critical peak in $\mathrm{CP(R)}$ *of the form* $(0,1)$ *or* $(1,0)$ *is complementary and each critical peak in* $\mathrm{CP(R)}$ *of the form* $(1,1)$ *is weakly complementary, then* $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

Note that both theorems are applicable[27] to the rule system of Example 2.3 where the subtraction on natural numbers is defined via a non-complete syntactic case distinction that does not yield critical peaks at all and where the member-predicate is defined by a syntactic case distinction followed (for the case of a nonempty list) by a semantic case distinction via condition literals which yields only critical peaks with complementary equations. To illustrate the possibility of partiality due to non-termination as well as the possibility of critical peaks with complementary predicate literals, here is another toy example to which we can apply Theorem 13.3 (but not Theorem 13.4).

Example 13.5 (continuing Example 2.3)

$$\begin{array}{lcl} \mathbb{C} & := & \{0, \mathsf{s}, \mathsf{true}, \mathsf{false}, \mathsf{nil}, \mathsf{cons}\} \\ \mathbb{N} & := & \{-, \mathsf{mbp}, \mathsf{while}\} \\ \mathbb{S} & := & \{\mathsf{nat}, \mathsf{bool}, \mathsf{list}\} \\ \mathrm{R}_{13.5} & : & \mathrm{R}_{2.3} \\ & & \vdots \\ & & \mathsf{while}(X,Y) = Y \qquad\qquad\qquad \longleftarrow X = \mathsf{false} \\ & & \mathsf{while}(X,Y) = \mathsf{while}(\ldots,\ldots) \quad \longleftarrow X = \mathsf{true},\ \ldots \\ & & \vdots \end{array}$$

We have added two rules to the system from Example 2.3 for a function 'while' with arity " $\mathsf{bool}\ \mathsf{nat} \rightarrow \mathsf{nat}$ " where X is meant to be a variable from $\mathrm{V}_{\mathrm{SIG},\mathsf{bool}}$ and Y from $\mathrm{V}_{\mathrm{SIG},\mathsf{nat}}$. The two resulting critical peaks are of the form $(1,1)$ and complementary. Furthermore, we assume that there are no rules with true, false, or a variable of the sort bool as left-hand sides, such that we have $\mathsf{true}, \mathsf{false} \in \mathcal{GT} \setminus \mathrm{dom}(\longrightarrow_{\mathrm{R}_{13.5},\mathrm{X}})$.

The main part of the following theorem is part (I). Parts (III) and (IV) only weaken the required ω-shallow noisy parallel joinability for critical peaks of the form $(1,1)$ to ω-shallow noisy *weak* parallel joinability but have to pay a considerable price for it. It would be of practical importance (cf. Example 10.6) to achieve this weakening for critical peaks of the form $(0,1)$, but this is not possible, cf. Example 10.5. Furthermore, the difference between (III) and (IV) is marginal since non-overlays of the form $(1,0)$ are pathological[28] anyway. (II) is rather interesting for the cases where it is possible to restrict the right-hand sides to be linear w.r.t. general variables; this severe restriction is necessary, however; cf. the second version of Example 10.7 or cf. Example 10.8. Besides these examples, also Example 10.4 may be able to discourage the search for a further generalization of the theorem. Finally note that the 'i' and 'j' in the theorem range over $\{0,1\}$.

[27]The careful reader may have noticed that the last two rules of $\mathrm{R}_{2.3}$ actually are lacking the required Def-literals. For practical specification, however, this Def-literal can be omitted here because it is tautological for $\longrightarrow_{\mathrm{R},\mathrm{X}}$ if $\mathrm{X} \subseteq \mathrm{V}_{\mathrm{SIG}}$. Note that in practice of specification one is only interested in $\longrightarrow_{\mathrm{R},\emptyset}$ and $\longrightarrow_{\mathrm{R},\mathrm{V}_{\mathrm{SIG}}}$ cf. Wirth & Gramlich (1994a) and Wirth & Gramlich (1994b). (This, however, does not mean that we do not need formulas containing $\mathrm{V}_{\mathcal{C}}$ for inductive theorem proving.)

[28]A critical peak of the form $(1,0)$ requires a non-constructor rule whose left-hand side has a constructor function symbol as top symbol, and also requires a constructor rule with a general variable in its left-hand side.

Theorem 13.6 (Syntactic Criterion for ω-Shallow Confluence)

Let R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume R *to have conservative constructors,* R, X *to be quasi-normal, and the following weak kind of left-linearity:*

$$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q\in\mathcal{POS}(l).\ \forall x\in\mathrm{V}.\ \left(\left(\begin{array}{ll} & l/p=x=l/q \\ \wedge & p\neq q\end{array}\right)\Rightarrow\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C) \\ \wedge & x\in\mathrm{V}_C\end{array}\right)\right).$$

Furthermore, assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent.*

(I) *Now if each critical peak in* CP(R) *of the form* $(i,1)$ *is* ω*-shallow noisy parallel joinable up to* $\omega{+}i{*}\omega$ *w.r.t.* R, X*, and each non-overlay in* CP(R) *of the form* $(1,j)$ *is* ω*-shallow parallel closed up to* $\omega{+}j{*}\omega$ *w.r.t.* R, X*, then* R, X *is* ω*-shallow confluent.*

(II) *If we have the following kind of right-linearity w.r.t. general variables*

$$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V_{SIG}}.\ \forall p,q\in\mathcal{POS}(r).\ (\ r/p=x=r/q\ \Rightarrow\ p=q\ ),$$

and if each critical peak in CP(R) *of the form* $(i,1)$ *is* ω*-shallow noisy strongly joinable up to* $\omega{+}i{*}\omega$ *w.r.t.* R, X*, and each non-overlay in* CP(R) *of the form* $(1,j)$ *is* ω*-shallow noisy anti-closed up to* $\omega{+}j{*}\omega$ *w.r.t.* R, X*, then* R, X *is* ω*-shallow confluent.*

Now additionally assume the following very weak kind of right-linearity of constructor rules:

$$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V_{SIG}}.\ \forall p,q\in\mathcal{POS}(r).\left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C) \\ \wedge & r/p=x=r/q\end{array}\right)\Rightarrow\ p=q\right).$$

Furthermore, additionally assume that each critical peak in CP(R) *of the form* $(0,1)$ *is* ω*-shallow noisy strongly joinable up to* ω*, that each critical peak in* CP(R) *of the form* $(1,1)$ *is* ω*-shallow noisy weak parallel joinable w.r.t.* R, X*, and that each non-overlay in* CP(R) *of the form* $(1,1)$ *is* ω*-shallow closed w.r.t.* R, X.

(III) *Now if each non-overlay in* CP(R) *of the form* $(1,0)$ *is* ω*-shallow parallel closed up to* ω *w.r.t.* R, X*, then* R, X *is* ω*-shallow confluent.*

Now additionally assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is strongly confluent.*

(IV) *Now if each non-overlay in* CP(R) *of the form* $(1,0)$ *is* ω*-shallow closed up to* ω *w.r.t.* R, X*, then* R, X *is* ω*-shallow confluent.*

If we consider all symbols to be non-constructor symbols, then each of the parts (I), (III), and (IV) of Theorem 13.6 is strong enough to imply Theorem 1 of Dershowitz &al. (1988) (which is taken from Bergstra & Klop (1986)). If we, moreover, restrict to unconditional rule systems, then Theorem 13.6(I) specializes to Corollary 3.2 of Toyama (1988) (which is stronger than the more restrictive corollary on page 815 in Huet (1980) which says that a left-linear and unconditional system is confluent if all its critical pairs are parallel closed). Moreover, Theorem 13.6(II) is a generalization of Theorem 5.2 of Avenhaus & Becker (1994) translated into our framework.

The proof of Theorem 13.6 is similar to that of Corollary 3.2 of Toyama (1988) for unconditional systems, but with a global induction loop on the depth of reduction for using the shallow joinability to get along with the conditions of the rules, and this whole proof twice due to our separation into constructors and non-constructors, and this again for each part of the theorem. Since it is very long, tedious, and uninteresting we have put most its lemmas into A and the proofs into D. The only lemmas we consider to be interesting are those which make clear why it is possible to generalize from normal to quasi-normal rule systems. The problematic case is always the variable-overlap case since it is not covered by critical peaks. The hard step in this case is to show that an equation "$u_0{=}u_1$" which had been joinable when instantiated with substitution μ is still joinable after the instantiations for its variables have been reduced, yielding a new substitution ν. Thus one has to show that for two natural numbers n_0 and n_1 with $u_0\mu{\downarrow}_{\mathrm{R,X},\omega+n_1}u_1\mu$ and $\forall x{\in}\mathrm{V}.\ x\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_0}x\nu$ we always have $u_0\nu{\downarrow}_{\mathrm{R,X},\omega+n_1}u_1\nu$. This means that the fulfilledness of the instantiated equation "$u_0{=}u_1$" is not changed by the reduction of its instantiating substitution. For showing this we may use the global induction hypothesis implying that $\mathrm{R,X}$ is ω-shallow confluent up to $n_0{+_\omega}n_1$. The reader may verify that we do not seem to have a chance for being successful here unless we require some kind of normality. Lemma 13.7(4) depicts the situation we are in (matching its s_i to $u_i\mu$ and its s'_i to $u_i\nu$) and shows that irreducibility of $u_1\nu$ (roughly speaking i.e. normality) is just as helpful as some literal "Def $u_1\mu$" in the condition list (i.e. an alternative allowed by quasi-normality) (because the latter implies the existence of some $t_1 \in \mathcal{GT}(\mathrm{cons})$ with $u_1\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1}t_1$). Finally, Lemma 13.8 states that the other alternative given by quasi-normality (i.e. that the equation contains no non-constructor variables) is no problem either, and that Def- and $\neq$-literals do not make any problems and therefore need not at all be restricted by normality requirements.

Since we consider the proofs of the following two lemmas to be interesting, we did not put them into the appendix but included them here. The form of presentation is very general. This enables the proof to present the idea of quasi-normality in its essential form and also enables more than a dozen of applications of Lemma 13.8 in the proofs of the theorems in this and the following sections. When reading the lemmas please note that the optional parts are only necessary for reusing the lemmas in the proofs of the theorems of the following sections where termination arguments will be included into the confluence criteria. Moreover for a first reading only the second cases of their initial disjunctive assumptions should be considered. The others are uninteresting special cases.

Lemma 13.7

[Let $\rhd$ be a wellfounded ordering.] Let $n_0, n_1 \prec \omega$. Let $\alpha \in \{0, \omega\}$. Assume that
$\forall i \prec 2. \left(\begin{array}{ll} & s_i = s'_i \\ \vee & \mathrm{R,X} \textit{ is } \alpha\textit{-shallow confluent up to } n_0 +_\alpha n_1 \textit{ [and } s_i \textit{ in } \lhd \textit{]} \end{array} \right)$. *Now:*

1. $n_0 \preceq n_1$ *and* $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0$ *implies* $s'_0 \downarrow_{\mathrm{R,X},\alpha+n_1} t_0$.

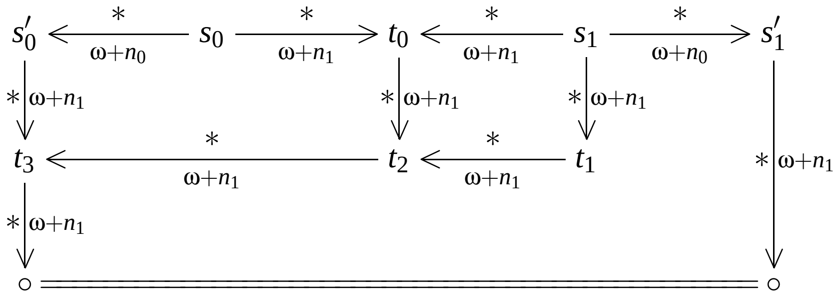
2. $n_0 \preceq n_1$ *and* $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_1} s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_0} s'_1$ *implies* $s'_0 \downarrow_{\mathrm{R,X},\alpha+n_1} s'_1$.

3. $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_2 \in \mathcal{GT}(\mathrm{cons})$ *implies* $\exists t_3 \in \mathcal{GT}(\mathrm{cons}).\ s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_3 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_2$.

4. $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_1} s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_0} s'_1$ *together with either* $s_1 \not\in \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})$ *or*
$\left(\begin{array}{ll} & \alpha = \omega \\ \wedge & s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_1 \in \mathcal{GT}(\mathrm{cons}) \\ \wedge & \forall \delta \prec n_0 +_\omega n_1.\ \mathrm{R,X} \textit{ is } \omega\textit{-shallow confluent up to } \delta \textit{ [and } s_1 \textit{ in } \lhd \textit{]} \end{array} \right)$ *implies*
$s'_0 \downarrow_{\mathrm{R,X},\alpha+n_1} s'_1$.

Proof of Lemma 13.7 1: Consider the peak $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0$. If $s_0 = s'_0$, then we are finished due to $s'_0 = s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0$. Otherwise: We have assumed that R, X is α-shallow confluent up to $n_0 +_\alpha n_1$ [and s_0 in $\lhd$]. Thus we get $s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_0$ and then due to $n_0 \preceq n_1$ and Lemma 2.12 we get $s'_0 \downarrow_{\mathrm{R,X},\alpha+n_1} t_0$.
2: By (1) we get $s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_1 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_1} t_0$ for some t_1. Finally, consider the peak $t_1 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_1} s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_0} s'_1$. By (1) again we get $s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_1 \downarrow_{\mathrm{R,X},\alpha+n_1} s'_1$ as desired.
3: Consider the peak $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_2$. If $s_0 = s'_0$, then we are finished due to $s'_0 = s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_2$. Otherwise: By α-shallow confluence up to $n_0 +_\alpha n_1$ [and s_0 in $\lhd$] we get $s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_3 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_2$ for some t_3. By $t_2 \in \mathcal{GT}(\mathrm{cons})$ and Lemma 2.10 we get $\mathcal{GT}(\mathrm{cons}) \ni t_3 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega} t_2$. Thus we have $s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_3 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_2$ as desired.
4: $s_1 \not\in \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})$: If $s_0 = s'_0$, then we are finished due to $s'_0 = s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0 = s_1 = s'_1$. Otherwise: Consider the peak $s'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_0$. By α-shallow confluence up to $n_0 +_\alpha n_1$ [and s_0 in $\lhd$] we get $s'_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} t_2 \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_0$ for some t_2. Since $s_1 \not\in \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})$ this finishes the proof in this case due to $t_2 = t_0 = s_1 = s'_1$.

$\underline{s_1 \in \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})}$: Then we have $\alpha=\omega$, $s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_1 \in \mathcal{G}\mathcal{T}(\mathrm{cons})$, and $\forall\delta\prec n_0+_\omega n_1.$ $\mathrm{R,X}$ is ω-shallow confluent up to δ [and s_1 in $\lhd$], cf. the diagram below. Consider the peak $t_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_1$. We may assume $n_1 \prec n_0$ because in case of $n_0 \preceq n_1$ the proof is finished due to (2). Then we have $n_1+_\omega n_1 \prec n_0+_\omega n_1$. Thus by ω-shallow confluence up to $n_1+_\omega n_1$ [and s_1 in $\lhd$] we get $t_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_2 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_1$ for some t_2. By $t_1 \in \mathcal{G}\mathcal{T}(\mathrm{cons})$ and Lemma 2.10 we get $\mathcal{G}\mathcal{T}(\mathrm{cons}) \ni t_2$. Consider the peak $s_0' \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_0} s_0 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_2$. Due to $t_2 \in \mathcal{G}\mathcal{T}(\mathrm{cons})$ and (3) there is some $t_3 \in \mathcal{G}\mathcal{T}(\mathrm{cons})$ with $s_0' \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_3 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_2$. By (3) again, the peak $t_3 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} s_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_0} s_1'$ implies $t_3 \downarrow_{\mathrm{R,X},\omega+n_1} s_1'$ as desired.



Q.e.d. (Lemma 13.7)

Lemma 13.8

[Let $\rhd$ be a wellfounded ordering.]
Let $\alpha \in \{0,\omega\}$. *Let* $n_0, n_1 \prec \omega$. *Let* $\mu,\nu \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$.
Let $((l,r),C) \in \mathrm{R}$ *with* $\alpha=0 \Rightarrow l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$.
Assume that $n_0 \preceq n_1$ *or that* $((l,r),C)$ *is* α*-quasi-normal w.r.t.* $\mathrm{R,X}$. *Assume that* $\forall L$ *in* $C.\ \forall u \in \mathcal{TERMS}(L).$

$$\left(\begin{array}{ll} & u\mu \notin \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha}) \\ \vee & \mathrm{R,X}\textit{ is }\alpha\textit{-shallow confluent up to } n_0+_\alpha n_1 \textit{ [and } u\mu \textit{ in } \lhd\textit{]} \\ \vee & \left(\begin{array}{ll} & \forall x \in \mathcal{V}(u).\ x\mu = x\nu \\ \wedge & \left(\begin{array}{ll} & \alpha=0 \\ \vee & \forall v.\ L \notin \{(u{=}v),(v{=}u)\} \\ \vee & \forall x \in \mathcal{V}(L).\ x\mu = x\nu \\ \vee & \forall\delta\prec n_0+_\alpha n_1. \\ & \mathrm{R,X}\textit{ is }\alpha\textit{-shallow confluent up to } \delta \textit{ [and } u\mu \textit{ in } \lhd\textit{]} \end{array}\right) \end{array}\right) \end{array}\right).$$

Now, if $C\mu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R,X},\alpha+n_1}$ *and* $\forall x \in \mathrm{V}.\ x\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_0} x\nu$,
then $C\nu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R,X},\alpha+n_1}$ *and* $l\nu \longrightarrow_{\mathrm{R,X},\alpha+n_1+1} r\nu$.

Proof of Lemma 13.8 Since $\alpha=0 \Rightarrow l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)$, it suffices to show that for each literal L in C: $L\nu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\alpha+n_1}$. Note that we already know that $L\mu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\alpha+n_1}$. In case of $u\mu\not\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})$ we get $u\mu=u\nu$ due to $u\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_0}u\nu$. In case of $\forall x\in\mathcal{V}(u).\,x\mu=x\nu$ we get $u\mu=u\nu$ again. Thus we may assume $\forall u\in\mathcal{TERMS}(L).$ ($u\mu=u\nu\ \vee$ R, X is α-shallow confluent up to $n_0+_\alpha n_1$ [and $u\mu$ in $\lhd$]).

<u>$L=(s_0{=}s_1)$:</u> We have $s_0\nu\overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0}s_0\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1}t_0\overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_1}s_1\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_0}s_1\nu$ for some t_0. In case of $n_0\preceq n_1$ we get the desired $s_0\nu\downarrow_{\mathrm{R,X},\alpha+n_1}s_1\nu$ by Lemma 13.7(2). Otherwise, by assumption of the lemma, $((l,r),C)$ must be α-quasi-normal. Since $C\mu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+\alpha}$, according to the definition of α-quasi-normality and the disjunctive assumption of the lemma we have two distinguish several cases here. First we treat the case in which $\exists i\prec 2.\ s_i\mu\not\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})$. W.l.o.g. say $s_1\mu\not\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\alpha})$. By Lemma 13.7(4) we get the desired $s_0\nu\downarrow_{\mathrm{R,X},\alpha+n_1}s_1\nu$. Second, in case of $\forall x\in\mathcal{V}(L).\,x\mu=x\nu$ we know that $L\nu=L\mu$ which is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\alpha+n_1}$. Note that now we may assume that $\alpha=\omega$ because the second case includes the only case left for 0-quasi-normality, namely $\mathcal{V}(s_0,s_1)\subseteq\emptyset$. Third, in case of $\mathcal{V}(s_0,s_1)\subseteq\mathrm{V}_C$ we have for all $x\in\mathcal{V}(s_0,s_1)$: $x\mu\in\mathcal{T}(\mathrm{cons},\mathrm{V}_C)$; and then $x\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}x\nu$ by Lemma 2.10. This means $s_i\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}s_i\nu$. By Lemma 13.7(2) (matching its n_0 to 0) due to $0+_\omega n_1\preceq n_0+_\omega n_1$ we get the desired $s_0\nu\downarrow_{\mathrm{R,X},\omega+n_1}s_1\nu$. Finally we come to the fourth case where w.l.o.g. $(\mathrm{Def}\,s_1\mu)$ occurs in $C\mu$. Then there is some $t_1\in\mathcal{GT}(\mathrm{cons})$ with $s_1\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1}t_1$. Since we may assume that we are not in any of the previous cases, the disjunctive assumption of the lemma now states that $\forall\delta\prec n_0+_\omega n_1$. R, X is ω-shallow confluent up to δ [and $u\mu$ in $\lhd$]. By Lemma 13.7(4) we get the desired $s_0\nu\downarrow_{\mathrm{R,X},\omega+n_1}s_1\nu$.

<u>$L=(\mathrm{Def}\,s)$:</u> We know the existence of $t\in\mathcal{GT}(\mathrm{cons})$ with $s\nu\overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0}s\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1}t$. By Lemma 13.7(3) there is some $t'\in\mathcal{GT}(\mathrm{cons})$ with $s\nu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1}t'\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1}t$.

<u>$L=(s_0{\neq}s_1)$:</u> There exist some $t_0,t_1\in\mathcal{GT}(\mathrm{cons})$ with $\forall i\prec 2.\ s_i\nu\overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0}s_i\mu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1}t_i$ and $t_0\not\downarrow_{\mathrm{R,X},\alpha+n_1}t_1$. Just like above we get t_0', $t_1'\in\mathcal{GT}(\mathrm{cons})$ with $\forall i\prec 2.\ s_i\nu\overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1}t_i'\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1}t_i$. Finally $t_0'\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1}t_0\not\downarrow_{\mathrm{R,X},\alpha+n_1}t_1\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1}t_1'$ implies $t_0'\not\downarrow_{\mathrm{R,X},\omega+n_1}t_1'$ since we have $\alpha=\omega$ due to $l\not\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)$ in this case of a negative literal. **Q.e.d. (Lemma 13.8)**

We do not have to discuss the following theorem in detail here, because it is very similar to Theorem 13.6, but weakens the required ω-shallow joinabilities to ω-level joinabilities wherever possible. Note that from Example 10.2 we can conclude that the ω-shallow joinabilities required for critical peaks of the form $(0,1)$ cannot be weakened to ω-level joinabilities in any of the four parts of the theorem.[29] However, the price we have to pay for weakening shallow to level joinability is to extend our requirement that the conditions contain constructor variables only, from constructor rules ("conservative constructors") to all rules! That this restriction is necessary indeed can be seen from Example 12.2. On the other hand, this restriction gives *quasi*-normality for free.

We prefer to discuss and apply Theorem 13.6 wherever possible because contrary to Theorem 13.9 it has interesting implications for the standard framework without the separation into constructor and non-constructor symbols where "only constructor variables in conditions" means "no variables in conditions" which again can (in general not effectively) be reduced to "no conditions" by removing the fulfilled conditions and the rules with non-fulfilled conditions.

The main part of the following theorem is part (I). Parts (III) and (IV) only weaken the required ω-level parallel joinability for critical peaks of the form $(1,1)$ to ω-level weak parallel joinability but have to pay a considerable price for it. Furthermore, the difference between (III) and (IV) is marginal since non-overlays of the form $(1,0)$ are pathological anyway. (II) is rather interesting for the cases where it is possible to restrict the right-hand sides to be linear w.r.t. general variables; this severe restriction is necessary, however; cf. the second version of Example 10.7 or cf. Example 10.8.

[29] Note that with the exception of part (II) of the theorem we could also use the first version of Example 10.7 for this conclusion.

Theorem 13.9 (Syntactic Criterion for ω-Level Confluence)

Let R *be a CRS over* sig/cons/V. *Let* X⊆V.
Assume the following important restriction on variables in conditions to hold:

$$\forall((l,r),C)\in\mathrm{R}.\ \mathcal{V}(C)\subseteq\mathrm{V}_C.$$

Moreover, assume the following weak kind of left-linearity:
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q\in\mathcal{POS}(l).\ \forall x\in\mathrm{V}.$

$$\left(\left(\begin{array}{l} l/p=x=l/q \\ \wedge\ \ p\neq q \end{array}\right)\Rightarrow\left(\begin{array}{l} l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C) \\ \wedge\ \ x\in\mathrm{V}_C \end{array}\right)\right).$$

Furthermore, assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent.*

(I) *Now if each critical peak in* CP(R) *of the form* $(0,1)$ *is* ω*-shallow parallel joinable up to* ω *w.r.t.* R, X*, each non-overlay in* CP(R) *of the form* $(1,0)$ *is* ω*-shallow parallel closed up to* ω *w.r.t.* R, X*, each critical peak in* CP(R) *of the form* $(1,1)$ *is* ω*-level parallel joinable w.r.t.* R, X*, and each non-overlay in* CP(R) *of the form* $(1,1)$ *is* ω*-level parallel closed w.r.t.* R, X*, then* R, X *is* ω*-level confluent.*

(II) *If we have the following kind of right-linearity w.r.t. general variables*

$$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V_{SIG}}.\ \forall p,q\in\mathcal{POS}(r).\ \left(\ r/p=x=r/q\ \Rightarrow\ p=q\ \right),$$

and if each critical peak in CP(R) *of the form* $(0,1)$ *is* ω*-shallow strongly joinable up to* ω *w.r.t.* R, X*, each non-overlay in* CP(R) *of the form* $(1,0)$ *is* ω*-shallow anti-closed up to* ω *w.r.t.* R, X*, each critical peak in* CP(R) *of the form* $(1,1)$ *is* ω*-level strongly joinable w.r.t.* R, X*, and each non-overlay in* CP(R) *of the form* $(1,1)$ *is* ω*-level anti-closed w.r.t.* R, X*, then* R, X *is* ω*-level confluent.*

Now additionally assume the following very weak kind of right-linearity of constructor rules:

$$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V_{SIG}}.\ \forall p,q\in\mathcal{POS}(r).\left(\left(\begin{array}{l} l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C) \\ \wedge\ \ r/p=x=r/q \end{array}\right)\ \Rightarrow\ p=q\ \right).$$

Furthermore, additionally assume that each critical peak in CP(R) *of the form* $(0,1)$ *is* ω*-shallow strongly joinable up to* ω*, that each critical peak in* CP(R) *of the form* $(1,1)$ *is* ω*-level weak parallel joinable w.r.t.* R, X*, and that each non-overlay in* CP(R) *of the form* $(1,1)$ *is* ω*-level closed w.r.t.* R, X.

(III) *Now if each non-overlay in* CP(R) *of the form* $(1,0)$ *is* ω*-shallow parallel closed up to* ω *w.r.t.* R, X*, then* R, X *is* ω*-level confluent.*

Now additionally assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is strongly confluent.*

(IV) *Now if each non-overlay in* CP(R) *of the form* $(1,0)$ *is* ω*-shallow closed up to* ω *w.r.t.* R, X*, then* R, X *is* ω*-level confluent.*

14 Criteria for Confluence of Terminating Systems

In this section we examine how we can relax our joinability requirements when we additionally require termination for our reduction relation. Note that in confluence criteria whose proof is by induction on an extension of the reduction relation the joinability requirement can be weakened to a *sub-connectedness* requirement, cf. Küchlin (1985). We here, however, present the simpler versions only, where the connectedness is required to have the form of a single "valley".

Due to its fundamental importance, we first repeat Theorem 7.17 of Wirth & Gramlich (1994a) here, which generalizes Theorem 3 of Dershowitz &al. (1988) by weakening decreasingness to compatibility with a termination-pair (defined in § 2.2) as well as joinability to $\rhd$-weak joinability (defined in § 5) which provides us with some confluence assumption when checking the fulfilledness of the condition of a critical peak.

Definition 14.1 (Compatibility with a Termination-Pair)
A rule $((l,r),C)$ is is R,X-*compatible* with a termination-pair $(>,\rhd)$ over sig/V if $\forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$.

$$\left(C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}} \;\Rightarrow\; \left(\begin{array}{ll} & l\tau > r\tau \\ \wedge & \forall u\in\mathcal{TERMS}(C).\; l\tau\rhd u\tau \end{array} \right) \right).^{30}$$

A CRS R over sig/cons/V is X-*compatible* with a termination-pair $(>,\rhd)$ over sig/V if $\forall((l,r),C)\in\mathrm{R}$. $((l,r),C)$ is R,X-compatible with $(>,\rhd)$.

Theorem 14.2 (Syntactic Test for Confluence)
Let R *be a CRS over* sig/cons/V *and* $\mathrm{X}\subseteq\mathrm{V}$.
Assume that R *is* X*-compatible with a termination-pair* $(>,\rhd)$ *over* sig/V.
[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$. *Define* ($p\in\mathbf{N}_+^*$) $A(p) := \{\, t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\omega,q}) \mid \emptyset\neq q\lll_t p \,\}$. *]*
The following two are logically equivalent:

1. *Each critical peak in* $\mathrm{CP}(\mathrm{R})$ *is* $\rhd$*-weakly joinable w.r.t.* R,X *[besides A].*
2. $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

[30] We could require the weaker $\forall u\in\mathcal{TERMS}(C).\left(\begin{array}{ll} & u\tau\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \\ \vee & l\tau\rhd u\tau \end{array} \right)$ instead of $\forall u\in\mathcal{TERMS}(C).\; l\tau\rhd u\tau$ here. Theorem 14.2 would still be true since its proof need not be modified. We did not do this because we did not see an interesting application that would justify the change of the notion already introduced in Wirth & Gramlich (1993), Wirth &al. (1993), and Wirth & Gramlich (1994a).

Due to a weakening of the notion of $\triangleright$-weak joinability, Theorem 14.2 actually differs from Theorem 7.17 of Wirth & Gramlich (1994a) in that it provides several irreducibility assumptions intended to restrict the number of substitutions φ for which for a critical peak

$$\big(\ (l_1[\,p\leftarrow r_0\,],C_0,\ldots),\ (r_1,C_1,\ldots),\ l_1,\ \sigma,\ p\ \big)$$

resulting from two rules $l_0{=}r_0{\longleftarrow}C_0$ and $l_1{=}r_1{\longleftarrow}C_1$ (with no variables in common) we have to show $l_1[\,p\leftarrow r_0\,]\sigma\varphi{\downarrow_{\mathrm{R,X}}}r_1\sigma\varphi$ in case of $(C_0C_1)\sigma\varphi$ being fulfilled. This means that Theorem 14.2 provides further means to tackle problem 4 of our § 1.

The first assumption allowed is that the substitution φ itself is normalized: $\forall x\in\mathrm{V}.\ x\varphi\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$.

The second allows to assume that for non-overlays (i.e. for $p\neq\emptyset$) even $\sigma\varphi$ is normalized on all variables occurring in the left-hand side l_1.

Moreover, by weakening "$\triangleright$-weak joinability" to "$\triangleright$-weak joinability besides A" with A defined as in the theorem via some family $\ggg\ =(\ggg_t)_{t\in\mathcal{T}(\mathrm{sig,X})}$ of arbitrary wellfounded orderings $\ggg_t$ on $\mathcal{POS}(t)$, we have added a new feature which allows to assume the instantiated peak term (or superposition term) $l_1\sigma\varphi$ to be irreducible at all nonempty positions which are $\lll_{l_1\sigma\varphi}$-smaller than the overlap position p. Generally, beyond our first two assumptions, we may use $\lll$ to further reduce the number of instantiations for which the joinability test must succeed in the following way: If we can choose $\lll_{l_1\sigma\varphi}$ such that

$$\left(p{=}\emptyset\ \Rightarrow\ \forall x\in\mathcal{V}(l_1).\left(\begin{array}{l}\quad x\sigma\neq x\\ \Rightarrow\ \ \exists q\in\mathcal{POS}(l_1).\left(\begin{array}{ll} & l_1/q=x\\ \wedge & \forall q'\in\mathcal{POS}(x\sigma\varphi).\ qq'\lll_{l_1\sigma\varphi}\ p\end{array}\right)\end{array}\right)\right)$$

as well as $\quad\forall x\in\mathcal{V}(l_0).\left(\begin{array}{l}\quad x\sigma\neq x\\ \Rightarrow\ \ \exists q\in\mathcal{POS}(l_0).\left(\begin{array}{ll} & l_0/q=x\\ \wedge & \forall q'\in\mathcal{POS}(x\sigma\varphi).\ pqq'\lll_{l_1\sigma\varphi}\ p\end{array}\right)\end{array}\right)$,

then we may assume $\sigma\varphi$ to be normalized: $\forall x\in\mathrm{V}.\ x\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$. This can be a considerable help for showing that $(C_0C_1)\sigma\varphi$ is not fulfilled when we have a certain knowledge on the normal forms of the terms of the sorts of the variables occurring in C_0C_1. E.g., when we define the depth of a term $t\in\mathcal{T}$ by $\mathrm{depth}(t):=\max\{\ |p'|\ \mid\ p'\in\mathcal{POS}(t)\ \}$ and then define ($p,q\in\mathcal{POS}(t)$) $q\lll_t p$ if $\mathrm{depth}(t)-|q|\prec\mathrm{depth}(t)-|p|$, then we can forget about all critical peaks which are called "composite" in § 2.3 of Kapur &al. (1988) — and even some more, namely all those whose peak term is reducible at some position that is longer than the overlap position of the critical peak. Kapur &al. (1988) already states in Corollary 5 that (unless $l_0\in\mathrm{V}$, which some authors generally disallow) the irreducibility of these positions implies the irreducibility of all terms introduced by the unifying substitution σ; more precisely, the joinability test may assume: $\forall x\in\mathrm{V}.\ \big(\ x\sigma\neq x\ \Rightarrow\ x\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})\ \big)$, which, by our first irreducibility assumption can be simplified to $\forall x\in\mathrm{V}.\ x\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$. If we, however, revert $\lll$ by defining $q\lll_t p$ if $|q|\prec|p|$, then we can forget about all critical peaks which are called "composite" in § 4.1 of Kapur &al. (1988) — and even some more, namely all those whose peak term is reducible at some nonempty position that is shorter than the overlap position of the critical peak.

The power of the combination of the two weakenings of the joinability requirement, i.e. the confluence and the irreducibility assumptions, is demonstrated by the following simple but non-trivial example whose predicate ‘nonnegp’ checks whether an integer number is non-negative:

Example 14.3

$$\begin{array}{lll}
\mathbb{C} & := & \{0, \mathsf{s}, \mathsf{p}, \mathsf{true}, \mathsf{false}\} \\
\mathbb{N} & := & \{\mathsf{nonnegp}\} \\
\mathrm{R}_{14.3} & : & \begin{array}{lcll}
\mathsf{s}(\mathsf{p}(y)) & = & y & \\
\mathsf{p}(\mathsf{s}(y)) & = & y & \\
\mathsf{nonnegp}(0) & = & \mathsf{true} & \\
\mathsf{nonnegp}(\mathsf{s}(x)) & = & \mathsf{true} & \longleftarrow \mathsf{nonnegp}(x) = \mathsf{true} \\
\mathsf{nonnegp}(\mathsf{p}(0)) & = & \mathsf{false} & \\
\mathsf{nonnegp}(\mathsf{p}(x)) & = & \mathsf{false} & \longleftarrow \mathsf{nonnegp}(x) = \mathsf{false}
\end{array}
\end{array}$$

Let 0, s, p be constructor symbols of the sort int and true, false constructor symbols of the sort bool. Let nonnegp be a non-constructor predicate with arity “ int $\rightarrow$ bool ”. Let x, y be constructor variables of the sort int.

Obviously, $\mathrm{R}_{14.3}, \mathrm{V}$ is V-compatible with the termination-pair $(\rhd, \rhd)$ where $\rhd$ is the lexicographic path ordering generated by nonnegp being bigger than true and false.

There are only the following two critical peaks which are both of the form $(0,1)$:

$$\begin{array}{ccc}
\mathsf{nonnegp}(\mathsf{s}(x))\sigma & \longrightarrow & \mathsf{true} \\
\Big\downarrow {\scriptstyle 1,1} & & \\
\mathsf{nonnegp}(y) & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathsf{nonnegp}(\mathsf{p}(x))\sigma' & \longrightarrow & \mathsf{false} \\
\Big\downarrow {\scriptstyle 1,1} & & \\
\mathsf{nonnegp}(y) & &
\end{array}$$

where $\sigma := \{x \mapsto \mathsf{p}(y)\}$ and $\sigma' := \{x \mapsto \mathsf{s}(y)\}$. Their respective condition lists are the following two lists containing each one literal only:

$$\mathsf{nonnegp}(x)\sigma = \mathsf{true} \qquad\qquad \mathsf{nonnegp}(x)\sigma' = \mathsf{false}$$

Now the following is easy to show: The irreducible constructor terms of the sort int are exactly the terms of the form $\mathsf{s}^n(z)$ or $\mathsf{p}^{n+1}(z)$ with $n \in \mathbf{N}$ and $z \in \mathrm{V}_{\mathcal{C},\mathsf{int}} \cup \{0\}$. The irreducible constructor terms of the sort bool are $\mathrm{V}_{\mathcal{C},\mathsf{bool}} \cup \{\mathsf{true}, \mathsf{false}\}$. Furthermore, by induction on $n \in \mathbf{N}$ one easily shows $\mathsf{nonnegp}(\mathsf{s}^n(0)) \overset{*}{\longrightarrow}_{\mathrm{R}_{14.3},\emptyset} \mathsf{true}$ and $\mathsf{nonnegp}(\mathsf{p}^{n+1}(0)) \overset{*}{\longrightarrow}_{\mathrm{R}_{14.3},\emptyset} \mathsf{false}$. Finally by induction on $n \in \mathbf{N}$ one easily shows that $\mathsf{nonnegp}(t) \overset{*}{\longrightarrow}_{\mathrm{R}_{14.3},\mathrm{V},\omega+n} \mathsf{true} \vee \mathsf{nonnegp}(t) \overset{*}{\longrightarrow}_{\mathrm{R}_{14.3},\mathrm{V},\omega+n} \mathsf{false}$ implies $\mathcal{V}(t) = \emptyset$, which we only need to show confluence besides ground confluence.

Define $\lll$ via $(p,q \in \mathcal{POS}(t))$: $q \lll_t p$ if $\mathrm{depth}(t) - |q| \prec \mathrm{depth}(t) - |p|$. Now the new combined weakening of joinability to $\rhd$-weak joinability w.r.t. $\mathrm{R}_{14.3}, \mathrm{V}$ besides A (with A defined as in the theorem) allows us to show joinability of the above critical peaks very easily. Since the second critical peak can be treated analogous to the first, we explain how to treat the first only: By the new additional feature for assuming irreducibility, our weakened joinability allows to assume that $x\sigma\varphi$ is irreducible for the first critical peak, which can be seen in two different ways: First, since the critical peak is a non-overlay and x occurs in the peak term $\mathsf{nonnegp}(\mathsf{s}(x))$. Second, since the overlap position is 1, $\mathsf{nonnegp}(\mathsf{s}(x))/1\,1 = x$ and $\forall q' \in \mathcal{POS}(x\sigma\varphi).\ 1\,1\,q' \lll_{\mathsf{nonnegp}(\mathsf{s}(x))\sigma\varphi} 1$. Furthermore, we are allowed to assume that the condition of the critical peak is fulfilled, i.e. that $\mathsf{nonnegp}(x)\sigma\varphi \xrightarrow{*}_{\mathrm{R}_{14.3},\mathrm{V}} \mathsf{true}$. Together with the irreducibility of $x\sigma\varphi = \mathsf{p}(y)\varphi$ this implies that $y\varphi$ is of the form $\mathsf{p}^n(0)$. This again implies $\mathsf{nonnegp}(x)\sigma\varphi \xrightarrow{*}_{\mathrm{R}_{14.3},\mathrm{V}} \mathsf{false}$. But since we may assume confluence below the condition term $\mathsf{nonnegp}(x)\sigma\varphi$ we get $\mathsf{true} \downarrow_{\mathrm{R}_{14.3},\mathrm{V}} \mathsf{false}$, which is impossible. Thus the properties that weak joinability allows us to assume for the joinability test are inconsistent and the critical pair need not be joined at all.

All in all, Theorem 14.2 implies confluence of $\longrightarrow_{\mathrm{R}_{14.3},\mathrm{V}}$ without solving the task of showing that for each arbitrary (not necessarily normalized) substitution φ either $\mathsf{nonnegp}(\mathsf{p}(y))\varphi \xrightarrow{*}_{\mathrm{R}_{14.3},\mathrm{V}} \mathsf{true}$ does not hold or $\mathsf{nonnegp}(y)\varphi \xrightarrow{*}_{\mathrm{R}_{14.3},\mathrm{V}} \mathsf{true}$ holds, which is more difficult to show than our simple properties above.

The following theorem is a generalization of Theorem 7.18 in Wirth & Gramlich (1994a). In comparison with Theorem 14.2 it offers for each condition term u of a rule $l{=}r{\longleftarrow}C$ the possibility to replace the requirement $l\tau \rhd u\tau$ (roughly speaking i.e. decreasingness) with $\mathcal{V}(u)\subseteq \mathrm{V}_{\mathcal{C}}$ (i.e. the absence of general variables). The basic idea of its proof is first to show ω-shallow confluence up to ω (i.e. commutation of $\longrightarrow_{\mathrm{R,X},\omega}$ and $\longrightarrow_{\mathrm{R,X}}$) with the usual argumentation on quasi-normality, left-linearity, termination and ω-shallow joinability (cf. Theorem 14.5), and then to use decreasingness argumentation for the confluence of $\longrightarrow_{\mathrm{R,X}}$.

Theorem 14.4 (Syntactic Test for Confluence)

Let R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume the following very weak kind of left-linearity of constructor rules w.r.t. general variables:
$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V}_{\mathrm{SIG}}.\ \forall p,q\in\mathcal{POS}(l).$

$$\left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ \wedge & l/p=x=l/q\end{array}\right)\Rightarrow\ p=q\right).$$

Furthermore, assume that constructor rules are quasi-normal w.r.t. R, X*:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ \wedge & C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega}\end{array}\right)\Rightarrow\ ((l,r),C) \text{ is quasi-normal w.r.t. } \mathrm{R,X}\right).$$

Moreover, assume the following compatibility property for a termination-pair $(>,\rhd)$ *over* sig/V*:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ \Rightarrow\left(\begin{array}{ll} & l\tau>r\tau\\ \wedge & \forall u\in\mathcal{TERMS}(C).\left(\begin{array}{ll} & l\tau\rhd u\tau\\ \vee & u\tau\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})\\ \vee & \mathcal{V}(u)\subseteq\mathrm{V}_{\mathcal{C}}\end{array}\right)\end{array}\right)\right).$$

Assume $\longrightarrow_{\mathrm{R,X},\omega}$ *to be confluent.*
Assume that each critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)\in\mathrm{CP}(\mathrm{R})$
(with $(\Lambda_0,\Lambda_1)\neq(1,1)$ *and (* $(\Lambda_0,\Lambda_1)\neq(0,0)\ \vee\ \mathcal{TERMS}(D_0\sigma D_1\sigma)\not\subseteq\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathcal{C}})$ *))*
is ω-shallow joinable up to ω w.r.t. R, X *and* $\lhd$.

[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$*. Define* ($p\in\mathbf{N}_+^*$)
$A(p):=\{\ t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\omega,q})\mid\emptyset\neq q\lll_t p\ \}\cup\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega})$*.]*

Now the following two are logically equivalent:

1. *Each critical peak* $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)\in\mathrm{CP}(\mathrm{R})$
   (with $\forall k\prec 2.\ (\ \Lambda_k=1\ \vee\ \mathcal{TERMS}(D_k\sigma)\not\subseteq\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathcal{C}})\ )$*)*
   is $\rhd$*-weakly joinable w.r.t.* R, X *[besides A].*
2. $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

The following theorem generalizes Theorem 2 in Dershowitz &al. (1988) by weakening normality to quasi-normality.

Theorem 14.5 (Syntactic Test for ω-Shallow Confluence)

Let R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume the following weak kind of left-linearity w.r.t. general variables:
$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V_{SIG}}.\ \forall p,q\in\mathcal{POS}(l).\ (\ l/p=x=l/q\ \Rightarrow\ p=q\ ).$

Furthermore, assume R, X *to be quasi-normal.*

Let $(>,\rhd)$ *be a termination-pair over* sig/V *such that the following compatibility property for constructor rules holds (which is always satisfied when* R *has conservative constructors):*
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}}) \\ \wedge & C\tau \textit{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega}\end{array}\right)\Rightarrow\ \forall u\in\mathcal{TERMS}(C).\left(\begin{array}{ll} & l\tau\rhd u\tau \\ \vee & u\tau\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \\ \vee & \mathcal{V}(u)\subseteq\mathrm{V}_{\mathcal{C}}\end{array}\right)\right).$$

Furthermore, assume that the system is terminating:

$$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ (\ (\ C\tau \textit{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ )\ \Rightarrow\ l\tau>r\tau\ ).$$

[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$. *Define (*$p\in\mathbf{N}_+^*$*,* $n\prec\omega$*)* $A(p,n):=\{\ t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+n,q})\mid\emptyset\neq q\lll_t p\ \}$*.]*

Now the following two are logically equivalent:

1. $\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent and*
   each critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)\in\mathrm{CP}(\mathrm{R})$
   (with ($(\Lambda_0,\Lambda_1)\neq(0,0)\ \vee\ \mathcal{TERMS}(D_0\sigma D_1\sigma)\not\subseteq\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathcal{C}})$ *))*
   is ω*-shallow joinable w.r.t.* R, X *and* $\lhd$ *[besides A].*
2. R, X *is* ω*-shallow confluent.*

The following theorem weakens the ω-shallow joinability requirement to that of ω-level joinability, but disallows general variables in conditions of rules. That this restriction is necessary indeed can be seen from Example 12.2.

Theorem 14.6 (Syntactic Test for ω-Level Confluence)

Let R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.

Assume $\forall((l,r),C)\in\mathrm{R}.\ \mathcal{V}(C)\subseteq\mathrm{V}_C$.

Let $(>,\rhd)$ *be a termination-pair over* sig/V. *Assume that the system is terminating:*

$$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \big(\ \big(\ C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ \big)\Rightarrow\ l\tau>r\tau\ \big).$$

[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$. *Define (*$p\in\mathbf{N}_+^*$, $n\prec\omega$*)* $A(p,n):=\{\ t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+n,q})\mid\emptyset\neq q\lll_t p\ \}$. *]*

Now the following two are logically equivalent:

1. $\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent and each critical peak in* CP(R)
   of the forms $(0,1)$, $(1,0)$, *or* $(1,1)$
   is ω*-level joinable w.r.t.* R, X *and* $\rhd$ *[besides A].*
2. R, X *is* ω*-level confluent.*

The following theorem generalizes Theorem 4 in Dershowitz &al. (1988) and Theorem 6.3 in Wirth & Gramlich (1994a) by weakening overlay joinability to $\rhd$-quasi overlay joinability. For a discussion of the notion of $\rhd$-quasi overlay joinability cf. § 9. The proof is discussed above the key lemma B.8.

Theorem 14.7 (Syntactic Confluence Criterion)

Let R *be a CRS over* sig/cons/V *and* X⊆V.

Assume either that $\longrightarrow_{\mathrm{R,X}}$ *is terminating*[31] *and* $\rhd = \rhd_{\mathrm{ST}}$ *or that* $\longrightarrow_{\mathrm{R,X}} \subseteq \rhd$, $\rhd_{\mathrm{ST}} \subseteq \rhd$, *and* $\rhd$ *is a wellfounded ordering on* $\mathcal{T}$.

Now, if all critical peaks in $\mathrm{CP(R)}$ *are* $\rhd$*-quasi overlay joinable w.r.t.* R, X,
then $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

Example 14.8

Let X⊆V. The following system is neither decreasing, nor left-linear, nor overlay joinable; but it is terminating and $\rhd_{\mathrm{ST}}$-quasi overlay joinable w.r.t. $\mathrm{R}_{14.8}$, X. Thus Theorem 14.7 is the only one that implies confluence of $\longrightarrow_{\mathrm{R}_{14.8},\mathrm{X}}$. Note that Theorem 14.4 becomes applicable when we replace the non-constructor variable in (p1) with a constructor variable. Moreover, if we additionally do the same with (p2), then Theorem 14.6 becomes applicable, too.

Even though it is irrelevant for Theorem 14.7, let $X, Y \in \mathrm{V_{SIG}}$, $0, \mathsf{s}, \mathsf{a}, \mathsf{true}, \mathsf{false} \in \mathbb{C}$, and $\mathsf{less}, \mathsf{p}, \mathsf{f}, \mathsf{g} \in \mathbb{F}$. Note that $0, \mathsf{s}, \mathsf{a}, \mathsf{less}$ model the ordinal number $\omega+1$.

$\mathrm{R}_{14.8}$:

(s1)	$\mathsf{s}(\mathsf{a})$	$= \mathsf{a}$	
(less1)	$\mathsf{less}(\mathsf{s}(X), \mathsf{s}(Y))$	$= \mathsf{less}(X, Y)$	
(less2)	$\mathsf{less}(X, X)$	$= \mathsf{false}$	
(less3)	$\mathsf{less}(0, \mathsf{s}(Y))$	$= \mathsf{true}$	
(less4)	$\mathsf{less}(X, 0)$	$= \mathsf{false}$	
(less5)	$\mathsf{less}(0, \mathsf{a})$	$= \mathsf{true}$	
(less6)	$\mathsf{less}(\mathsf{a}, \mathsf{s}(Y))$	$= \mathsf{less}(\mathsf{a}, Y)$	
(less7)	$\mathsf{less}(\mathsf{s}(X), \mathsf{a})$	$= \mathsf{less}(X, \mathsf{a})$	
(p1)	$\mathsf{p}(X)$	$= \mathsf{true}$	$\longleftarrow \mathsf{p}(\mathsf{s}(X))=\mathsf{true}$
(p2)	$\mathsf{p}(X)$	$= \mathsf{true}$	$\longleftarrow \mathsf{less}(\mathsf{f}(X), \mathsf{g}(X))=\mathsf{true}$
(fi)	$\mathsf{f}(X)$	$= \ldots$	
(gi)	$\mathsf{g}(X)$	$= \ldots$	

[31] Actually innermost termination is enough here when we require overlay joinability instead of $\rhd$-quasi overlay joinability, cf. Gramlich (1995a).

The critical peaks are the following:

From (s1) into (less1) we get:

$$\begin{array}{ccc} \mathsf{less}(\mathsf{s}(\mathsf{a}),\mathsf{s}(Y)) & \xrightarrow[\omega+1]{} & \mathsf{less}(\mathsf{a},Y) \\ \downarrow{\scriptstyle 1,1} & & \| \\ \mathsf{less}(\mathsf{a},\mathsf{s}(Y)) & \xrightarrow[\omega+1,\emptyset]{} & \mathsf{less}(\mathsf{a},Y) \end{array}$$

$$\begin{array}{ccc} \mathsf{less}(\mathsf{s}(X),\mathsf{s}(\mathsf{a})) & \xrightarrow[\omega+1]{} & \mathsf{less}(X,\mathsf{a}) \\ \downarrow{\scriptstyle 1,2} & & \| \\ \mathsf{less}(\mathsf{s}(X),\mathsf{a}) & \xrightarrow[\omega+1,\emptyset]{} & \mathsf{less}(X,\mathsf{a}) \end{array}$$

$$\begin{array}{ccc} \mathsf{less}(\mathsf{s}(\mathsf{a}),\mathsf{s}(\mathsf{a})) & \xrightarrow[\omega+1]{} & \mathsf{less}(\mathsf{a},\mathsf{a}) \\ \not\downarrow{\scriptstyle 1,\{1,2\}} & & \| \\ \mathsf{less}(\mathsf{a},\mathsf{a}) & = & \mathsf{less}(\mathsf{a},\mathsf{a}) \end{array}$$

From (s1) into (less3) we get:

$$\begin{array}{ccc} \mathsf{less}(0,\mathsf{s}(\mathsf{a})) & \xrightarrow[\omega+1]{} & \mathsf{true} \\ \downarrow{\scriptstyle 1,2} & & \| \\ \mathsf{less}(0,\mathsf{a}) & \xrightarrow[\omega+1,\emptyset]{} & \mathsf{true} \end{array}$$

The criticial peaks resulting from (s1) into (less6) and (less7) are trivial.
From (less1) into (less2) we get:

$$\begin{array}{ccc} \mathsf{less}(\mathsf{s}(X),\mathsf{s}(X)) & \xrightarrow[\omega+1]{} & \mathsf{false} \\ \downarrow{\scriptstyle \omega+1,\emptyset} & & \| \\ \mathsf{less}(X,X) & \xrightarrow[\omega+1,\emptyset]{} & \mathsf{false} \end{array}$$

From (less2) into (less1) we get:

$$\begin{array}{ccc} \mathsf{less}(\mathsf{s}(X),\mathsf{s}(X)) & \xrightarrow[\omega+1]{} & \mathsf{less}(X,X) \\ \downarrow{\scriptstyle \omega+1,\emptyset} & & \downarrow{\scriptstyle \omega+1,\emptyset} \\ \mathsf{false} & = & \mathsf{false} \end{array}$$

The criticial peaks resulting from (less2) into (less4), (less4) into (less2), (p1) into (p2), and (p2) into (p1) are trivial.

15 Criteria for Confluence of the Constructor Sub-System

Define the *constructor sub-system* of a rule system R to be

$$\mathrm{R}_{\mathcal{C}} := \{\ ((l,r),C) \in \mathrm{R} \mid l \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})\ \},$$

i.e. the system of the constructor rules of R. In this section we discuss the problem how to find out that $\longrightarrow_{\mathrm{R,X},\omega} = \longrightarrow_{\mathrm{R}_{\mathcal{C}},\mathrm{X},\omega}$ is confluent. Note that this is a necessary ingredient for achieving confluence via any of the theorems 13.3, 13.4, 13.6, 13.9, 14.4, 14.5, and 14.6.

The easiest way to achieve confluence of $\longrightarrow_{\mathrm{R,X},\omega}$ is to have no constructor rules at all, i.e. $\mathrm{R}_{\mathcal{C}} = \emptyset$. While it is rather restrictive, this case of *free constructors* is very important in practice since a lot of data structures can be specified this way. Moreover, it is economic to restrict to this case because non-free constructors make a lot of trouble when working with the specification, e.g., most techniques for proving inductive validity get into tremendous trouble with non-free constructors — if they are able to handle them at all.

The second case where confluence of $\longrightarrow_{\mathrm{R,X},\omega}$ is immediate is when for each rule $l{=}r{\longleftarrow}C$ in $\mathrm{R}_{\mathcal{C}}$ also $r{=}l{\longleftarrow}C$ is an instance of a rule of R, and then also of $\mathrm{R}_{\mathcal{C}}$ due to the restriction on the constructor rule $l{=}r{\longleftarrow}C$ given by Definition 2.2. An example for this is the commutativity rule which is equal to a renamed version of the reverse of itself. In this case it may be worthwhile to consider reduction modulo a constructor congruence as described in Avenhaus & Becker (1992) and Avenhaus & Becker (1994).

A third way to achieve confluence of $\longrightarrow_{\mathrm{R,X},\omega}$ is to use semantic confluence criteria in the style of Plaisted (1985), cf. also Theorem 6.5 in Wirth & Gramlich (1994a). While this semantic argumentation is very powerful when one has sufficient knowledge about the constructor domain, it is, however, not at all obvious how to formalize or even automate such semantic considerations. Above that, these semantic confluence criteria are based on the existence of normal forms and therefore require termination of the constructor sub-system (at least in some weak form).

Termination of the constructor sub-system, of course, does not mean termination of the whole rule system. We may, e.g., apply Theorem 14.2 to infer confluence of a terminating constructor sub-system containing the associativity rule of Example 10.8 (whose confluence can hardly be inferred without termination) and then infer the confluence of the whole non-terminating rule system by some of the theorems of § 13. This case where a terminating constructor sub-system is part of a non-terminating rule system seems to be important in practice since confluence of non-free constructors often can hardly be inferred without termination whereas termination is usually not needed for then inferring confluence of the whole system because the non-constructor rules can be chosen in such a way that their critical peaks are complementary, cf. Theorem 13.3. Moreover note that the reverse case, i.e. that of a non-terminating constructor sub-system of a terminating rule system, is impossible in our framework but not in the abovementioned one of Avenhaus & Becker (1992) and Avenhaus & Becker (1994) where the notion of reduction is different, namely reduction via $\mathrm{R} \setminus \mathrm{R}_{\mathcal{C}}$ modulo $\mathrm{R}_{\mathcal{C}}$.

In the rest of this section we will present syntactic criteria for confluence of $\longrightarrow_{\mathrm{R,X},\omega}$.

First note that the theorems 14.2 and 14.7 can directly be applied to infer confluence of $\longrightarrow_{\mathrm{R},\mathrm{X},\omega}$ simply by instantiating the 'R' of these theorems with $\mathrm{R}_{\mathcal{C}}$.

The other theorems we will present in the following are nothing but informal corollaries of other theorems of the sections 13 and 14. To apply the latter theorems to our special case here, it is not sufficient only to throw away the non-constructor rules, but we also have to transform the constructor function symbols of the constructor rules into non-constructor function symbols. For consistency we then also have to rename their constructor variables with general variables. Then the constructor sub-system of the transformed system is empty and therefore trivially confluent, such that these theorems can be applied. If the constructor rules contain general variables or Def-literals, then, however, this transformation brings us beyond the two layered framework presented in this paper: As we translate constructor variables (level 0) into general variables (level 1), then, for consistency, since $\longrightarrow_{\mathrm{R},\mathrm{X},\omega}$ is a relation on the terms of the whole signature, we also have to translate general variables (level 1) into some kind of variables of level 2, and non-constructor function symbols (level 1) into some kind of function symbols of level 2. Symbols of level 2, however, are not present in the framework presented in this paper. Moreover we have to translate our Def-literals (which test for reducibility to a ground term of level 0) into predicate literals that test for reducibility to a ground term of level 1, which are also not present in our framework. While it would be possible and beautiful to present our confluence criteria of the sections 13 and 14 in a framework with a special signature and variable-system for the level of each natural number, we have decided not to do so for the following reasons: First, it would make the paper even more technically and conceptually difficult as it is. Second, the infinitely layered framework may be of little importance (since its only useful application so far is this section). Third, the step of level 0 we want to treat here may in principle allow of more powerful criteria than an arbitrary level *i* and therefore it does not seem to be a good idea to achieve its confluence criteria as corollaries of the theorems for an arbitrary level. Fourth, by proving the theorems of this section separately, we provide the reader interested only in the standard positive conditional rule systems without constructor sub-signature and constructor sub-system with a direct approach to this special case. This can clearly be seen when one translates a system of the standard positive conditional framework into our framework by simply saying that all its symbols are constructor symbols.

For all the following theorems let $\mathrm{R}_{\mathcal{C}}$ be the constructor sub-system of a CRS R over sig/cons/V as defined above, and let $\mathrm{X}\subseteq\mathrm{V}$. Note that the critical peaks in $\mathrm{CP}(\mathrm{R}_{\mathcal{C}})$ are exactly the critical peaks of the form $(0,0)$ in $\mathrm{CP}(\mathrm{R})$.

The following is the analogue of parts (I) and (II) of Theorem 13.6. Note that we do not present the analogues of parts (III) and (IV) because they are subsumed[32] by the analogue of part (I).

Theorem 15.1 (Syntactic Criterion for 0-Shallow Confluence)
Assume R,X *to be* 0*-quasi-normal and* R_C *to be left-linear.*

(I) *Now if each critical peak in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$ *is* 0*-shallow noisy parallel joinable w.r.t.* R,X*, and each non-overlay in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$ *is* 0*-shallow parallel closed w.r.t.* R,X*, then* R,X *is* 0*-shallow confluent.*

(IIa) *If* R_C *is right-linear and if each critical peak in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$ *is* 0*-shallow noisy strongly joinable w.r.t.* R,X*, and each non-overlay in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$ *is* 0*-shallow noisy anti-closed w.r.t.* R,X*, then* R,X *is* 0*-shallow confluent.*

(IIb) *If* R_C *is right-linear and if each critical peak in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$ *is* 0*-shallow strongly joinable w.r.t.* R,X*, and each non-overlay in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$ *is* 0*-shallow anti-closed w.r.t.* R,X*, then* $\longrightarrow_{\mathrm{R},\mathrm{X},\omega}$ *is strongly confluent.*

Corollary 15.2 *If* R,X *is* 0*-shallow confluent, then* $\longrightarrow_{\mathrm{R},\mathrm{X},\omega}$ *is confluent.*

We omit the analogue of Theorem 13.9 here because it requires that the conditions of the constructor rules do not contain any variables. In this case R_C can (in general not effectively) be transformed into an unconditional system with identical reduction relation (with possibly different depths) to which we can then apply Theorem 15.1 instead.

The following is the analogue of Theorem 13.3.

Theorem 15.3 (Syntactic Confluence Criterion)
If R_C *is left-linear and normal and all critical peaks of* R_C *are complementary, then* $\longrightarrow_{\mathrm{R},\mathrm{X},\omega}$ *is confluent.*

The analogue of theorems 14.2 and 14.4 is just Theorem 14.2 with ‘R’ instantiated with R_C.

[32]This is because the notion of 0-shallow [noisy] *weak* parallel joinability (when defined analogous to the notion of ω-shallow [noisy] *weak* parallel joinability) is identical to the notion of 0-shallow [noisy] parallel joinability.

The following is the analogue of Theorem 14.5.

Theorem 15.4 (Syntactic Test for 0-Shallow Confluence)
Let $(>,\rhd)$ be a termination-pair over sig/V.
Assume R,X *to be* 0*-quasi-normal and* $\mathrm{R}_{\mathcal{C}}$ *to be left-linear.*
Furthermore, assume that $\longrightarrow_{\mathrm{R},\mathrm{X},\omega}$ *is terminating:*
$$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \left(\ \left(\begin{array}{rl} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ \wedge & C\tau\text{ fulfilled w.r.t. }\longrightarrow_{\mathrm{R},\mathrm{X},\omega}\end{array}\right)\ \Rightarrow\ l\tau>r\tau\ \right).$$
[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ assume $\lll_t$ to be a wellfounded ordering on $\mathcal{POS}(t)$. Define ($p\in\mathbf{N}_+^*$, $n\prec\omega$) $A(p,n):=\{\,t\in\mathrm{dom}(\longrightarrow_{\mathrm{R},\mathrm{X},n,q})\mid\emptyset\neq q\lll_t p\,\}$.]
Now the following two are logically equivalent:

1. *Each critical peak in* $\mathrm{CP}(\mathrm{R})$ *of the form* $(0,0)$
   is 0*-shallow joinable w.r.t.* R,X *and* $\lhd$ *[besides A].*

2. R,X *is* 0*-shallow confluent.*

We omit the analogue of Theorem 14.6 here because it requires that the conditions of the constructor rules do not contain any variables. In this case $\mathrm{R}_{\mathcal{C}}$ can be transformed into an unconditional system with identical reduction relation to which we can then apply Theorem 14.2 with ‘R’ instantiated with $\mathrm{R}_{\mathcal{C}}$.

The analogue of Theorem 14.7 is just Theorem 14.7 with ‘R’ instantiated with $\mathrm{R}_{\mathcal{C}}$.

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Acknowledgements: I would like to thank Bernhard Gramlich for many fruitful discussions and Jürgen Avenhaus, Roland Fettig, Klaus Madlener, Birgit Reinert, and Andrea Sattler-Klein for some useful hints. I also would like to thank Thomas Deiß for providing me with a TEX-version with huge semantic stack size and Paul Taylor for his support with his diagram typesetting TEX-package.

A Further Lemmas for Section 13

Lemma A.1 *Let* R *be a CRS over* sig/cons/V. *Let* X$\subseteq$V.
Assume R *to have conservative constructors,* R, X *to be quasi-normal, and the following weak kind of left-linearity:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q\in\mathcal{POS}(l).\ \forall x\in\mathrm{V}.$
$$\left(\left(\begin{array}{ll} & l/p=x=l/q\\ \wedge & p\neq q\end{array}\right)\Rightarrow\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_C)\\ \wedge & x\in\mathrm{V}_C\end{array}\right)\right).$$
Furthermore, assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent, that each critical peak from* CP(R) *of the form* $(0,1)$ *is* ω*-shallow* [*noisy*] *parallel joinable up to* ω *w.r.t.* R, X*, and that each non-overlay from* CP(R) *of the form* $(1,0)$ *is* ω*-shallow parallel closed up to* ω *w.r.t.* R, X.

Now for each $n\prec\omega$*:* $\mathrel{\|\!\!\!\longrightarrow}_{\mathrm{R,X},\omega+n}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega[+(n\dot{-}1)]}$ *strongly commutes over* $\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}$.
A fortiori R, X *is* ω*-shallow confluent up to* ω.

Lemma A.2 *Let* R *be a CRS over* sig/cons/V. *Let* X$\subseteq$V.
Assume R *to have conservative constructors,* R, X *to be quasi-normal, and the following very weak kind of left-linearity:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q\in\mathcal{POS}(l).\ \forall x\in\mathrm{V}.$
$$\left(\left(\begin{array}{ll} & l/p=x=l/q\\ \wedge & p\neq q\end{array}\right)\Rightarrow\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_C)\\ \vee & x\in\mathrm{V}_C\end{array}\right)\right).$$
Furthermore, assume that for each $n\prec\omega$*:*
$$\mathrel{\|\!\!\!\longrightarrow}_{\mathrm{R,X},\omega+n}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+(n\dot{-}1)}\ \textit{strongly commutes over}\ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}.$$
Moreover, assume that each critical peak from CP(R) *of the form* $(1,1)$ *is* ω*-shallow noisy parallel joinable w.r.t.* R, X*, and that each non-overlay from* CP(R) *of the form* $(1,1)$ *is* ω*-shallow parallel closed w.r.t.* R, X.

Now for all $n_0\preceq n_1\prec\omega$*:*
$$\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}\circ\mathrel{\|\!\!\!\longrightarrow}_{\mathrm{R,X},\omega+n_1}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+(n_1\dot{-}1)}\ \textit{strongly commutes over}\ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_0}.$$
A fortiori R, X *is* ω*-shallow confluent.*

Lemma A.3 *Let* R *be a CRS over* sig/cons/V. *Let* X$\subseteq$V.
Assume R *to have conservative constructors,* R, X *to be quasi-normal, and the following very weak kind of left-linearity:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q\in\mathcal{POS}(l).\ \forall x\in\mathrm{V}.$
$$\left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_C)\\ \wedge & l/p=x=l/q\\ \wedge & p\neq q\end{array}\right)\Rightarrow x\in\mathrm{V}_C\right).$$
Furthermore, assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is strongly confluent, that each critical peak from* CP(R) *of the form* $(0,1)$ *is* ω*-shallow* [*noisy*] *weak parallel joinable up to* ω *w.r.t.* R, X*, and that each non-overlay from* CP(R) *of the form* $(1,0)$ *is* ω*-shallow closed up to* ω *w.r.t.* R, X.

Now for each $n\prec\omega$*:*
$$\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}\circ\mathrel{\|\!\!\!\longrightarrow}_{\mathrm{R,X},\omega+n}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega[+(n\dot{-}1)]}\ \textit{strongly commutes over}\ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}.$$
A fortiori R, X *is* ω*-shallow confluent up to* ω.

Lemma A.4 *Let* R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume R *to have conservative constructors,* R,X *to be quasi-normal, and the following weak kinds of left- and right-linearity:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V}.$

$$\left(\begin{array}{ll} & \forall p,q\in\mathcal{POS}(l).\ \left(\left(\begin{array}{ll} & l/p=x=l/q\\ \wedge & p\neq q\end{array}\right)\Rightarrow\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\\ \wedge & x\in\mathrm{V}_C\end{array}\right)\right)\\ \wedge & \forall p,q\in\mathcal{POS}(r).\ \left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\\ \wedge & r/p=x=r/q\\ \wedge & p\neq q\end{array}\right)\Rightarrow x\in\mathrm{V}_C\right)\end{array}\right).$$

Furthermore, assume that $\longrightarrow_{\mathrm{R,X},\omega}$ *is confluent, that each critical peak from* $\mathrm{CP(R)}$ *of the form* $(0,1)$ *is* ω*-shallow* [*noisy*] *strongly joinable up to* ω *w.r.t.* R,X, *and that each non-overlay from* $\mathrm{CP(R)}$ *of the form* $(1,0)$ *is* ω*-shallow* [*noisy*] *anti-closed up to* ω *w.r.t.* R,X.
Now for each $n\prec\omega$*:* $\longrightarrow_{\mathrm{R,X},\omega+n}\circ\xrightarrow{*}_{\mathrm{R,X},\omega[+(n\dot{-}1)]}$ *strongly commutes over* $\xrightarrow{*}_{\mathrm{R,X},\omega}$.
A fortiori R,X *is* ω*-shallow confluent up to* ω.

Lemma A.5 *Let* R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume R *to have conservative constructors,* R,X *to be quasi-normal, and the following very weak kind of left-linearity*
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q\in\mathcal{POS}(l).\ \forall x\in\mathrm{V}.$

$$\left(\left(\begin{array}{ll} & l/p=x=l/q\\ \wedge & p\neq q\end{array}\right)\Rightarrow\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\\ \vee & x\in\mathrm{V}_C\end{array}\right)\right).$$

Furthermore, assume that for each $n\prec\omega$*:*

$$\longrightarrow_{\mathrm{R,X},\omega+n}\circ\xrightarrow{*}_{\mathrm{R,X},\omega+(n\dot{-}1)}\ \text{strongly commutes over}\ \xrightarrow{*}_{\mathrm{R,X},\omega}.$$

$$\xleftarrow{*}_{\mathrm{R,X},\omega}\circ\mathrel{+\!\!\!\!\rightarrow}_{\mathrm{R,X},\omega+n}\circ\xrightarrow{*}_{\mathrm{R,X},\omega+(n\dot{-}1)}\ \subseteq\ \xrightarrow{*}_{\mathrm{R,X},\omega}\circ\mathrel{+\!\!\!\!\rightarrow}_{\mathrm{R,X},\omega+n}\circ\xrightarrow{*}_{\mathrm{R,X},\omega+(n\dot{-}1)}\circ\xleftarrow{*}_{\mathrm{R,X},\omega}.$$

Moreover, assume that that each critical peak from $\mathrm{CP(R)}$ *of the form* $(1,1)$ *is* ω*-shallow noisy weak parallel joinable w.r.t.* R,X, *and that each non-overlay from* $\mathrm{CP(R)}$ *of the form* $(1,1)$ *is* ω*-shallow closed w.r.t.* R,X.
Now for all $n_0\preceq n_1\prec\omega$*:*

$$\xrightarrow{*}_{\mathrm{R,X},\omega}\circ\mathrel{+\!\!\!\!\rightarrow}_{\mathrm{R,X},\omega+n_1}\circ\xrightarrow{*}_{\mathrm{R,X},\omega+(n_1\dot{-}1)}\ \text{strongly commutes over}\ \xrightarrow{*}_{\mathrm{R,X},\omega+n_0}.$$

A fortiori R,X *is* ω*-shallow confluent.*

Lemma A.6 *Let* R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume R *to have conservative constructors,* R,X *to be quasi-normal, and the following very weak kind of left- and right-linearity:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q.\ \forall x\in\mathrm{V}.$

$$\left(\left(\begin{array}{ll} & l/p=x=l/q\\ \vee & r/p=x=r/q\end{array}\right)\Rightarrow\left(\begin{array}{ll} & p=q\\ \vee & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\\ \vee & x\in\mathrm{V}_C\end{array}\right)\right).$$

Furthermore, assume that for each $n\prec\omega$*:*

$$\longrightarrow_{\mathrm{R,X},\omega+n}\circ\xrightarrow{*}_{\mathrm{R,X},\omega+(n\dot{-}1)}\ \text{strongly commutes over}\ \xrightarrow{*}_{\mathrm{R,X},\omega}.$$

Moreover, assume that each critical peak from $\mathrm{CP(R)}$ *of the form* $(1,1)$ *is* ω*-shallow noisy strongly joinable w.r.t.* R,X, *and that each non-overlay from* $\mathrm{CP(R)}$ *of the form* $(1,1)$ *is* ω*-shallow noisy anti-closed w.r.t.* R,X.
Now for all $n_0\preceq n_1\prec\omega$*:*

$$\xrightarrow{*}_{\mathrm{R,X},\omega}\circ\longrightarrow_{\mathrm{R,X},\omega+n_1}\circ\xrightarrow{*}_{\mathrm{R,X},\omega+(n_1\dot{-}1)}\ \text{strongly commutes over}\ \xrightarrow{*}_{\mathrm{R,X},\omega+n_0}.$$

A fortiori R,X *is* ω*-shallow confluent.*

Lemma A.7
Let $n_0, n_1 \prec \omega$. *Let* $\mu, \nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$. *Let* $((l,r),C) \in \mathrm{R}$.
Assume that $n_0 \preceq n_1$ *or that* $\mathcal{V}(C) \subseteq \mathrm{V}_C$. *Assume that*
R, X *is* ω*-level confluent up to* n_1.

Now, if $C\mu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R},\mathrm{X},\omega+n_1}$ *and* $\forall x \in \mathrm{V}.\ x\mu \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n_0} x\nu$,
then $C\nu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R},\mathrm{X},\omega+n_1}$ *and* $l\nu \longrightarrow_{\mathrm{R},\mathrm{X},\omega+n_1+1} r\nu$.

Lemma A.8 *Let* R *be a CRS over* sig/cons/V. *Let* $\mathrm{X} \subseteq \mathrm{V}$.
Assume $\forall ((l,r),C) \in \mathrm{R}.\ \mathcal{V}(C) \subseteq \mathrm{V}_C$ *and the following very weak kind of left-linearity:*
$\forall ((l,r),C) \in \mathrm{R}.\ \forall p,q \in \mathcal{POS}(l).\ \forall x \in \mathrm{V}.$

$$\left(\left(\begin{array}{rl} & l/p = x = l/q \\ \wedge & p \neq q \end{array} \right) \Rightarrow \left(\begin{array}{rl} & l \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_C) \\ \vee & x \in \mathrm{V}_C \end{array} \right) \right).$$

Furthermore, assume[33] *that for each* $n \prec \omega$*:*

$$\mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega}.$$

Moreover, assume that each critical peak from $\mathrm{CP}(\mathrm{R})$ *of the form* $(1,1)$ *is* ω*-level parallel joinable w.r.t.* R, X*, and that each non-overlay from* $\mathrm{CP}(\mathrm{R})$ *of the form* $(1,1)$ *is* ω*-level parallel closed w.r.t.* R, X.

Now for all $n \prec \omega$*:*

$$\stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n}.$$

A fortiori R, X *is* ω*-level confluent.*

Lemma A.9 *Let* R *be a CRS over* sig/cons/V. *Let* $\mathrm{X} \subseteq \mathrm{V}$.
Assume $\forall ((l,r),C) \in \mathrm{R}.\ \mathcal{V}(C) \subseteq \mathrm{V}_C$*, and the following very weak kind of left-linearity*
$\forall ((l,r),C) \in \mathrm{R}.\ \forall p,q \in \mathcal{POS}(l).\ \forall x \in \mathrm{V}.$

$$\left(\left(\begin{array}{rl} & l/p = x = l/q \\ \wedge & p \neq q \end{array} \right) \Rightarrow \left(\begin{array}{rl} & l \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_C) \\ \vee & x \in \mathrm{V}_C \end{array} \right) \right).$$

Furthermore, assume that for each $n \prec \omega$*:*

$$\longrightarrow_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega}.$$

$$\stackrel{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \subseteq \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \stackrel{*}{\longleftarrow}_{\mathrm{R},\mathrm{X},\omega}.$$

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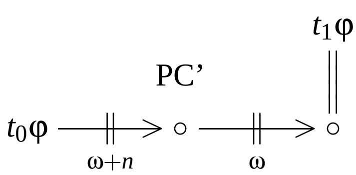
[33] Contrary to analogous lemma for shallow joinability (i.e. Lemma A.2), this strong commutation assumption is not really essential for this lemma if we are confident with the result that $\mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega}$ (instead of $\stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega} \circ \mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega}$) strongly commutes over $\stackrel{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n}$ (which directly allows to get rid of the application of the strong commutation assumption in the proof of Claim 2). Then it is sufficient to assume that R, X is ω-shallow confluent up to ω (which means that Claim 0 of the proof holds directly), that $\mathrel{\longleftarrow\!\!\!\!\!\!\parallel}_{\omega} \circ \mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\omega+n} \subseteq \mathrel{\parallel\!\!\!\!\!\!\longrightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega} \circ \stackrel{*}{\longleftarrow}_{\omega+n}$ (which replaces the application of the strong commutation assumption in the proof of Claim 5), and that the non-overlays of the form $(1,1)$ satisfy



instead of ω-level parallel closedness (which allows to replace the application of the strong commutation assumption at the end of "The critical peak case").

Moreover, assume that that each
critical peak from $\mathrm{CP}(\mathrm{R})$ *of the form* $(1,1)$ *is* ω*-level weak parallel joinable w.r.t.* R,X*,*
and that each non-overlay from $\mathrm{CP}(\mathrm{R})$ *of the form* $(1,1)$ *is* ω*-level closed w.r.t.* R,X.

Now for all $n \prec \omega$*:*

$$\xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega} \circ \mathrel{-\!\!\!\|\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n} \circ \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega} \text{ strongly commutes over } \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega+n}.$$

A fortiori R,X *is* ω*-level confluent.*

Lemma A.10 *Let* R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$.
Assume $\forall((l,r),C)\in\mathrm{R}.\ \mathcal{V}(C)\subseteq\mathrm{V}_C$ *and the following very weak kind of left- and right- linearity:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall p,q.\ \forall x\in\mathrm{V}.$

$$\left(\left(\begin{array}{ll} & l/p=x=l/q \\ \vee & r/p=x=r/q \end{array}\right) \Rightarrow \left(\begin{array}{ll} & p=q \\ \vee & l\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_C) \\ \vee & x\in\mathrm{V}_C \end{array}\right)\right).$$

Furthermore, assume[34] *that for each* $n \prec \omega$*:*

$$\longrightarrow_{\mathrm{R},\mathrm{X},\omega+n} \circ \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega} \text{ strongly commutes over } \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega}.$$

Moreover, assume that each critical peak from $\mathrm{CP}(\mathrm{R})$ *of the form* $(1,1)$ *is* ω*-level strongly joinable w.r.t.* R,X*, and that each non-overlay from* $\mathrm{CP}(\mathrm{R})$ *of the form* $(1,1)$ *is* ω*-level anti-closed w.r.t.* R,X.
Now for all $n \prec \omega$*:*

$$\xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega} \circ \longrightarrow_{\mathrm{R},\mathrm{X},\omega+n} \circ \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega} \text{ strongly commutes over } \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega+n}.$$

A fortiori R,X *is* ω*-level confluent.*

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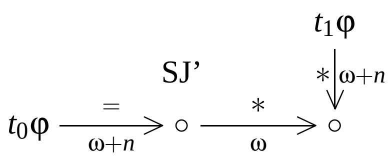
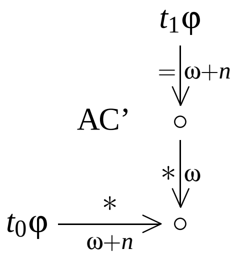
[34] Contrary to analogous lemma for shallow joinability (i.e. Lemma A.6), this strong commutation assumption is not really essential for this lemma if we are confident with the result that $\longrightarrow_{\mathrm{R},\mathrm{X},\omega+n} \circ \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega}$ (instead of $\xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega} \circ \longrightarrow_{\mathrm{R},\mathrm{X},\omega+n} \circ \xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega}$) strongly commutes over $\xrightarrow{*}_{\mathrm{R},\mathrm{X},\omega+n}$ (which directly allows to get rid of the application of the strong commutation assumption in the proof of Claim 2). Then it is sufficient to assume that R,X is ω-shallow confluent up to ω (which means that Claim 0 of the proof holds directly), that $\longleftarrow_{\omega} \circ \longrightarrow_{\omega+n} \subseteq \xrightarrow{=}_{\omega+n} \circ \xrightarrow{*}_{\omega} \circ \xleftarrow{*}_{\omega+n}$ (which replaces the application of the strong commutation assumption in the proof of Claim 5), that the critical peaks of the form $(1,1)$ satisfy



instead of ω-level strong joinability (which allows to complete "The second critical peak case" for the new induction hypothesis), that the non-overlays of the form $(1,1)$ satisfy



instead of ω-level anti-closedness (which allows to complete "The critical peak case" for the new induction hypothesis).

B Further Lemmas for Section 14

Lemma B.1 *Let* R *be a CRS over* sig/cons/V. *Let* X⊆V. *Let* $\alpha \in \{0, \omega\}$.
Let $(>, \rhd)$ *be a termination-pair over* sig/V.
If $\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\left(\begin{array}{c} C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+\alpha} \\ \wedge\ \left(\ \alpha=0\ \Rightarrow l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\ \right)\end{array}\right)\Rightarrow\ l\tau > r\tau\right),$$

then $\longrightarrow_{\mathrm{R,X},\omega+\alpha}\ \subseteq\ \rhd$.

Lemma B.2
Let $\mu,\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$. *Let* $((l,r),C)\in\mathrm{R}$.
Let $(>, \rhd)$ *be a termination-pair over* sig/V *such that:*
$\forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ \Rightarrow\left(\begin{array}{c} l\tau > r\tau \\ \wedge\ \ \forall u\in\mathcal{TERMS}(C).\left(\begin{array}{c} l\tau\rhd u\tau \\ \vee\ \ u\tau\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \\ [\vee\ \ \mathcal{V}(u)\subseteq\mathrm{V}_{\mathcal{C}}\ ]\end{array}\right)\end{array}\right)\right).$$

Assume that $\forall u\lhd l\mu.\ \longrightarrow_{\mathrm{R,X}}$ *is confluent below* u.
[Assume that $\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X}}\ \subseteq\ \downarrow_{\mathrm{R,X}}$. *]*

Now, if $C\mu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R,X}}$ *and* $\forall x\in\mathrm{V}.\ x\mu\overset{*}{\longrightarrow}_{\mathrm{R,X}}x\nu$,
then $C\nu$ *is fulfilled w.r.t.* $\longrightarrow_{\mathrm{R,X}}$ *and* $l\nu\longrightarrow_{\mathrm{R,X}}r\nu$.

Lemma B.3
Let R *be a CRS over* sig/cons/V. *Let* X⊆V.
Let $(>, \rhd)$ *be a termination-pair over* sig/V *such that:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ \Rightarrow\left(\begin{array}{c} l\tau > r\tau \\ \wedge\ \ \forall u\in\mathcal{TERMS}(C).\left(\begin{array}{c} l\tau\rhd u\tau \\ \vee\ \ u\tau\notin\mathrm{dom}(\longrightarrow_{\mathrm{R,X}}) \\ [\vee\ \ \mathcal{V}(u)\subseteq\mathrm{V}_{\mathcal{C}}\ ]\end{array}\right)\end{array}\right)\right).$$

For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$. *Define* $(p\in\mathbf{N}_+^*)$
$A(p) := \{\ t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+\omega,q})\mid\emptyset\neq q\lll_t p\ \}\ [\ \cup\ \mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega})]$.
[Assume that $\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega}\circ\overset{*}{\longrightarrow}_{\mathrm{R,X}}\ \subseteq\ \downarrow_{\mathrm{R,X}}$. *]*
Assume that each critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)\in\mathrm{CP(R)}$
[with $\forall k\prec 2.\ (\ \Lambda_k=1\ \vee\ \mathcal{TERMS}(D_k\sigma)\not\subseteq\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathcal{C}})\ )$ *]*
is $\rhd$*-weakly joinable w.r.t.* R, X *besides* A.

Now: $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

Lemma B.4

Let R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$. *Let* $\beta\preceq\omega$. *Let* $\hat{s}\in\mathcal{T}$.
Assume the following very weak kind of left-linearity:
$\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V_{SIG}}.\ \forall p,q\in\mathcal{POS}(l).$

$$\left(\left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\\ \wedge & l/p=x=l/q\end{array}\right)\Rightarrow\ p=q\right).$$

Furthermore, assume the following compatibility property for a termination-pair $(>,\rhd)$ *over* sig/V*:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\begin{array}{ll} & \left(\begin{array}{ll} & l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\\ \wedge & C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega}\end{array}\right)\\ \Rightarrow & \left(\begin{array}{ll} & ((l,r),C) \text{ is quasi-normal w.r.t. } \mathrm{R,X}\\ \wedge & \forall u\in\mathcal{TERMS}(C).\left(\begin{array}{ll} & l\tau\rhd u\tau\\ \vee & u\tau\not\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})\\ \vee & \mathcal{V}(u)\subseteq\mathrm{V}_C\end{array}\right)\end{array}\right)\end{array}\right)$$

and
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\ \left(\ \left(\ C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\ \right)\ \Rightarrow\ l\tau>r\tau\ \right).$
[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$*. Define (*$p\in\mathbf{N}_+^*$*,* $n\prec\omega$*)* $A(p,n):=\{\ t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\omega+n,q})\mid\emptyset\neq q\lll_t p\ \}$*.]*

Assume $\longrightarrow_{\mathrm{R,X},\omega}$ *to be confluent.*
Assume that each critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)\in\mathrm{CP(R)}$ *with* $(\Lambda_0,\Lambda_1)\neq(1,1)$ *and* $\left(\ (\Lambda_0,\Lambda_1)\neq(0,0)\ \vee\ \mathcal{TERMS}(D_0\sigma D_1\sigma)\not\subseteq\mathcal{T}(\mathrm{cons},\mathrm{V}_C)\ \right)$ *is* ω*-shallow joinable up to* β *and* $\hat{s}$ *w.r.t.* R, X *and* $\lhd$ *[besides A].*

Now: R, X *is* ω*-shallow confluent up to* β *and* $\hat{s}$ *in* $\lhd$*.*

Lemma B.5

Let R *be a CRS over* sig/cons/V. *Let* $\mathrm{X}\subseteq\mathrm{V}$. *Let* $\alpha\in\{0,\omega\}$. *Let* $\beta\preceq\omega+\alpha$. *Let* $\hat{s}\in\mathcal{T}$.
Assume the following weak kind of left-linearity: $\forall((l,r),C)\in\mathrm{R}.\ \forall x\in\mathrm{V}.\ \forall p,q\in\mathcal{POS}(l).$

$$\left(\left(\begin{array}{ll} & l/p=x=l/q\\ \wedge & (\ \alpha=0\ \Rightarrow\ l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\ )\\ \wedge & (\ \alpha=\omega\ \Rightarrow\ x\in\mathrm{V_{SIG}}\ )\end{array}\right)\Rightarrow\ p=q\right).$$

Furthermore, assume R, X *to be* α*-quasi-normal.*
Let $(>,\rhd)$ *be a termination-pair over* sig/V *such that the following compatibility property holds:*
$\forall((l,r),C)\in\mathrm{R}.\ \forall\tau\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).$

$$\left(\left(\begin{array}{ll} & C\tau \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X},\omega+\alpha}\\ \wedge & (\ \alpha=0\ \Rightarrow l\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\ )\end{array}\right)\Rightarrow\ l\tau>r\tau\right)$$

[For each $t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$*. Define (*$p\in\mathbf{N}_+^*$*,* $n\prec\omega$*)* $A(p,n):=\{\ t\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X},\alpha+n,q})\mid\emptyset\neq q\lll_t p\ \}$*.]*

Assume R, X *to be* α*-shallow confluent up to* α*.*
Assume that each critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ \sigma,\ p)\in\mathrm{CP(R)}$
with $\forall k\prec 2.\left(\begin{array}{ll} & (\ \alpha=0\ \Rightarrow\ \Lambda_k=0\ )\\ \wedge & (\ \alpha=\omega\ \Rightarrow\ (\ \Lambda_k=1\ \vee\ \mathcal{TERMS}(D_k\sigma)\not\subseteq\mathcal{T}(\mathrm{cons},\mathrm{V}_C)\ )\ )\end{array}\right)$
is α*-shallow joinable up to* β *and* $\hat{s}$ *w.r.t.* R, X *and* $\lhd$ *[besides A].*

Now: R, X *is* α*-shallow confluent up to* β *and* $\hat{s}$ *in* $\lhd$*.*

Lemma B.6
Let R *be a CRS over* sig/cons/V. *Let* $X\subseteq V$. *Let* $\beta\preceq\omega$. *Let* $\hat{s}\in\mathcal{T}$.
Assume $\forall((l,r),C)\in R.\ \mathcal{V}(C)\subseteq V_C$.
Let $(>,\rhd)$ *be a termination-pair over* sig/V *such that the following compatibility property holds:*

$$\forall((l,r),C)\in R.\ \forall\tau\in\mathcal{SUB}(V,\mathcal{T}(X)).\ \left(\ \left(\ C\tau \text{ fulfilled w.r.t. } \longrightarrow_{R,X}\ \right)\Rightarrow\ l\tau>r\tau\ \right).$$

[For each $t\in\mathcal{T}(\mathrm{sig},X)$ *assume* $\lll_t$ *to be a wellfounded ordering on* $\mathcal{POS}(t)$. *Define* $(p\in\mathbf{N}_+^*$, $n\prec\omega)$ $A(p,n):=\{\ t\in\mathrm{dom}(\longrightarrow_{R,X,\omega+n,q})\mid\emptyset\neq q\lll_t p\ \}$. *]*

Assume $\longrightarrow_{R,X,\omega}$ *to be confluent. Assume that each critical peak in* CP(R) *of the forms* $(0,1)$, $(1,0)$, *or* $(1,1)$ *is* ω*-level joinable up to* β *and* $\hat{s}$ *w.r.t.* R, X *and* $\lhd$ *[besides A].*
Now: R, X *is* ω*-level confluent up to* β *and* $\hat{s}$ *in* $\lhd$.

The following lemma generalizes Lemma 7.6 of Wirth & Gramlich (1994a) by requiring $\rightrightarrows$ to be terminating only below a restricted set of terms T:

Lemma B.7
Let $T\subseteq\mathcal{T}$. *Let* $\unrhd_{ST}[T]$ *denote the set of subterms of* T. *Let* $\rightrightarrows$ *be a sort-invariant (This can always be achieved by identifying all sorts.) and* T*-monotonic relation on* $\mathcal{T}$. *Define* $\rhd\ :=\ {}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\circ(\rightrightarrows\cup\rhd_{ST})^+$. *Now:*

1. $${}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\circ\rightrightarrows\ =\ {}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\circ\rightrightarrows\circ{}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\ ;$$
   $${}_{T}{\upharpoonright}\mathrm{id}\circ\rightrightarrows\ =\ {}_{T}{\upharpoonright}\mathrm{id}\circ\rightrightarrows\circ{}_{T}{\upharpoonright}\mathrm{id}\ .$$

2. ${}_{T}{\upharpoonright}\mathrm{id}\circ\rhd_{ST}\circ\rightrightarrows\ \subseteq\ {}_{T}{\upharpoonright}\mathrm{id}\circ\rightrightarrows\circ{}_{T}{\upharpoonright}\mathrm{id}\circ\rhd_{ST}$.
   Moreover, for $T=\mathcal{T}$*:* $\rhd_{ST}\circ\rightrightarrows\ \subseteq\ \rightrightarrows\circ\rhd_{ST}$.

3. $$\rhd\ \subseteq\ \unlhd_{ST}\circ{}_{T}{\upharpoonright}\mathrm{id}\circ(\rightrightarrows\cup\rhd_{ST})^+\ ;$$
   $$\rhd\ =\ \left(\ ({}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\circ\ \rightrightarrows)\cup({}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\circ\rhd_{ST})\ \right)^+\circ{}_{\unrhd_{ST}[T]}{\upharpoonright}\mathrm{id}\ ;$$
   $${}_{T}{\upharpoonright}\mathrm{id}\circ(\rightrightarrows\cup\rhd_{ST})^+\ =\ \left(\ {}_{T}{\upharpoonright}\mathrm{id}\circ\rhd_{ST}\ \right)\cup\left(\ \left(\ {}_{T}{\upharpoonright}\mathrm{id}\circ\rightrightarrows\ \right)^+\circ{}_{T}{\upharpoonright}\mathrm{id}\circ\unrhd_{ST}\ \right)\ .$$
   Moreover, for $T=\mathcal{T}$*:* $\rhd\ =\ \rhd_{ST}\cup(\rightrightarrows^+\circ\unrhd_{ST})$.

4. *If* $\rightrightarrows$ *is terminating (below all* $t\in T$*) [and* $\rightrightarrows$ *and* T *are* X*-stable], then* $\rhd$ *is a wellfounded [and* X*-stable] ordering on* $\unrhd_{ST}[T]$ *(which does not need to be sort-invariant or* T*-monotonic).*

5. *(4) does not hold in general if one of the two conditions "*$\rightrightarrows$ *sort-invariant" or "*$\rightrightarrows$ T*-monotonic" is removed. Moreover, (4) does not hold in general for* $(\rightrightarrows\cup\rhd_{ST})^+$ *instead of* $\rhd$.

The proof of the following lemma and its far more restrictive predecessors has an interesting history. After its first occurrence in Dershowitz &al. (1988) for overlay joinable positive conditional systems, in our proof for quasi overlay joinable positive/negative-conditional systems in Wirth & Gramlich (1994a) we changed the third component of the induction ordering from $\xrightarrow{+}_{\mathrm{R,X}}$ to $\succ$, the ordering of the ordinals. This change was done because it allowed us to check for generalizations more easily but did not result in a stronger criterion at first. Later, however, this change of the induction ordering turned out to be essential for Theorem 21 of Gramlich (1995a) saying that an innermost terminating overlay joinable positive conditional rule system is terminating and confluent: Due to the mutual dependency of the termination and the confluence proof, when proving confluence it was not possible to assume global termination but local termination only. And it was especially impossible to assume termination for that part of $\longrightarrow_{\mathrm{R,X}}$ which was necessary for the third component of the induction ordering. The following lemma (just like Theorem 7 of Gramlich (1995a)) requires local termination instead of global termination, which is not really necessary for proving Theorem 14.7 but again allows us to check for future generalizations more easily. Moreover, note that the form of the proof has been considerably improved compared to any previous publication: Claim 0 of the proof does not only provide us with the new irreducibility assumptions we have included into the notion of $\rhd$-quasi overlay joinability but also subsumes the whole second case of the global case distinction of the proof (as presented in Dershowitz (1987) as well as presented in Wirth & Gramlich (1994a)). As a consequence, in the whole new proof now the second and the third component of the induction ordering are used only once.

Lemma B.8 (Syntactic Confluence Criterion)
Let R *be a CRS over* sig/cons/V *and* $\mathrm{X}\subseteq\mathrm{V}$. *Let* $\hat{s}\in\mathcal{T}(\mathrm{sig},\mathrm{X})$. *Define* $\mathrm{T}:=\xrightarrow{*}_{\mathrm{R,X}}[\{\hat{s}\}]$.
Assume either that ${}_{\mathrm{T}}\upharpoonright\!\longrightarrow_{\mathrm{R,X}}$ *is terminating and* $\rhd=\rhd_{\mathrm{ST}}$
or that ${}_{\trianglerighteq[\mathrm{T}]}\upharpoonright\!\longrightarrow_{\mathrm{R,X}}\subseteq\rhd,\ \rhd_{\mathrm{ST}}\subseteq\rhd,$ *and* $\rhd$ *is a wellfounded ordering on* $\mathcal{T}$.
Now, if all critical peaks in $\mathrm{CP}(\mathrm{R})$ *are* $\rhd$*-quasi overlay joinable w.r.t.* R,X,
then ${}_{\trianglerighteq[\mathrm{T}]}\upharpoonright\!\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

C ω-Coarse Level Joinability

Using the following notions for ω-coarse level joinability one can work out a whole analogue of Theorem 13.9. We did not do so because this analogue does not allow of a corollary theorem analogous to Theorem 13.4 because the information on confluence provided by the joinability notion for testing the conditions of critical peaks is to poor for practically applicable reasoning. To those who are interested in this notion, however, we present here the analogues of Definition 8.1, Definition 8.2, Lemma A.7, and Lemma A.8, for which we also have included the proofs.

Definition C.1 (ω-Coarse Level Parallel Closed)
A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$
is ω-*coarse level parallel closed w.r.t.* R,X if

$$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\left(\begin{array}{l}\left(\begin{array}{ll} & \forall i\prec 2.\ D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\\ \wedge & \longrightarrow_{\mathrm{R,X}} \text{ and } \longrightarrow_{\mathrm{R,X},\omega} \text{ are commuting}\end{array}\right)\\ \Rightarrow\quad t_0\varphi \mathrel{\|\!\!\!\!\longrightarrow}_{\mathrm{R,X}}\circ\xrightarrow{*}_{\mathrm{R,X},\omega}\circ\xleftarrow{*}_{\mathrm{R,X},\omega}t_1\varphi\end{array}\right).$$

Definition C.2 (ω-Coarse Level Parallel Joinable)
A critical peak $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$
is ω-*coarse level parallel joinable w.r.t.* R,X if

$$\forall\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X})).\left(\begin{array}{l}\left(\begin{array}{ll} & \forall i\prec 2.\ D_i\varphi \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}}\\ \wedge & \longrightarrow_{\mathrm{R,X}} \text{ and } \longrightarrow_{\mathrm{R,X},\omega} \text{ are commuting}\end{array}\right)\\ \Rightarrow\quad t_0\varphi \mathrel{\|\!\!\!\!\longrightarrow}_{\mathrm{R,X}}\circ\xrightarrow{*}_{\mathrm{R,X},\omega}\circ\xleftarrow{*}_{\mathrm{R,X}}t_1\varphi\end{array}\right).$$

Now:

$$\xrightarrow{*}_{\mathrm{R,X},\omega} \circ \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\mathrm{R,X}} \circ \xrightarrow{*}_{\mathrm{R,X},\omega} \textit{ strongly commutes over } \xrightarrow{*}_{\mathrm{R,X}}.$$

A fortiori $\longrightarrow_{\mathrm{R,X}}$ *is confluent.*

Lemma C.4

Let $\mu,\nu \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$. *Let* $((l,r),C) \in \mathrm{R}$.
Assume that $\mathcal{V}(C) \subseteq \mathrm{V}_{\mathcal{C}}$.
Assume $\xleftarrow{*}_{\mathrm{R,X}} \circ \xrightarrow{*}_{\mathrm{R,X},\omega} \subseteq \downarrow_{\mathrm{R,X}}$.

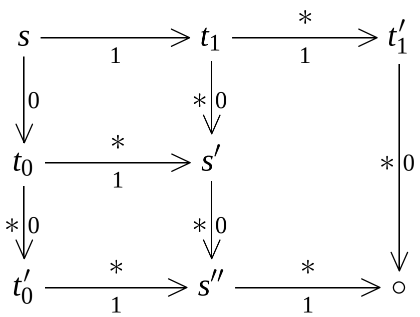
Now, if $C\mu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$ and $\forall x\in \mathrm{V}.\ x\mu \overset{*}{\longrightarrow}_{\mathrm{R,X}} x\nu$, then $C\nu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$ and $l\nu \longrightarrow_{\mathrm{R,X}} r\nu$.

D The Proofs

Proof of Lemma 3.2

Assume $\longrightarrow_0$ and $\longrightarrow_1$ to be locally commuting.

For the first claim we assume that $\longrightarrow_0 \cup \longrightarrow_1$ is terminating. We show commutation by induction over the wellfounded ordering $\longrightarrow_0 \cup \longrightarrow_1{}^+$. Suppose $t'_0 \overset{*}{\longleftarrow}_0 s \overset{*}{\longrightarrow}_1 t'_1$. We have to show $t'_0 \overset{*}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t'_1$. In case there is some $i \prec 2$ with $t'_i = s$ the proof is finished due to $t'_i = s \overset{*}{\longrightarrow}_{1-i} t'_{1-i} \overset{*}{\longleftarrow}_i t'_{1-i}$. Otherwise $t'_0 \overset{*}{\longleftarrow}_0 t_0 \longleftarrow_0 s \longrightarrow_1 t_1 \overset{*}{\longrightarrow}_1 t'_1$ for some t_0, t_1 (cf. diagram below). By local commutation there is some s' with $t_0 \overset{*}{\longrightarrow}_1 s' \overset{*}{\longleftarrow}_0 t_1$. Due to $s \longrightarrow_0 \cup \longrightarrow_1{}^+ t_0$, by induction hypothesis we get some s'' with $t'_0 \overset{*}{\longrightarrow}_1 s'' \overset{*}{\longleftarrow}_0 s'$. Due to $s \longrightarrow_0 \cup \longrightarrow_1{}^+ t_1$, by induction hypothesis we get $s'' \overset{*}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t'_1$.

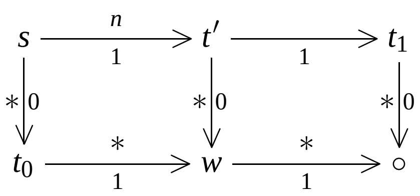


For the second claim we now assume that $\longrightarrow_0$ or $\longrightarrow_1$ is transitive. W.l.o.g. (due to symmetry in 0 and 1) say $\longrightarrow_0$ is transitive. It is sufficient to show

$$\forall n\in\mathbf{N}.\ \forall s,t_0,t_1.\ (t_0 \overset{*}{\longleftarrow}_0 s \overset{n}{\longrightarrow}_1 t_1 \ \Rightarrow\ t_0 \overset{*}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t_1).$$

<u>$n=0$:</u> $t_0 \overset{*}{\longrightarrow}_1 t_0 \overset{*}{\longleftarrow}_0 s = t_1$.

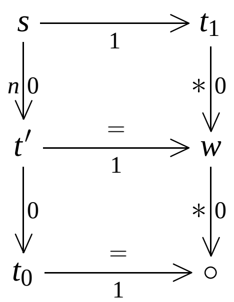
<u>$n \Rightarrow (n+1)$:</u> Assume $t_0 \overset{*}{\longleftarrow}_0 s \overset{n}{\longrightarrow}_1 t' \longrightarrow_1 t_1$ (cf. diagram below). By induction hypothesis there is some w with $t_0 \overset{*}{\longrightarrow}_1 w \overset{*}{\longleftarrow}_0 t'$. In case of $w = t'$ the proof is finished by $t_0 \overset{*}{\longrightarrow}_1 w = t' \longrightarrow_1 t_1 \overset{*}{\longleftarrow}_0 t_1$. Otherwise, since $\longrightarrow_0$ is transitive, we have $w \longleftarrow_0 t' \longrightarrow_1 t_1$. By the local commutation of $\longrightarrow_0$ and $\longrightarrow_1$ this implies $w \overset{*}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t_0$.



Proof of Lemma 3.3
That (3) (or else (2)) implies (1) is trivial. For (1) implying (2) and (3) it is sufficient to show under the assumption of (1) that

$$\forall n\in\mathbf{N}.\ \forall s,t_0,t_1.\ (t_0 \overset{n}{\longleftarrow}_0 s \longrightarrow_1 t_1 \ \Rightarrow\ t_0 \overset{=}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 t_1).$$

<u>$n=0$:</u> $t_0 = s \longrightarrow_1 t_1 \overset{*}{\longleftarrow}_0 t_1$.
<u>$n \Rightarrow (n+1)$:</u> Suppose $t_0 \longleftarrow_0 t' \overset{n}{\longleftarrow}_0 s \longrightarrow_1 t_1$ (cf. diagram below). By induction hypothesis there is some w with $t' \overset{=}{\longrightarrow}_1 w \overset{*}{\longleftarrow}_0 t_1$. In case of $t' = w$ the proof is finished due to $t_0 \overset{=}{\longrightarrow}_1 t_0 \longleftarrow_0 t' = w \overset{*}{\longleftarrow}_0 t_1$. Otherwise we have $t_0 \longleftarrow_0 t' \longrightarrow_1 w$ and get by the assumed strong commutation $t_0 \overset{=}{\longrightarrow}_1 \circ \overset{*}{\longleftarrow}_0 w$.



For proving the final implication of the lemma, we may assume that $\longrightarrow_1$ strongly commutes over $\overset{+}{\longrightarrow}_0$. A fortiori $\overset{+}{\longrightarrow}_0$ and $\longrightarrow_1$ are locally commuting. By Lemma 3.2 they are commuting. Therefore $\longrightarrow_0$ and $\longrightarrow_1$ are commuting, too.

Proof of Lemma 3.4
It is trivial to show $\forall n\in\mathbf{N}.\ \overset{n}{\longleftrightarrow} \subseteq \downarrow$ by induction on n.

Proof of Lemma 5.1
Just like the proof of Lemma 6.3 when the depth considerations are omitted.

Proof of Lemma 6.3
For $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p) \in \mathrm{CP}(\mathrm{R})$ there are two rules $l_0{=}r_0{\longleftarrow}C_0$ and $l_1{=}r_1{\longleftarrow}C_1$ in R (assuming $\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0) \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$ w.l.o.g.) and $\sigma \in \mathcal{SUB}(\mathrm{V},\mathcal{T})$ with $l_0\sigma = l_1\sigma/p$; $(t_0,\ D_0,\ t_1,\ D_1,\ \hat{t}) = (l_1[p \leftarrow r_0],\ C_0,\ r_1,\ C_1,\ l_1)\sigma$ and $\Lambda_i = \left\{ \begin{array}{ll} 0 & \text{if } l_i \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C}) \\ 1 & \text{otherwise} \end{array} \right\}$. Let $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; $n_0,n_1 \prec \omega$; and assume $[(n_0{+_\alpha}n_1,\ \hat{t}\varphi) \underline{\preceq\lhd} (\beta,\ s)$ and] for all $i \prec 2$: $(\ \alpha{=}0 \Rightarrow \Lambda_i{=}0{\prec}n_i\ )$; $(\ \alpha{=}\omega \Rightarrow \Lambda_i{\preceq}n_i\ )$; $D_i\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\alpha+(n_i\dot{-}1)}$; i.e. $C_i\sigma\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\alpha+(n_i\dot{-}1)}$. In case of $n_i{=}0$ we have $\Lambda_i{=}0$ and $\alpha{=}\omega$ and therefore by Corollary 2.6 $l_i\sigma\varphi{\longrightarrow}_{\mathrm{R,X},\alpha+n_i} r_i\sigma\varphi$. In case of $n_i{\succ}0$ we have $n_i{=}(n_i{\dot{-}}1){+}1$ and therefore $l_i\sigma\varphi{\longrightarrow}_{\mathrm{R,X},\alpha+n_i} r_i\sigma\varphi$ again due to $\alpha{=}0 \Rightarrow \Lambda_i{=}0$. Then

$$t_0\varphi = l_1\sigma\varphi[p \leftarrow r_0\sigma\varphi] {\longleftarrow}_{\mathrm{R,X},\alpha+n_0} l_1\sigma\varphi {\longrightarrow}_{\mathrm{R,X},\alpha+n_1} r_1\sigma\varphi = t_1\varphi.$$

By α-shallow confluence [up to β [and s in $\lhd$]] we have $t_0\varphi \overset{*}{\longrightarrow}_{\mathrm{R,X},\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\alpha+n_0} t_1\varphi$.

Proof of Lemma 6.4
The proof is analogous to the proof of Lemma 6.3.

Proof of Lemma 9.1
In case of $(\hat{t}/p')\sigma\varphi{=}(\hat{t}/\emptyset)\sigma\varphi$ we get $p'{=}\emptyset$. Thus $\Delta \subseteq \mathcal{POS}(\hat{t})\backslash\{\emptyset\}$ together with $\forall p'{\in}\Delta.$ $(\hat{t}/p')\sigma\varphi{=}(\hat{t}/\emptyset)\sigma\varphi$ implies $\Delta{=}\emptyset$. If there is some $\bar{u}_1$ with $t_0\sigma\mu \overset{*}{\longrightarrow} \bar{u}_1 \overset{*}{\longleftarrow} t_1\sigma\mu$; define $\bar{n} := 1$; $\bar{u}_0 := t_1\sigma\mu$; $\bar{p}_0 := \emptyset$; and note that $t_1\sigma\varphi{\longleftarrow}\hat{t}\sigma\varphi$ when $D_1\sigma\varphi$ is fulfilled.

Proof of Lemma 13.2
If R has conservative constructors we get $\mathcal{V}(C){\subseteq}\mathrm{V}_\mathcal{C}$ (since $l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})$). If $\mathcal{V}(C) \subseteq \mathrm{V}_\mathcal{C}$,
then $\mathcal{TERMS}(C\mu){\subseteq}\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})$ (since $l \in \mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})$).
Thus we can always assume $\mathcal{TERMS}(C\mu){\subseteq}\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})$. Then we have $\forall x{\in}\mathcal{V}(C).$ $x\mu{\in}\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})$ and thus $\forall x{\in}\mathcal{V}(C).\ x\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega} x\nu$ by Lemma 2.10. Moreover $C\mu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega}$ by Lemma 2.10. By confluence of $\longrightarrow_{\mathrm{R,X},\omega}$ and Lemma 2.10 $C\nu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega}$. By Corollary 2.6 we finally get $l\nu{\longrightarrow}_{\mathrm{R,X},\omega} r\nu$.

Proof of Theorem 13.3 and Theorem 13.4

Due to Corollary 3.8, it suffices to show that the conditions of Theorem 13.6(I) or else (in case of Theorem 13.4) Theorem 13.9(I) are satisfied. The only non-trivial part are the joinability requirements for the critical pairs. We just have to show that the conjunctive condition lists of the joinability notions are never satisfied. Assume $((t_0,D_0,\Lambda_0),\ (t_1,D_1,\Lambda_1),\ \hat{t},\ p)$ to be a critical peak.

We first treat the critical peaks of the form $(0,1)$ or $(1,0)$, and, in case of Theorem 13.3, also of the form $(1,1)$. For these we have to show ω-shallow parallel joinability or else ω-shallow parallel closedness. Thus, assume $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ and $n_0,n_1 \prec \omega$ such that $\forall i \prec 2.\ (\ D_i\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+(n_i \dot{-} 1)}\ )$ and $\forall \delta \prec n_0 +_\omega n_1.\ (\ \mathrm{R,X}$ is ω-shallow confluent up to $\delta\ )$. By the assumed complementarity there must be complementary equation literals in D_0 and D_1. Due to our symmetry in 0 and 1 so far, we may w.l.o.g. assume that $(u{=}v)$ occurs in D_0 and $(u{\neq}v)$ occurs in D_1 or else that $(p{=}\mathsf{true})$ occurs in D_0 and $(p{=}\mathsf{false})$ occurs in D_1. We treat the first case first. Then there are $\hat{u},\hat{v} \in \mathcal{GT}(\mathrm{cons})$ with $\hat{u} \stackrel{*}{\longleftarrow}_{\omega+(n_1 \dot{-} 1)} u\varphi \downarrow_{\omega+(n_0 \dot{-} 1)} v\varphi \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \hat{v}$ and $\hat{u} \not\downarrow_\omega \hat{v}$. In case of $n_0,n_1 \preceq 1$ this contradicts the required confluence of $\longrightarrow_\omega$, cf. Lemma 3.4. Otherwise, in case of $n_0 \succeq 1$ we have $(n_0 \dot{-} 1) +_\omega (n_1 \dot{-} 1) \prec n_0 +_\omega n_1$ and thus by our above assumption $\mathrm{R,X}$ is ω-shallow confluent up to $(n_0 \dot{-} 1) +_\omega (n_1 \dot{-} 1)$. Due to the assumption of the theorem at least one of $u\varphi$, $v\varphi$, w.l.o.g. say $v\varphi$, must be either irreducible or have a $v' \in \mathcal{GT}(\mathrm{cons})$ with $v\varphi \stackrel{*}{\longrightarrow}_{\omega+(n_0 \dot{-} 1)} v'$. Now Lemma 13.7(4) implies $\hat{u} \downarrow_{\omega+(n_0 \dot{-} 1)} \hat{v}$, and then Lemma 2.11 implies the contradicting $\hat{u} \downarrow_\omega \hat{v}$. Now we treat the case that that $(p{=}\mathsf{true})$ occurs in D_0 and $(p{=}\mathsf{false})$ occurs in D_1. Due to the definition of complementarity, true and false are distinct irreducible ground terms. Thus we have $p\varphi \stackrel{*}{\longrightarrow}_{\omega+(n_0 \dot{-} 1)} \mathsf{true}$ and $p\varphi \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \mathsf{false}$. In case of $n_0,n_1 \preceq 1$ this contradicts the required confluence of $\longrightarrow_\omega$. Otherwise, in case of $n_0 \succeq 1$ we have $(n_0 \dot{-} 1) +_\omega (n_1 \dot{-} 1) \prec n_0 +_\omega n_1$ and thus by our above assumption $\mathrm{R,X}$ is ω-shallow confluent up to $(n_0 \dot{-} 1) +_\omega (n_1 \dot{-} 1)$. This again implies the contradicting $\mathsf{true} \downarrow \mathsf{false}$.

Finally we treat the critical peaks of the form $(1,1)$ in case of Theorem 13.4. For these we have to show ω-level parallel joinability or else ω-level parallel closedness. Thus, assume $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ and $n \prec \omega$ with $0 \prec n$ such that $\forall i \prec 2.\ (\ D_i\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+(n \dot{-} 1)}\ )$ and $\forall \delta \prec n.\ (\ \mathrm{R,X}$ is ω-level confluent up to $\delta\ )$. Due to $0 \prec n$ we have $n \dot{-} 1 \prec n$ and thus $\mathrm{R,X}$ is ω-level confluent up to $n \dot{-} 1$. By the assumed weak complementarity there must be complementary equation literals in D_0D_1. First we treat the case that $(u{=}v)$ and $(u{\neq}v)$ occur in D_0D_1. Then there are $\hat{u},\hat{v} \in \mathcal{GT}(\mathrm{cons})$ and $v' \in \mathcal{T}(\mathrm{sig},\mathrm{X})$ with $\hat{u} \stackrel{*}{\longleftarrow}_{\omega+(n \dot{-} 1)} u\varphi \stackrel{*}{\longrightarrow}_{\omega+(n \dot{-} 1)} v' \stackrel{*}{\longleftarrow}_{\omega+(n \dot{-} 1)} v\varphi \stackrel{*}{\longrightarrow}_{\omega+(n \dot{-} 1)} \hat{v}$ and $\hat{u} \not\downarrow_\omega \hat{v}$. Now, by ω-level confluence up to $n \dot{-} 1$, there is some u' with $\hat{u} \stackrel{*}{\longrightarrow}_{\omega+(n \dot{-} 1)} u' \stackrel{*}{\longleftarrow}_{\omega+(n \dot{-} 1)} v'$ and then by ω-level confluence up to $n \dot{-} 1$ again $u' \downarrow_{\omega+(n \dot{-} 1)} \hat{v}$, and then Lemma 2.11 implies the contradicting $\hat{u} \downarrow_\omega \hat{v}$. Now we treat the case that that $(p{=}\mathsf{true})$ and $(p{=}\mathsf{false})$ occur in D_0D_1. Due to the definition of weak complementarity, true and false are distinct irreducible ground terms. Thus we have $\mathsf{true} \stackrel{*}{\longleftarrow}_{\omega+(n \dot{-} 1)} p\varphi \stackrel{*}{\longrightarrow}_{\omega+(n \dot{-} 1)} \mathsf{false}$. By ω-level confluence up to $n \dot{-} 1$ this again implies the contradicting $\mathsf{true} \downarrow \mathsf{false}$. **Q.e.d. (Theorem 13.3 and Theorem 13.4)**

Proof of Theorem 13.6
(I) follows from the lemmas A.1 and A.2.
(II) follows from the lemmas A.4 and A.6.
(III) follows from the lemmas A.1, A.4, and A.5, since for critical peaks of the form $(0,1)$ ω-shallow noisy strong joinability up to ω implies ω-shallow noisy parallel joinability up to ω (cf. Corollary 7.7) and for non-overlays of the form $(1,0)$ ω-shallow parallel closedness up to ω implies ω-shallow noisy anti-closedness up to ω (cf. Corollary 7.8).
(IV) follows from the lemmas A.3, A.4, and A.5, since for critical peaks of the form $(0,1)$ ω-shallow noisy strong joinability up to ω implies ω-shallow noisy weak parallel joinability up to ω (cf. Corollary 7.7) and for critical peaks of the form $(1,0)$ ω-shallow closedness up to ω implies ω-shallow anti-closedness up to ω (cf. Corollary 7.8).

Proof of Theorem 13.9
(I) follows from the lemmas A.1 and A.8.
(II) follows from the lemmas A.4 and A.10
(III) follows from the lemmas A.1, A.4, and A.9, since for critical peaks of the form $(0,1)$ ω-shallow strong joinability up to ω implies ω-shallow parallel joinability up to ω (cf. Corollary 7.7) and for non-overlays of the form $(1,0)$ ω-shallow parallel closedness up to ω implies ω-shallow anti-closedness up to ω (cf. Corollary 7.8).
(IV) follows from the lemmas A.3, A.4, and A.9, since for critical peaks of the form $(0,1)$ ω-shallow strong joinability up to ω implies ω-shallow weak parallel joinability up to ω (cf. Corollary 7.7) and for critical peaks of the form $(1,0)$ ω-shallow closedness up to ω implies ω-shallow anti-closedness up to ω (cf. Corollary 7.8).

Proof of Theorem 14.2
<u>1 ⇒ 2:</u> By Lemma B.3. <u>2 ⇒ 1:</u> By Lemma 5.1.

Proof of Theorem 14.4
<u>1 ⇒ 2:</u> Directly by the lemmas B.4 and B.3. <u>2 ⇒ 1:</u> By Lemma 5.1.

Proof of Theorem 14.5
<u>1 ⇒ 2:</u> Directly by the lemmas B.4 and B.5. <u>2 ⇒ 1:</u> By Corollary 3.9 and Lemma 6.3.

Proof of Theorem 14.6
<u>1 ⇒ 2:</u> Directly by Lemma B.6. <u>2 ⇒ 1:</u> By Corollary 3.9 and Lemma 6.4.

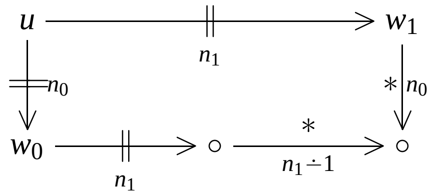
Proof of Theorem 14.7
Directly by Lemma B.8.

Proof of Theorem 15.1(I)

Claim 1: If $\mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{n_0}$, then $\longrightarrow_{n_1}$ and $\longrightarrow_{n_0}$ are commuting.
Proof of Claim 1: $\mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}$ and $\stackrel{*}{\longrightarrow}_{n_0}$ are commuting by Lemma 3.3. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{n_1}\subseteq\mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}\subseteq\stackrel{*}{\longrightarrow}_{n_1}$, now $\longrightarrow_{n_1}$ and $\longrightarrow_{n_0}$ are commuting, too. Q.e.d. (Claim 1)

For $n_0 \preceq n_1 \prec \omega$ we are going to show by induction on $n_0{+}n_1$ the following property:

$$w_0 \mathrel{\leftarrow\!\!\parallel\!\!-}_{n_0} u \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1} w_1 \quad\Rightarrow\quad w_0 \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}\circ\stackrel{*}{\longleftarrow}_{n_0} w_1.$$



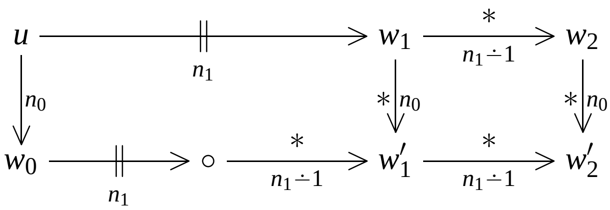
Claim 2: Let $\delta \prec \omega$. If

$$\forall n_0,n_1 \prec \omega.\left(\begin{array}{l}\left(\begin{array}{l} n_0 \preceq n_1 \\ \wedge\ n_0{+}n_1 \preceq \delta\end{array}\right)\\ \Rightarrow\ \forall w_0,w_1,u.\left(\begin{array}{l} w_0 \mathrel{\leftarrow\!\!\parallel\!\!-}_{n_0} u \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1} w_1 \\ \Rightarrow\ w_0 \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}\circ\stackrel{*}{\longleftarrow}_{n_0} w_1\end{array}\right)\end{array}\right),$$

then

$$\forall n_0,n_1 \prec \omega.\left(\begin{array}{l}\left(\begin{array}{l} n_0 \preceq n_1 \\ \wedge\ n_0{+}n_1 \preceq \delta\end{array}\right)\\ \Rightarrow\ \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{n_0}\end{array}\right),$$

and R,X is 0-shallow confluent up to δ.
Proof of Claim 2: By induction on δ in $\prec$. First we show the strong commutation. Assume $n_0 \preceq n_1 \prec \omega$ with $n_0{+}n_1 \preceq \delta$. By Lemma 3.3 it suffices to show that $\mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}$ strongly commutes over $\longrightarrow_{n_0}$. Assume $w_0 \longleftarrow_{n_0} u \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1} w_1 \stackrel{*}{\longrightarrow}_{n_1\dot{-}1} w_2$ (cf. diagram below). By the above property there is some w_1' with $w_0 \mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1} w_1' \stackrel{*}{\longleftarrow}_{n_0} w_1$. Next we show that we can close the peak $w_1' \stackrel{*}{\longleftarrow}_{n_0} w_1 \stackrel{*}{\longrightarrow}_{n_1\dot{-}1} w_2$ according to $w_1' \stackrel{*}{\longrightarrow}_{n_1\dot{-}1} w_2' \stackrel{*}{\longleftarrow}_{n_0} w_2$ for some w_2'. In case of $n_1 = 0$ this is possible due $w_1 = w_2$. Otherwise we have $n_0{+}(n_1\dot{-}1) \prec n_0{+}n_1 \preceq \delta$ and due to our induction hypothesis (saying that R,X is 0-shallow confluent up to all $\delta' \prec \delta$) this is possible again.



Finally we show 0-shallow confluence up to δ. Assume $n_0{+}n_1 \preceq \delta$ and $w_0 \stackrel{*}{\longleftarrow}_{n_0} u \stackrel{*}{\longrightarrow}_{n_1} w_1$. Due to symmetry in n_0 and n_1 we may assume $n_0 \preceq n_1$. Above we have shown that $\mathrel{-\!\!\parallel\!\!\rightarrow}_{n_1}\circ\stackrel{*}{\longrightarrow}_{n_1\dot{-}1}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{n_0}$. By Claim 1 we finally get $w_0 \stackrel{*}{\longrightarrow}_{n_1}\circ\stackrel{*}{\longleftarrow}_{n_0} w_1$ as desired.
Q.e.d. (Claim 2)

Note that for $n_0 = 0$ our property follows from $\mathrel{\leftarrow\!\!\shortparallel\!\!-}_0 \subseteq \mathrm{id}$.

The benefit of Claim 2 is twofold: First, it says that our theorem is valid if the above property holds for all $n_0 \preceq n_1 \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n_i{+}1$ instead of n_i since we may assume $0 \prec n_0 \preceq n_1$) it now suffices to show for $n_0 \preceq n_1 \prec \omega$ that

$$w_0 \mathrel{\leftarrow\!\!\shortparallel\!\!-}_{n_0+1,\Pi_0} u \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_1+1,\Pi_1} w_1$$

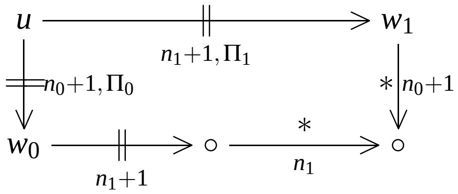
together with our induction hypotheses that

$$\forall \delta \prec (n_0{+}1)+(n_1{+}1).\ \mathrm{R},\mathrm{X} \text{ is 0-shallow confluent up to } \delta$$

and (due to $n_0 \preceq n_1{+}1$ and $n_0+(n_1{+}1) \prec (n_0{+}1)+(n_1{+}1)$)

$$\mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_1+1} \circ \stackrel{*}{\longrightarrow}_{n_1} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{n_0}$$

implies
$$w_0 \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_1+1} \circ \stackrel{*}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longleftarrow}_{n_0+1} w_1 .$$



Note that for the availability of our second induction hypothesis it is important that we have imposed the restriction "$n_0 \preceq n_1$" in opposition to the restriction "$n_0 \succeq n_1$". In the latter case the availability of our second induction hypothesis would require $n_0{+}1 \succeq n_1{+}1 \Rightarrow n_0 \succeq n_1{+}1$ which is not true for $n_0 = n_1$. The additional hypothesis

$$\mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_1} \circ \stackrel{*}{\longrightarrow}_{n_1 \dot{-} 1} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{n_0+1}$$

of the latter restriction is useless for our proof.

W.l.o.g. let the positions of Π_i be maximal in the sense that for any $p \in \Pi_i$ and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $u \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_i+1,(\Pi_i \setminus \{p\}) \cup \Xi} w_i$ anymore. Then for each $i \prec 2$ and $p \in \Pi_i$ there are $((l_{i,p}, r_{i,p}), C_{i,p}) \in \mathrm{R}$ and $\mu_{i,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $l_{i,p} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, $u/p = l_{i,p}\mu_{i,p}$, $r_{i,p}\mu_{i,p} = w_i/p$, $C_{i,p}\mu_{i,p}$ fulfilled w.r.t. $\longrightarrow_{n_i}$. Finally, for each $i \prec 2$: $w_i = u[\, p \leftarrow r_{i,p}\mu_{i,p} \mid p \in \Pi_i \,]$.

Define the set of inner overlapping positions by

$$\Omega(\Pi_0, \Pi_1) := \bigcup_{i \prec 2} \{\, p \in \Pi_{1-i} \mid \exists q \in \Pi_i.\ \exists q' \in \mathbf{N}^*.\ p = qq' \,\},$$

and the length of a term by $\lambda(f(t_0, \ldots, t_{m-1})) := 1 + \sum_{j \prec m} \lambda(t_j)$.

Now we start a second level of induction on $\displaystyle\sum_{p' \in \Omega(\Pi_0,\Pi_1)} \lambda(u/p')$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\, p \in \Pi_0 \cup \Pi_1 \mid \neg \exists q \in \Pi_0 \cup \Pi_1.\ \exists q' \in \mathbf{N}^+.\ p = qq' \,\}.$$

Since the prefix ordering is wellfounded we have $\forall i \prec 2.\ \forall p \in \Pi_i.\ \exists q \in \Theta.\ \exists q' \in \mathbf{N}^*.\ p = qq'$. Then $\forall i \prec 2.\ w_i = w_i[\, q \leftarrow w_i/q \mid q \in \Theta \,] = u[\, p \leftarrow r_{i,p}\mu_{i,p} \mid p \in \Pi_i \,][\, q \leftarrow w_i/q \mid q \in \Theta \,] = u[\, q \leftarrow w_i/q \mid q \in \Theta \,]$. Thus, it now suffices to show for all $q \in \Theta$

$$w_0/q \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_1+1} \circ \stackrel{*}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longleftarrow}_{n_0+1} w_1/q$$

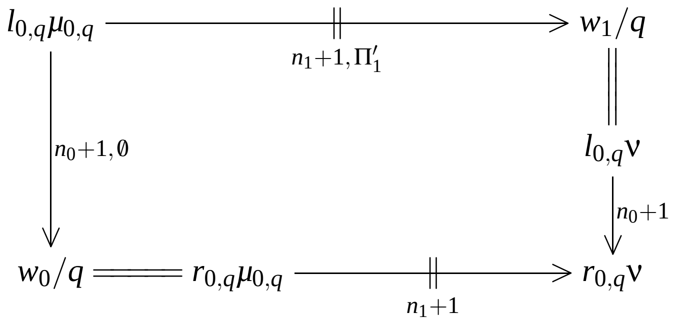
because then we have

$$w_0 = u[\, q \leftarrow w_0/q \mid q \in \Theta \,] \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{n_1+1} \circ \stackrel{*}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longleftarrow}_{n_0+1} u[\, q \leftarrow w_1/q \mid q \in \Theta \,] = w_1.$$

Therefore we are left with the following two cases for $q \in \Theta$:

$q \notin \Pi_1$: Then $q \in \Pi_0$. Define $\Pi'_1 := \{\, p \mid qp \in \Pi_1 \,\}$. We have two cases:

“The variable overlap (if any) case”: $\forall p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}$:



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\, (p', p'') \mid l_{0,q}/p' = x \wedge p'p'' \in \Pi'_1 \,\}$.

Claim 7: There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}.\left(\begin{array}{ll} & x\mu_{0,q} \;{+\!\!\!\!\!\longleftrightarrow}_{n_1+1} x\nu \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu = x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \end{array}\right).$$

Proof of Claim 7:

In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$x\mu_{0,q} = x\mu_{0,q}[\, p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \;{+\!\!\!\!\!\longleftrightarrow}_{n_1+1}$$
$$x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] = x\nu.$$

Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x, which contradicts the left-linearity assumption of the theorem. Q.e.d. (Claim 7)

Claim 8: $l_{0,q}\nu = w_1/q$.

Proof of Claim 8:

By Claim 7 we get $w_1/q = u/q[\, p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x) \,] =$
$l_{0,q}[\, p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V} \,][\, p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x) \,] =$
$l_{0,q}[\, p' \leftarrow x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \mid l_{0,q}/p' = x \in \mathrm{V} \,] =$
$l_{0,q}[\, p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V} \,] = l_{0,q}\nu$. Q.e.d. (Claim 8)

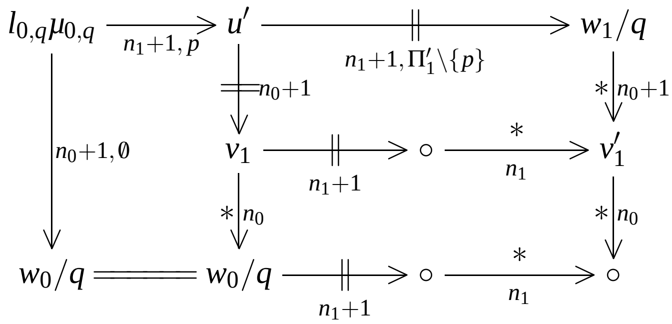
Claim 9: $w_0/q \;{+\!\!\!\!\!\longleftrightarrow}_{n_1+1} r_{0,q}\nu$.

Proof of Claim 9: Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. Q.e.d. (Claim 9)

By claims 8 and 9 it now suffices to show $l_{0,q}\nu \longrightarrow_{n_0+1} r_{0,q}\nu$, which again follows from Lemma 13.8 since $((l_{0,q}, r_{0,q}), C_{0,q})$ is 0-quasi-normal w.r.t. R, X (due to $l_{0,q} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ and the assumption of our theorem), since R, X is 0-shallow confluent up to $(n_1{+}1){+}n_0$ (by our induction hypothesis), and since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \overset{*}{\longrightarrow}_{n_1+1} x\nu$ by Claim 7 and Corollary 2.14.

Q.e.d. (“The variable overlap (if any) case”)

"The critical peak case": There is some $p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \not\in \mathrm{V}$:



Claim 10: $p \neq \emptyset$.
Proof of Claim 10: If $p = \emptyset$, then $\emptyset \in \Pi'_1$, then $q \in \Pi_1$, which contradicts our global case assumption. Q.e.d. (Claim 10)

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp},r_{1,qp}),C_{1,qp}))] \cap \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp},r_{1,qp}),C_{1,qp}))] \cup \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{Bmatrix}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\upharpoonright(\sigma\varphi) = {}_{\mathrm{Y}}\upharpoonright\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1\setminus\{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \mathrel{+\!\!\!\!\!\longrightarrow}_{n_1+1,\Pi'_1\setminus\{p\}} u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1] = w_1/q.$$

If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \mathrel{+\!\!\!\!\!\longrightarrow}_{n_1+1,\Pi'_1\setminus\{p\}} w_1/q.$$

Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi], C_{1,qp}\xi, 0), (r_{0,q}, C_{0,q}, 0), l_{0,q}, \sigma, p) \in \mathrm{CP}(\mathrm{R})$; $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{n_1}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{n_0}$. Since $\forall\delta \prec (n_1+1)+(n_0+1)$. R, X is 0-shallow confluent up to δ (by our induction hypothesis) due to our assumed 0-shallow parallel closedness (matching the definition's n_0 to our n_1+1 and its n_1 to our n_0+1) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \mathrel{+\!\!\!\!\!\longrightarrow}_{n_0+1} v_1 \overset{*}{\longrightarrow}_{n_0} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v_1. We then have $v_1 \mathrel{\longleftarrow\!\!\!\!\!+}_{n_0+1,\Pi''} u' \mathrel{+\!\!\!\!\!\longrightarrow}_{n_1+1,\Pi'_1\setminus\{p\}} w_1/q$ for some Π''. By

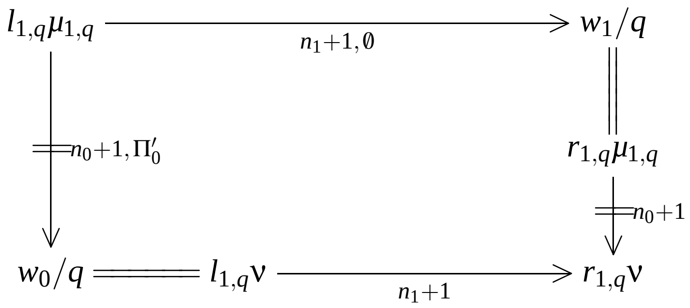
$$\sum_{p'' \in \Omega(\Pi'',\Pi'_1\setminus\{p\})} \lambda(u'/p'') \preceq \sum_{p'' \in \Pi'_1\setminus\{p\}} \lambda(u'/p'') = \sum_{p'' \in \Pi'_1\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p'' \in \Pi'_1} \lambda(u/qp'') =$$
$$\sum_{p' \in q\Pi'_1} \lambda(u/p') = \sum_{p' \in \Omega(\{q\},\Pi_1)} \lambda(u/p') \preceq \sum_{p' \in \Omega(\Pi_0,\Pi_1)} \lambda(u/p'),$$

due to our second induction level we get some v'_1 with $v_1 \mathrel{+\!\!\!\!\!\longrightarrow}_{n_1+1} \circ \overset{*}{\longrightarrow}_{n_1} v'_1 \overset{*}{\longleftarrow}_{n_0+1} w_1/q$. Finally by our induction hypothesis that $\mathrel{+\!\!\!\!\!\longrightarrow}_{n_1+1} \circ \overset{*}{\longrightarrow}_{n_1}$ strongly commutes over $\overset{*}{\longrightarrow}_{n_0}$ the peak at v_1 can be closed according to $w_0/q \mathrel{+\!\!\!\!\!\longrightarrow}_{n_1} \circ \overset{*}{\longrightarrow}_{n_1} \circ \overset{*}{\longleftarrow}_{n_0} v'_1$.

Q.e.d. ("The critical peak case") Q.e.d. ("$q \not\in \Pi_1$")

$q\in\Pi_1$: Define $\Pi'_0 := \{\ p \mid qp\in\Pi_0\ \}$. We have two cases:

"The second variable overlap (if any) case": $\forall p\in\Pi'_0\cap\mathcal{POS}(l_{1,q}).\ l_{1,q}/p\in\mathrm{V}$:



Define a function Γ on V by ($x\in\mathrm{V}$): $\Gamma(x) := \{\ (p',p'') \mid l_{1,q}/p'=x \wedge p'p''\in\Pi'_0\ \}$.

Claim 11: There is some $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with

$$\forall x\in\mathrm{V}.\left(\begin{array}{ll} & x\nu \leftarrow\!\!\!\!+\!\!\!\!+\!\!\!-_{n_0+1} x\mu_{1,q} \\ \wedge & \forall p'\in\mathrm{dom}(\Gamma(x)).\ x\mu_{1,q}[p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)] = x\nu \end{array}\right).$$

Proof of Claim 11:
In case of $\mathrm{dom}(\Gamma(x))=\emptyset$ we define $x\nu := x\mu_{1,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{1,q}[p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)]$. This is appropriate since due to $\forall(p',p'')\in\Gamma(x).\ x\mu_{1,q}/p''=l_{1,q}\mu_{1,q}/p'p''=u/qp'p''=l_{0,qp'p''}\mu_{0,qp'p''}$ we have

$$x\mu_{1,q} = x\mu_{1,q}[p''\leftarrow l_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)] \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\!\rightarrow}_{n_0+1}$$
$$x\mu_{1,q}[p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)] = x\nu.$$

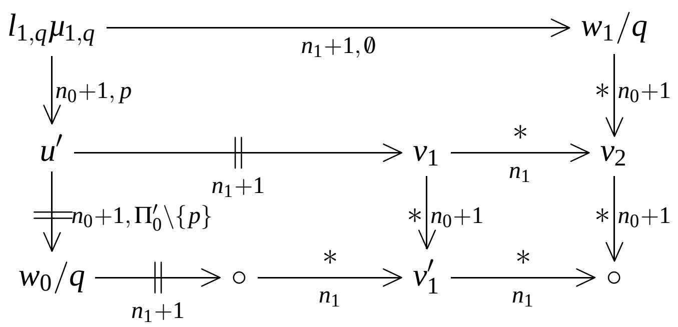
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{1,q}$ is not linear in x, which contradicts the left-linearity assumption of the theorem. Q.e.d. (Claim 11)

Claim 12: $w_0/q = l_{1,q}\nu$.
Proof of Claim 12:
By Claim 11 we get $w_0/q = u/q[p'p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x\in\mathrm{V}.\ (p',p'')\in\Gamma(x)] =$
$l_{1,q}[p'\leftarrow x\mu_{1,q} \mid l_{1,q}/p'=x\in\mathrm{V}][p'p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x\in\mathrm{V}.\ (p',p'')\in\Gamma(x)] =$
$l_{1,q}[p'\leftarrow x\mu_{1,q}[p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)] \mid l_{1,q}/p'=x\in\mathrm{V}] =$
$l_{1,q}[p'\leftarrow x\nu \mid l_{1,q}/p'=x\in\mathrm{V}] = l_{1,q}\nu$. Q.e.d. (Claim 12)

Claim 13: $r_{1,q}\nu \leftarrow\!\!\!\!+\!\!\!\!+\!\!\!-_{n_0+1} w_1/q$.
Proof of Claim 13: Since $r_{1,q}\mu_{1,q}=w_1/q$, this follows directly from Claim 11. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{n_1+1} r_{1,q}\nu$, which again follows from Claim 11, Corollary 2.14, Lemma 13.8 (matching its n_0 to our n_0+1 and its n_1 to our n_1), and our induction hypothesis that R, X is 0-shallow confluent up to $(n_0+1)+n_1$.
Q.e.d. ("The second variable overlap (if any) case")

“The second critical peak case”: There is some $p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q})$ with $l_{1,q}/p \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,qp},r_{0,qp}),C_{0,qp}))] \cap \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,qp},r_{0,qp}),C_{0,qp}))] \cup \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q})) \\ x\xi^{-1}\mu_{0,qp} & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_{0,qp}\xi\rho = l_{0,qp}\xi\xi^{-1}\mu_{0,qp} = u/qp = l_{1,q}\mu_{1,q}/p = l_{1,q}\rho/p = (l_{1,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,qp}\xi, l_{1,q}/p)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{1,q}\mu_{1,q}[p \leftarrow r_{0,qp}\mu_{0,qp}]$. We get

$$w_0/q = u/q[p' \leftarrow r_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0] \leftarrow\!\!\!\!+\!\!\!\!+\!\!\!-_{n_0+1,\Pi'_0\setminus\{p\}}$$
$$u/q[p' \leftarrow l_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0\setminus\{p\}][p \leftarrow r_{0,qp}\mu_{0,qp}] = u'.$$

If $l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q \leftarrow\!\!\!\!+\!\!\!\!+\!\!\!-_{n_0+1,\Pi'_0\setminus\{p\}} u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

Otherwise we have $((l_{1,q}[p \leftarrow r_{0,qp}\xi], C_{0,qp}\xi, 0), (r_{1,q}, C_{1,q}, 0), l_{1,q}, \sigma, p) \in \mathrm{CP}(\mathrm{R})$; $C_{0,qp}\xi\sigma\varphi = C_{0,qp}\mu_{0,qp}$ is fulfilled w.r.t. $\longrightarrow_{n_0}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{n_1}$. Since $\forall\delta \prec (n_0{+}1){+}(n_1{+}1)$. R, X is 0-shallow confluent up to δ (by our induction hypothesis) due to our assumed 0-shallow noisy parallel joinability (matching the definition’s n_0 to our $n_0{+}1$ and its n_1 to our $n_1{+}1$) we have $u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{n_1+1} v_1 \overset{*}{\longrightarrow}_{n_1} v_2 \overset{*}{\longleftarrow}_{n_0+1} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$ for some v_1, v_2. We then have $w_0/q \leftarrow\!\!\!\!+\!\!\!\!+\!\!\!-_{n_0+1,\Pi'_0\setminus\{p\}} u' \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{n_1+1,\Pi''} v_1$ for some Π''. Since $\sum_{p''\in\Omega(\Pi'_0\setminus\{p\},\Pi'')} \lambda(u'/p'') \preceq \sum_{p''\in\Pi'_0\setminus\{p\}} \lambda(u'/p'') = \sum_{p''\in\Pi'_0\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p''\in\Pi'_0} \lambda(u/qp'') = \sum_{p'\in q\Pi'_0} \lambda(u/p') = \sum_{p'\in\Omega(\Pi_0,\{q\})} \lambda(u/p') \preceq \sum_{p'\in\Omega(\Pi_0,\Pi_1)} \lambda(u/p')$ due to our second induction level we get some v'_1 with $w_0/q \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{n_1+1} \circ \overset{*}{\longrightarrow}_{n_1} v'_1 \overset{*}{\longleftarrow}_{n_0+1} v_1$. Finally the peak at v_1 can be closed according to $v'_1 \overset{*}{\longrightarrow}_{n_1} \circ \overset{*}{\longleftarrow}_{n_0+1} v_2$ by our induction hypothesis saying that R, X is 0-shallow confluent up to $(n_0{+}1){+}n_1$.
Q.e.d. (“The second critical peak case”) **Q.e.d. (Theorem 15.1(I))**

Proof of Theorem 15.1(II)

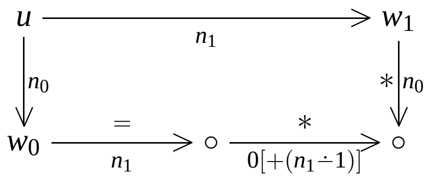
The parts in the following proof which are only for Theorem 15.1(IIa) are in optional brackets.

Claim 1: If $\longrightarrow_{n_1} \circ \overset{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]}$ strongly commutes over $\overset{*}{\longrightarrow}_{n_0}$, then $\longrightarrow_{n_1}$ and $\longrightarrow_{n_0}$ are commuting.

Proof of Claim 1: $\longrightarrow_{n_1} \circ \overset{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]}$ and $\overset{*}{\longrightarrow}_{n_0}$ are commuting by Lemma 3.3. Since by Lemma 2.12 we have $\longrightarrow_{n_1} \subseteq \longrightarrow_{n_1} \circ \overset{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} \subseteq \overset{*}{\longrightarrow}_{n_1}$, now $\longrightarrow_{n_1}$ and $\longrightarrow_{n_0}$ are commuting, too. Q.e.d. (Claim 1)

For $n_0 \preceq n_1 \prec \omega$ we are going to show by induction on $n_0{+}n_1$ the following property:

$$w_0 \longleftarrow_{n_0} u \longrightarrow_{n_1} w_1 \quad \Rightarrow \quad w_0 \stackrel{=}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} \circ \stackrel{*}{\longleftarrow}_{n_0} w_1 .$$



Claim 2: Let $\delta \prec \omega$. If

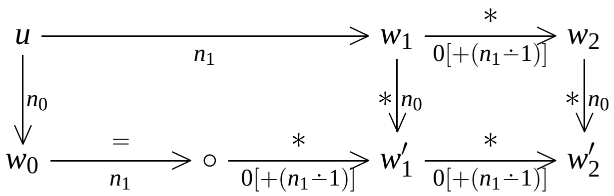
$$\forall n_0, n_1 \prec \omega . \left(\begin{array}{l} \left(\begin{array}{c} n_0 \preceq n_1 \\ \wedge \quad n_0{+}n_1 \preceq \delta \end{array} \right) \\ \Rightarrow \quad \forall w_0, w_1, u . \left(\begin{array}{l} w_0 \longleftarrow_{n_0} u \longrightarrow_{n_1} w_1 \\ \Rightarrow \quad w_0 \stackrel{=}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} \circ \stackrel{*}{\longleftarrow}_{n_0} w_1 \end{array} \right) \end{array} \right),$$

then

$$\forall n_0, n_1 \prec \omega . \left(\begin{array}{l} \left(\begin{array}{c} n_0 \preceq n_1 \\ \wedge \quad n_0{+}n_1 \preceq \delta \end{array} \right) \\ \Rightarrow \quad \longrightarrow_{n_1} \circ \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{n_0} \end{array} \right),$$

and R, X is 0-shallow confluent up to δ.

Proof of Claim 2: By induction on δ in $\prec$. First we show the strong commutation. Assume $n_0 \preceq n_1 \prec \omega$ with $n_0{+}n_1 \preceq \delta$. By Lemma 3.3 it suffices to show that $\longrightarrow_{n_1} \circ \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]}$ strongly commutes over $\longrightarrow_{n_0}$. Assume $w_0 \longleftarrow_{n_0} u \longrightarrow_{n_1} w_1 \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} w_2$ (cf. diagram below). By the above property there is some w_1' with $w_0 \stackrel{=}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} w_1' \stackrel{*}{\longleftarrow}_{n_0} w_1$. Next we show that we can close the peak $w_1' \stackrel{*}{\longleftarrow}_{n_0} w_1 \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} w_2$ according to $w_1' \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]} w_2' \stackrel{*}{\longleftarrow}_{n_0} w_2$ for some w_2'. In case of $n_1 = 0$ this is possible due to $w_1 = w_2$. Otherwise we have $n_0{+}(0[+(n_1 \dot{-} 1)]) \prec n_0{+}n_1 \preceq \delta$ and due to our induction hypothesis (saying that R, X is 0-shallow confluent up to all $\delta' \prec \delta$) this is possible again.



Finally we show 0-shallow confluence up to δ. Assume $n_0{+}n_1 \preceq \delta$ and $w_0 \stackrel{*}{\longleftarrow}_{n_0} u \stackrel{*}{\longrightarrow}_{n_1} w_1$. Due to symmetry in n_0 and n_1 we may assume $n_0 \preceq n_1$. Above we have shown that $\longrightarrow_{n_1} \circ \stackrel{*}{\longrightarrow}_{0[+(n_1 \dot{-} 1)]}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{n_0}$. By Claim 1 we finally get $w_0 \stackrel{*}{\longrightarrow}_{n_1} \circ \stackrel{*}{\longleftarrow}_{n_0} w_1$ as desired.

Q.e.d. (Claim 2)

Note that for $n_0 = 0$ our property follows from $\longleftarrow_{n_0} \subseteq \mathrm{id}$.

The benefit of Claim 2 is twofold: First, it says that our theorem is valid if the above property holds for all $n_0 \preceq n_1 \prec \omega$. For part (IIb) this is because then by Lemma 3.3 $\longrightarrow_{n_1}$ strongly commutes over $\longrightarrow_{n_0}$ for all $n_0 \preceq n_1 \prec \omega$, i.e. $\longrightarrow_{\omega}$ strongly commutes over $\longrightarrow_{n_0}$, i.e. $\longrightarrow_{\omega}$ strongly commutes over $\longrightarrow_{\omega}$, i.e. $\longrightarrow_{\omega}$ is strongly confluent. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n_i{+}1$ instead of n_i since we may assume $0 \prec n_0 \preceq n_1$) it now suffices to show for $n_0 \preceq n_1 \prec \omega$ that

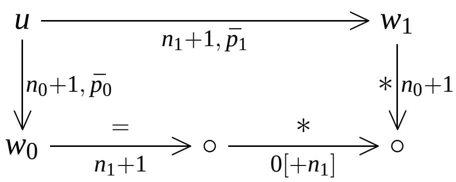
$$w_0 \longleftarrow_{n_0+1,\bar{p}_0} u \longrightarrow_{n_1+1,\bar{p}_1} w_1$$

together with our induction hypotheses that

$$\forall \delta \prec (n_0{+}1)+(n_1{+}1).\ \mathrm{R}, \mathrm{X} \text{ is 0-shallow confluent up to } \delta$$

implies

$$w_0 \overset{=}{\longrightarrow}_{n_1+1} \circ \overset{*}{\longrightarrow}_{0[+n_1]} \circ \overset{*}{\longleftarrow}_{n_0+1} w_1 .$$



Now for each $i \prec 2$ there are $((l_i, r_i), C_i) \in \mathrm{R}$ and $\mu_i \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/\bar{p}_i = l_i\mu_i$, $w_i = u[\,\bar{p}_i \leftarrow r_i\mu_i\,]$, $C_i\mu_i$ fulfilled w.r.t. $\longrightarrow_{n_i}$, and $l_i \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$.

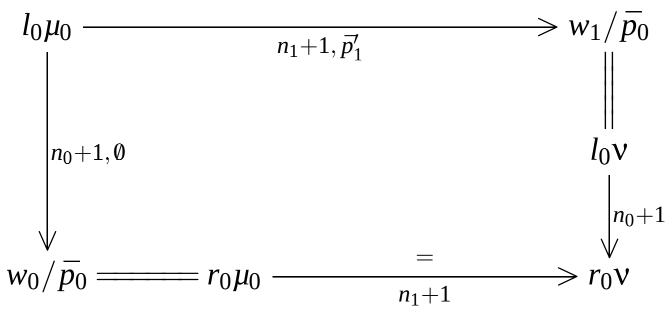
In case of $\bar{p}_0 \parallel \bar{p}_1$ we have $w_i/\bar{p}_{1-i} = u[\,\bar{p}_i \leftarrow r_i\mu_i\,]/\bar{p}_{1-i} = u/\bar{p}_{1-i} = l_{1-i}\mu_{1-i}$ and therefore $w_i \longrightarrow_{n_i+1} u[\,\bar{p}_k \leftarrow r_k\mu_k \mid k \prec 2\,]$, i.e. our proof is finished. Thus, according to whether $\bar{p}_0$ is a prefix of $\bar{p}_1$ or vice versa, we have the following two cases left:

<u>There is some $\overline{p}'_1$ with $\overline{p}_0\overline{p}'_1 = \overline{p}_1$ and $\overline{p}'_1 \neq \emptyset$:</u>

We have two cases:

<u>“The variable overlap case”:</u>

<u>There are $x \in \mathrm{V}$ and p', p'' such that $l_0/p' = x \wedge p'p'' = \overline{p}'_1$:</u>



<u>Claim 6:</u> We have $x\mu_0/p'' = l_1\mu_1$.
<u>Proof of Claim 6:</u> We have $x\mu_0/p'' = l_0\mu_0/p'p'' = u/\overline{p}_0p'p'' = u/\overline{p}_0\overline{p}'_1 = u/\overline{p}_1 = l_1\mu_1$.
<u>Q.e.d. (Claim 6)</u>

<u>Claim 7:</u> We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_0[\,p'' \leftarrow r_1\mu_1\,]$ and $\forall y \in \mathrm{V}\setminus\{x\}.\ y\nu = y\mu_0$.
Then we have $x\mu_0 \longrightarrow_{n_1+1} x\nu$.
<u>Proof of Claim 7:</u> This follows directly from Claim 6. <u>Q.e.d. (Claim 7)</u>

<u>Claim 8:</u> $l_0\nu = w_1/\overline{p}_0$.
<u>Proof of Claim 8:</u> By the left-linearity assumption of our theorem we may assume $\{\ p''' \mid l_0/p''' = x\ \} = \{p'\}$. Thus, by Claim 7 we get $w_1/\overline{p}_0 = u/\overline{p}_0[\,\overline{p}'_1 \leftarrow r_1\mu_1\,] =$
$l_0[\,p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V}\,][\,\overline{p}'_1 \leftarrow r_1\mu_1\,] =$
$l_0[\,p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x\,][\,p' \leftarrow x\mu_0\,][\,p'p'' \leftarrow r_1\mu_1\,] =$
$l_0[\,p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x\,][\,p' \leftarrow x\mu_0[\,p'' \leftarrow r_1\mu_1\,]\,] =$
$l_0[\,p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V}\,] = l_0\nu$. <u>Q.e.d. (Claim 8)</u>

<u>Claim 9:</u> $w_0/\overline{p}_0 \overset{=}{\longrightarrow}_{n_1+1} r_0\nu$.
<u>Proof of Claim 9:</u> By the right-linearity assumption of our theorem we may assume $|\{\ p''' \mid r_0/p''' = x\ \}| \preceq 1$. Thus by Claim 7 we get: $w_0/\overline{p}_0 = r_0\mu_0 =$
$r_0[\,p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}\,][\,p''' \leftarrow x\mu_0 \mid r_0/p''' = x\,] \overset{=}{\longrightarrow}_{n_1+1}$
$r_0[\,p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}\,][\,p''' \leftarrow x\nu \mid r_0/p''' = x\,] =$
$r_0[\,p''' \leftarrow y\nu \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}\,][\,p''' \leftarrow x\nu \mid r_0/p''' = x\,] = r_0\nu$. <u>Q.e.d. (Claim 9)</u>

By claims 8 and 9 it now suffices to show $l_0\nu \longrightarrow_{n_0+1} r_0\nu$, which again follows from Lemma 13.8 (matching its n_0 to our $n_1{+}1$ and its n_1 to our n_0) since R, X is 0-quasi-normal and 0-shallow confluent up to $(n_1{+}1){+}n_0$ by our induction hypothesis, and since $\forall y \in \mathrm{V}.\ y\mu_0 \overset{*}{\longrightarrow}_{n_1+1} y\nu$ by Claim 7. <u>Q.e.d. (“The variable overlap case”)</u>

<u>“The critical peak case”: $\bar{p}'_1 \in \mathcal{POS}(l_0) \wedge l_0/\bar{p}'_1 \notin \mathrm{V}$:</u>

$$\begin{array}{ccccc}
l_0\mu_0 & & \xrightarrow[n_1+1,\,\bar{p}'_1]{} & & w_1/\bar{p}_0 \\
\Big\downarrow{\scriptstyle n_0+1,\,\emptyset} & & & & {\scriptstyle *}\Big\downarrow{\scriptstyle n_0+1} \\
w_0/\bar{p}_0 & \xrightarrow[n_1+1]{=} & \circ & \xrightarrow[0[+n_1]]{*} & \circ
\end{array}$$

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_1,r_1),C_1))] \cap \mathcal{V}(((l_0,r_0),C_0)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_1,r_1),C_1))] \cup \mathcal{V}(((l_0,r_0),C_0))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_0 & \text{if } x \in \mathcal{V}(((l_0,r_0),C_0)) \\ x\xi^{-1}\mu_1 & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_1\xi\rho = l_1\xi\xi^{-1}\mu_1 = u/\bar{p}_1 = u/\bar{p}_0\bar{p}'_1 = l_0\mu_0/\bar{p}'_1 = l_0\rho/\bar{p}'_1 = (l_0/\bar{p}'_1)\rho$
let $\sigma := \mathrm{mgu}(\{(l_1\xi, l_0/\bar{p}'_1)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonleft(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonleft\rho$.
If $l_0[\bar{p}'_1 \leftarrow r_1\xi]\sigma = r_0\sigma$, then the proof is finished due to

$$w_0/\bar{p}_0 = r_0\mu_0 = r_0\sigma\varphi = l_0[\bar{p}'_1 \leftarrow r_1\xi]\sigma\varphi = l_0\mu_0[\bar{p}'_1 \leftarrow r_1\mu_1] = w_1/\bar{p}_0.$$

Otherwise we have $((l_0[\bar{p}'_1 \leftarrow r_1\xi], C_1\xi, 0), (r_0, C_0, 0), l_0, \sigma, \bar{p}'_1) \in \mathrm{CP(R)}$; $\bar{p}'_1 \neq \emptyset$ (due the global case assumption); $C_1\xi\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{n_1}$; $C_0\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{n_0}$. Since $\forall\delta \prec (n_1+1)+(n_0+1)$. R,X is 0-shallow confluent up to δ (by our induction hypothesis), due to our assumed 0-shallow [noisy] anti-closedness (matching the definition's n_0 to our n_1+1 and its n_1 to n_0+1) we have $w_1/\bar{p}_0 = l_0\mu_0[\bar{p}'_1 \leftarrow r_1\mu_1] = l_0[\bar{p}'_1 \leftarrow r_1\xi]\sigma\varphi \overset{*}{\longrightarrow}_{n_0+1} \circ \overset{*}{\longleftarrow}_{0[+n_1]} \circ \overset{=}{\longleftarrow}_{n_1+1} r_0\sigma\varphi = r_0\mu_0 = w_0/\bar{p}_0$.

<u>Q.e.d. (“The critical peak case”)</u>

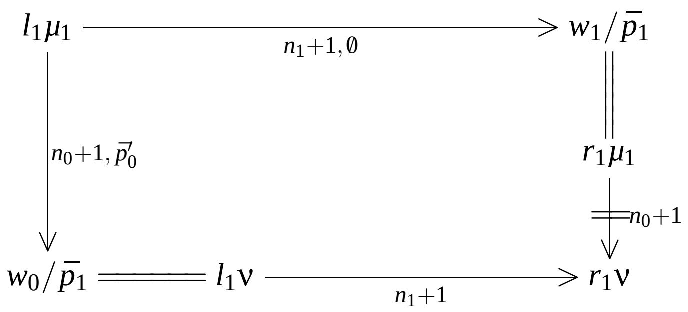
<u>Q.e.d. (“There is some $\bar{p}'_1$ with $\bar{p}_0\bar{p}'_1 = \bar{p}_1$ and $\bar{p}'_1 \neq \emptyset$ ”)</u>

There is some $\bar{p}'_0$ with $\bar{p}_1\bar{p}'_0 = \bar{p}_0$:

We have two cases:

"The second variable overlap case":

There are $x \in \mathrm{V}$ and p', p'' such that $l_1/p' = x \wedge p'p'' = \bar{p}'_0$:



Claim 11a: We have $x\mu_1/p'' = l_0\mu_0$.
Proof of Claim 11a: We have $x\mu_1/p'' = l_1\mu_1/p'p'' = u/\bar{p}_1 p'p'' = u/\bar{p}_1\bar{p}'_0 = u/\bar{p}_0 = l_0\mu_0$.
Q.e.d. (Claim 11a)

Claim 11b: We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_1[\,p'' \leftarrow r_0\mu_0\,]$ and $\forall y \in \mathrm{V}\setminus\{x\}.\, y\nu = y\mu_1$. Then we have $x\mu_1 \longrightarrow_{n_0+1} x\nu$.
Proof of Claim 11b: This follows directly from Claim 11a. Q.e.d. (Claim 11b)

Claim 12: $w_0/\bar{p}_1 = l_1\nu$.
Proof of Claim 12:
By the left-linearity assumption of our theorem we may assume $\{\, p''' \mid l_1/p''' = x \,\} = \{p'\}$.
Thus, by Claim 11b we get $w_0/\bar{p}_1 = u/\bar{p}_1[\,\bar{p}'_0 \leftarrow r_0\mu_0\,] =$
$l_1[\,p''' \leftarrow y\mu_1 \mid l_1/p''' = y \in \mathrm{V}\,][\,\bar{p}'_0 \leftarrow r_0\mu_0\,] =$
$l_1[\,p''' \leftarrow y\mu_1 \mid l_1/p''' = y \in \mathrm{V} \wedge y \neq x\,][\,p' \leftarrow x\mu_1\,][\,p'p'' \leftarrow r_0\mu_0\,] =$
$l_1[\,p''' \leftarrow y\nu \mid l_1/p''' = y \in \mathrm{V} \wedge y \neq x\,][\,p' \leftarrow x\mu_1[\,p'' \leftarrow r_0\mu_0\,]\,] =$
$l_1[\,p''' \leftarrow y\nu \mid l_1/p''' = y \in \mathrm{V}\,] = l_1\nu$. Q.e.d. (Claim 12)

Claim 13: $r_1\nu \leftarrow\!\!\!+\!\!\!\!\!-\!\!\!+_{n_0+1} w_1/\bar{p}_1$.
Proof of Claim 13: Since $r_1\mu_1 = w_1/\bar{p}_1$, this follows directly from Claim 11b. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_1\nu \longrightarrow_{n_1+1} r_1\nu$, which again follows from Claim 11b, Lemma 13.8 (matching its n_0 to our n_0+1 and its n_1 to our n_1), and our induction hypothesis that R, X is 0-shallow confluent up to $(n_0+1)+n_1$.
Q.e.d. ("The second variable overlap case")

"The second critical peak case": $\overline{p}'_0 \in \mathcal{POS}(l_1) \wedge l_1/\overline{p}'_0 \notin \mathrm{V}$:

$$\begin{array}{ccccc}
l_1\mu_1 & & \xrightarrow[n_1+1,\emptyset]{} & & w_1/\overline{p}_1 \\
\downarrow{\scriptstyle n_0+1,\,\overline{p}'_0} & & & & {\scriptstyle *}\downarrow{\scriptstyle n_0+1} \\
w_0/\overline{p}_1 & \xrightarrow[n_1+1]{=} & \circ & \xrightarrow[0[+n_1]]{*} & \circ
\end{array}$$

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_0,r_0),C_0))] \cap \mathcal{V}(((l_1,r_1),C_1)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_0,r_0),C_0))] \cup \mathcal{V}(((l_1,r_1),C_1))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_1 & \text{if } x \in \mathcal{V}(((l_1,r_1),C_1)) \\ x\xi^{-1}\mu_0 & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_0\xi\rho = l_0\xi\xi^{-1}\mu_0 = u/\overline{p}_0 = u/\overline{p}_1\overline{p}'_0 = l_1\mu_1/\overline{p}'_0 = l_1\rho/\overline{p}'_0 = (l_1/\overline{p}'_0)\rho$
let $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/\overline{p}'_0)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\upharpoonleft(\sigma\varphi) = {}_{\mathrm{Y}}\upharpoonleft\rho$.
If $l_1[\overline{p}'_0 \leftarrow r_0\xi]\sigma = r_1\sigma$, then the proof is finished due to

$$w_0/\overline{p}_1 = l_1\mu_1[\overline{p}'_0 \leftarrow r_0\mu_0] = l_1[\overline{p}'_0 \leftarrow r_0\xi]\sigma\varphi = r_1\sigma\varphi = r_1\mu_1 = w_1/\overline{p}_1.$$

Otherwise we have $((l_1[\overline{p}'_0 \leftarrow r_0\xi], C_0\xi, 0), (r_1, C_1, 0), l_1, \sigma, \overline{p}'_0) \in \mathrm{CP}(\mathrm{R})$; $C_0\xi\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{n_0}$; $C_1\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{n_1}$. Since $\forall\delta \prec (n_0+1)+(n_1+1)$. R,X is 0-shallow confluent up to δ (by our induction hypothesis) due to our assumed 0-shallow [noisy] strong joinability (matching the definition's n_0 to our n_0+1 and its n_1 to our n_1+1) we have $w_0/\overline{p}_1 = l_1\mu_1[\overline{p}'_0 \leftarrow r_0\mu_0] = l_1[\overline{p}'_0 \leftarrow r_0\xi]\sigma\varphi \overset{=}{\longrightarrow}_{n_1+1} \circ \overset{*}{\longrightarrow}_{0[+n_1]} \circ \overset{*}{\longleftarrow}_{n_0+1} r_1\sigma\varphi = r_1\mu_1 = w_1/\overline{p}_1$.
Q.e.d. ("The second critical peak case") **Q.e.d. (Theorem 15.1(II))**

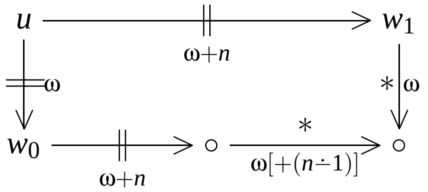
Proof of Theorem 15.3 Due to Corollary 15.2 it suffices to show that the conditions of Theorem 15.1 are satisfied. Since $\mathrm{R}_{\mathcal{C}}$ is normal, R,X is 0-quasi-normal. Thus we only have to show that the conjunctive condition lists of the 0-shallow joinability notions are never satisfied for critical peaks of the form $(0,0)$. Thus, assume $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ and $n_0, n_1 \prec \omega$ such that $\forall i \prec 2.$ ($D_i\varphi$ fulfilled w.r.t. $\longrightarrow_{\mathrm{R},\mathrm{X},n_i \dot{-} 1}$) and $\forall\delta \prec n_0+n_1$. (R,X is 0-shallow confluent up to δ). By the assumed complementarity there must be complementary equation literals in D_0 and D_1. Due to our symmetry in 0 and 1 so far, we may w.l.o.g. assume that $(u{=}v)$ occurs in D_0 and $(u{\neq}v)$ occurs in D_1 or else that $(p{=}\mathsf{true})$ occurs in D_0 and $(p{=}\mathsf{false})$ occurs in D_1. Since negative conditions are not allowed for constructor rules we must be in the latter case here. Due to the definition of complementarity, true and false are distinct irreducible ground terms. Thus we have $p\varphi \overset{*}{\longrightarrow}_{n_0 \dot{-} 1} \mathsf{true}$ and $p\varphi \overset{*}{\longrightarrow}_{n_1 \dot{-} 1} \mathsf{false}$. In case of $n_0, n_1 \preceq 1$ this implies the contradicting $\mathsf{true} = p\varphi = \mathsf{false}$. Otherwise, in case of $n_0 \succeq 1$ we have $(n_0 \dot{-} 1)+(n_1 \dot{-} 1) \prec n_0+n_1$ and thus by our above assumption R,X is 0-shallow confluent up to $(n_0 \dot{-} 1)+(n_1 \dot{-} 1)$. This implies the contradicting $\mathsf{true} \downarrow \mathsf{false}$.
Q.e.d. (Theorem 15.3)

Proof of Theorem 15.4
1 ⇒ 2: Directly by Lemma B.5. 2 ⇒ 1: Directly by Lemma 6.3.

Proof of Lemma A.1

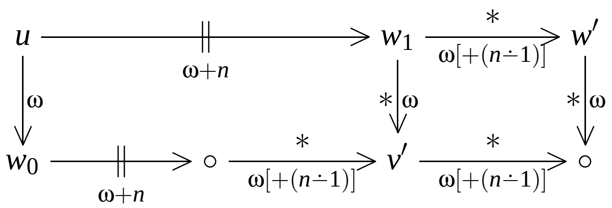
For $n \prec \omega$ we are going to show by induction on n the following property:

$$w_0 \mathrel{\leftarrow\!\!\|\!\!-}_{\omega} u \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} w_1 \quad \Rightarrow \quad w_0 \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \circ \stackrel{*}{\longleftarrow}_{\omega} w_1.$$



Claim 1: If the above property holds for a fixed $n \prec \omega$, and $\forall k \prec n.\ (\mathrm{R},\mathrm{X}$ is ω-shallow confluent up to $k)$, then $\mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{\omega}$.

Proof of Claim 1: By Lemma 3.3 it suffices to show that $\mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ strongly commutes over $\longrightarrow_{\omega}$. Assume $w_0 \longleftarrow_{\omega} u \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} w_1 \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w'$ (cf. diagram below). By the above property there is some v' with $w_0 \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} v' \stackrel{*}{\longleftarrow}_{\omega} w_1$. We only have to show that we can close the peak $v' \stackrel{*}{\longleftarrow}_{\omega} w_1 \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w'$ according to $v' \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \circ \stackrel{*}{\longleftarrow}_{\omega} w'$. [In case of $n=0$:] This is possible due to confluence of $\longrightarrow_{\omega}$. [Otherwise we have $n\dot{-}1 \prec n$ and due to the assumed ω-shallow confluence up to $n\dot{-}1$ this is possible again.]



Q.e.d. (Claim 1)

Claim 2: If the above property holds for a fixed $n \prec \omega$, and $\forall k \prec n.\ (\mathrm{R},\mathrm{X}$ is ω-shallow confluent up to $k)$, then $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega}$ are commuting.

Proof of Claim 2: $\mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ and $\stackrel{*}{\longrightarrow}_{\omega}$ are commuting by Lemma 3.3 and Claim 1. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{\omega+n} \subseteq \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \subseteq \stackrel{*}{\longrightarrow}_{\omega+n}$, now $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega}$ are commuting, too. Q.e.d. (Claim 2)

Claim 3: If the above property holds for all $n \preceq m$ for some $m \prec \omega$, then R,X is ω-shallow confluent up to m.

Proof of Claim 3: By induction on m in $\prec$. Assume $i +_{\omega} n \preceq m$ and $w_0 \stackrel{*}{\longleftarrow}_{\omega+i} u \stackrel{*}{\longrightarrow}_{\omega+n} w_1$. By definition of '$+_{\omega}$' and $i +_{\omega} n \prec \omega$ w.l.o.g. we have $i=0$ and $n \preceq m$. By Claim 2 and our induction hypothesis we finally get $w_0 \stackrel{*}{\longrightarrow}_{\omega+n} \circ \stackrel{*}{\longleftarrow}_{\omega} w_1$ as desired. Q.e.d. (Claim 3)

Note that our property for is trivial for $n=0$ since then by Corollary 2.14 we have $\mathrel{\parallel\!\!\!\!\!\longrightarrow}_{\omega+n} = \mathrel{\parallel\!\!\!\!\!\longrightarrow}_{\omega} \subseteq \stackrel{*}{\longrightarrow}_{\omega}$ and $\longrightarrow_{\omega}$ is confluent.

The benefit of claims 1 and 3 is twofold: First, they say that our lemma is valid if the above property holds for all $n \prec \omega$. Second, they strengthen the property when used as induction hypothesis. Thus (writing $n+1$ instead of n since we may assume $0 \prec n$) it now suffices to show for $n \prec \omega$ that

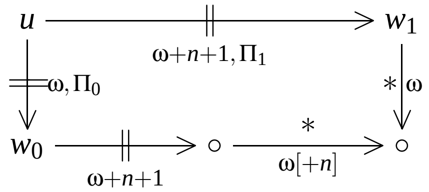
$$w_0 \mathrel{\longleftarrow\!\!\!\!\!\parallel}_{\omega,\Pi_0} u \mathrel{\parallel\!\!\!\!\!\longrightarrow}_{\omega+n+1,\Pi_1} w_1$$

together with our induction hypothesis that

$$\mathrm{R},\mathrm{X} \text{ is } \omega\text{-shallow confluent up to } n$$

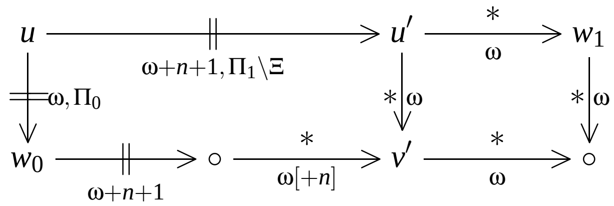
implies

$$w_0 \mathrel{\parallel\!\!\!\!\!\longrightarrow}_{\omega+n+1} \circ \stackrel{*}{\longrightarrow}_{\omega[+n]} \circ \stackrel{*}{\longleftarrow}_{\omega} w_1 .$$



W.l.o.g. let the positions of Π_0 (and Π_1) be maximal in the sense that for any $p \in \Pi_0$ (or else $p \in \Pi_1$) and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $w_0 \mathrel{\longleftarrow\!\!\!\!\!\parallel}_{\omega,(\Pi_0\setminus\{p\})\cup\Xi} u$ (or else $u \mathrel{\parallel\!\!\!\!\!\longrightarrow}_{\omega+n+1,(\Pi_1\setminus\{p\})\cup\Xi} w_1$) anymore. Then for each $i \prec 2$ and $p \in \Pi_i$ there are $((l_{i,p}, r_{i,p}), C_{i,p}) \in \mathrm{R}$ and $\mu_{i,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{i,p}\mu_{i,p}$, $r_{i,p}\mu_{i,p} = w_i/p$. Moreover, for each $p \in \Pi_0$: $l_{0,p} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ and $C_{0,p}\mu_{0,p}$ is fulfilled w.r.t. $\longrightarrow_{\omega}$. Similarly, for each $p \in \Pi_1$: $C_{1,p}\mu_{1,p}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Finally, for each $i \prec 2$: $w_i = u[p \leftarrow r_{i,p}\mu_{i,p} \mid p \in \Pi_i]$.

<u>Claim 5:</u> We may assume $\forall p\in\Pi_1.\ l_{1,p}\notin\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})$.
<u>Proof of Claim 5:</u> Define $\Xi := \{\ p\in\Pi_1 \mid l_{1,p}\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})\ \}$ and $u' := u[\,p\leftarrow r_{1,p}\mu_{1,p} \mid p\in\Pi_1\setminus\Xi\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $w_0 \mathrel{+\!\!\!\!\!\Vert\!\!\rightarrow}_{\omega+n+1}\circ\overset{*}{\longrightarrow}_{\omega[+n]}v'\overset{*}{\longleftarrow}_{\omega}u'$ for some v' (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{1,p}$) we get $\forall p\in\Xi.\ l_{1,p}\mu_{1,p}\longrightarrow_{\omega}r_{1,p}\mu_{1,p}$. Thus from $v'\overset{*}{\longleftarrow}_{\omega}u'\overset{*}{\longrightarrow}_{\omega}w_1$ we get $v'\overset{*}{\longrightarrow}_{\omega}\circ\overset{*}{\longleftarrow}_{\omega}w_1$ by confluence of $\longrightarrow_{\omega}$.



<u>Q.e.d. (Claim 5)</u>

Define the set of inner overlapping positions by

$$\Omega(\Pi_0,\Pi_1) := \bigcup_{i\prec 2}\{\ p\in\Pi_{1-i} \mid \exists q\in\Pi_i.\ \exists q'\in\mathbf{N}^*.\ p=qq'\ \},$$

and the length of a term by $\lambda(f(t_0,\ldots,t_{m-1})) := 1+\sum_{j\prec m}\lambda(t_j)$.

Now we start a second level of induction on $\displaystyle\sum_{p'\in\Omega(\Pi_0,\Pi_1)}\lambda(u/p')$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\ p\in\Pi_0\cup\Pi_1 \mid \neg\exists q\in\Pi_0\cup\Pi_1.\ \exists q'\in\mathbf{N}^+.\ p=qq'\ \}.$$

Since the prefix ordering is wellfounded we have $\forall i\prec 2.\ \forall p\in\Pi_i.\ \exists q\in\Theta.\ \exists q'\in\mathbf{N}^*.\ p=qq'$. Then $\forall i\prec 2.\ w_i=w_i[\,q\leftarrow w_i/q \mid q\in\Theta\,]=u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i\,][\,q\leftarrow w_i/q \mid q\in\Theta\,]=u[\,q\leftarrow w_i/q \mid q\in\Theta\,]$. Thus, it now suffices to show for all $q\in\Theta$

$$w_0/q \mathrel{+\!\!\!\!\!\Vert\!\!\rightarrow}_{\omega+n+1}\circ\overset{*}{\longrightarrow}_{\omega[+n]}\circ\overset{*}{\longleftarrow}_{\omega}w_1/q$$

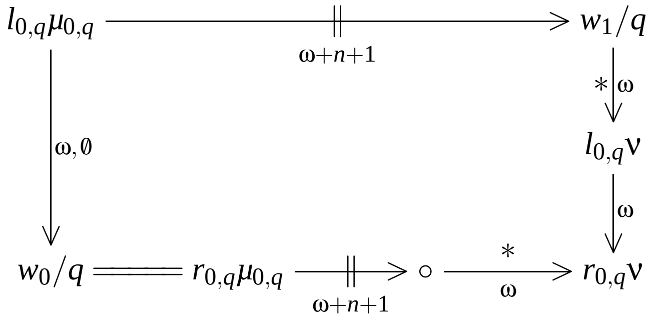
because then we have

$$w_0=u[\,q\leftarrow w_0/q \mid q\in\Theta\,]\mathrel{+\!\!\!\!\!\Vert\!\!\rightarrow}_{\omega+n+1}\circ\overset{*}{\longrightarrow}_{\omega[+n]}\circ\overset{*}{\longleftarrow}_{\omega}u[\,q\leftarrow w_1/q \mid q\in\Theta\,]=w_1.$$

Therefore we are left with the following two cases for $q\in\Theta$:

$\underline{\underline{q \notin \Pi_1}}$: Then $q \in \Pi_0$. Define $\Pi_1' := \{\, p \mid qp \in \Pi_1 \,\}$. We have two cases:

$\underline{\underline{\text{“The variable overlap (if any) case”: } \forall p \in \Pi_1' \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}\text{:}}}$



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\, (p', p'') \mid l_{0,q}/p' = x \ \wedge\ p'p'' \in \Pi_1' \,\}$.

Claim 7: There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}.\ \left(\begin{array}{l} x\mu_{0,q} \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n+1} \circ \xrightarrow{*}_{\omega} x\nu \\ \wedge\ \ \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu \xleftarrow{*}_{\omega} x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \end{array} \right).$$

Proof of Claim 7:

In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$x\mu_{0,q} = x\mu_{0,q}[\, p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n+1}$$
$$x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] = x\nu.$$

Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x. By the conditions of our lemma, this implies $x \in \mathrm{V}_{\mathcal{C}}$. Therefore $x\mu_{0,q} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$. Together with

$\forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q} \xrightarrow{*}_{\omega+n+1} x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,]$ this implies

$\forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q} \xrightarrow{*}_{\omega} x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$

by Lemma 2.10. By confluence of $\longrightarrow_{\omega}$ and Lemma 2.10 again, there is some $t \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ with

$\forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \xrightarrow{*}_{\omega} t$. Therefore we can define $x\nu := t$ in this case. This is appropriate since by $\exists p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q} \xrightarrow{*}_{\omega} x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \xrightarrow{*}_{\omega} x\nu$ we have $x\mu_{0,q} \xrightarrow{*}_{\omega} x\nu$. Q.e.d. (Claim 7)

Claim 8: $l_{0,q}\nu \xleftarrow{*}_{\omega} w_1/q$.

Proof of Claim 8:

By Claim 7 we get $w_1/q = u/q[\, p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x) \,] =$
$l_{0,q}[\, p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V} \,][\, p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x) \,] =$
$l_{0,q}[\, p' \leftarrow x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \mid l_{0,q}/p' = x \in \mathrm{V} \,] \xrightarrow{*}_{\omega}$
$l_{0,q}[\, p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V} \,] = l_{0,q}\nu$. Q.e.d. (Claim 8)

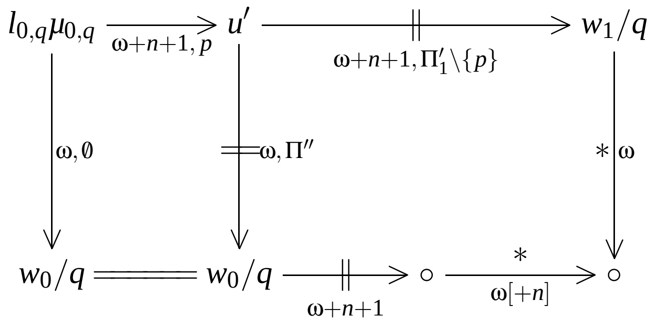
Claim 9: $w_0/q \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n+1} \circ \xrightarrow{*}_{\omega} r_{0,q}\nu$.

Proof of Claim 9: Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. Q.e.d. (Claim 9)

By claims 8 and 9 it now suffices to show $r_{0,q}\nu \longleftarrow_{\omega} l_{0,q}\nu$, which again follows from Lemma 13.2 since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \xrightarrow{*}_{\omega+n+1} x\nu$ by Claim 7 and Corollary 2.14.

Q.e.d. (“The variable overlap (if any) case”)

<u>“The critical peak case”: There is some $p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:</u>



<u>Claim 10:</u> $p \neq \emptyset$.
<u>Proof of Claim 10:</u> If $p = \emptyset$, then $\emptyset \in \Pi'_1$, then $q \in \Pi_1$, which contradicts our global case assumption. <u>Q.e.d. (Claim 10)</u>

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cap \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cup \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1\setminus\{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n+1, \Pi'_1\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1] = w_1/q.$$

If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n+1, \Pi'_1\setminus\{p\}} w_1/q.$$

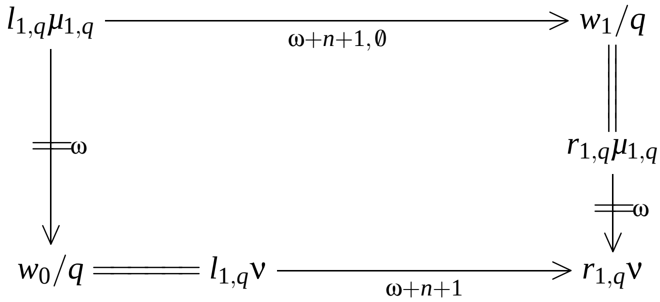
Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 0), l_{0,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega}$. Since R, X is ω-shallow confluent up to n (by our induction hypothesis), due to our assumed ω-shallow parallel closedness up to ω (matching the definition's n_0 to our $n{+}1$ and its n_1 to 0) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \mathrel{+\!\!\!\!\longrightarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$. We then have $w_0/q \mathrel{\longleftarrow\!\!\!\!+}_{\omega,\Pi''} u' \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n+1, \Pi'_1\setminus\{p\}} w_1/q$ for some Π''. We can finish the proof in this case due to our second induction level since

$$\sum_{p'' \in \Omega(\Pi'', \Pi'_1\setminus\{p\})} \lambda(u'/p'') \;\preceq\; \sum_{p'' \in \Pi'_1\setminus\{p\}} \lambda(u'/p'') \;=\; \sum_{p'' \in \Pi'_1\setminus\{p\}} \lambda(u/qp'')$$
$$\prec \sum_{p'' \in \Pi'_1} \lambda(u/qp'') \;=\; \sum_{p' \in q\Pi'_1} \lambda(u/p') \;=\; \sum_{p' \in \Omega(\{q\}, \Pi_1)} \lambda(u/p') \;\preceq\; \sum_{p' \in \Omega(\Pi_0, \Pi_1)} \lambda(u/p').$$

<u>Q.e.d. (“The critical peak case”)</u> <u>Q.e.d. (“$q \notin \Pi_1$”)</u>

$\underline{\underline{q\in\Pi_1}}$: Define $\Pi'_0 := \{\ p \mid qp\in\Pi_0\ \}$. We have two cases:

$\underline{\underline{\text{"The second variable overlap (if any) case": } \forall p\in\Pi'_0\cap\mathcal{POS}(l_{1,q}).\ l_{1,q}/p\in\mathrm{V}:}}$



Define a function Γ on V by ($x\in\mathrm{V}$): $\Gamma(x) := \{\ (p',p'') \mid l_{1,q}/p'=x\ \wedge\ p'p''\in\Pi'_0\ \}$.

<u>Claim 11:</u> There is some $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with

$$\forall x\in\mathrm{V}.\ \left(\begin{array}{ll} & x\nu \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!-}_{\omega} x\mu_{1,q} \\ \wedge & \forall p'\in\mathrm{dom}(\Gamma(x)).\ x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]=x\nu \end{array}\right).$$

<u>Proof of Claim 11:</u>
In case of $\mathrm{dom}(\Gamma(x))=\emptyset$ we define $x\nu := x\mu_{1,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x))=\{p'\}$ we define $x\nu := x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]$. This is appropriate since due to $\forall(p',p'')\in\Gamma(x).\ x\mu_{1,q}/p''=l_{1,q}\mu_{1,q}/p'p''=u/qp'p''=l_{0,qp'p''}\mu_{0,qp'p''}$ we have

$$x\mu_{1,q}=x\mu_{1,q}[\,p''\leftarrow l_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]\mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega}$$
$$x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]=x\nu.$$

Finally, in case of $|\mathrm{dom}(\Gamma(x))|\succ 1$, $l_{1,q}$ is not linear in x. By the conditions of our lemma, this contradicts Claim 5. <u>Q.e.d. (Claim 11)</u>

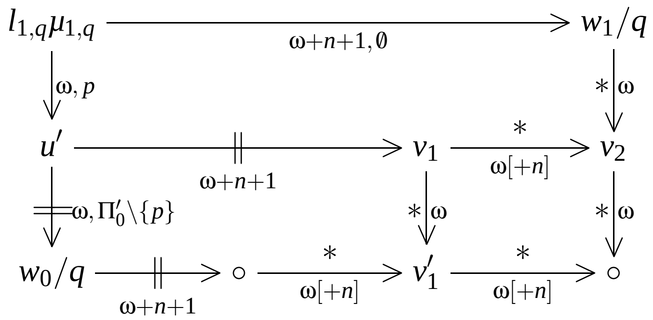
<u>Claim 12:</u> $w_0/q=l_{1,q}\nu$.
<u>Proof of Claim 12:</u>
By Claim 11 we get $w_0/q=u/q[\,p'p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x\in\mathrm{V}.\ (p',p'')\in\Gamma(x)\,]=$
$l_{1,q}[\,p'\leftarrow x\mu_{1,q} \mid l_{1,q}/p'=x\in\mathrm{V}\,][\,p'p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x\in\mathrm{V}.\ (p',p'')\in\Gamma(x)\,]=$
$l_{1,q}[\,p'\leftarrow x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,] \mid l_{1,q}/p'=x\in\mathrm{V}\,]=$
$l_{1,q}[\,p'\leftarrow x\nu \mid l_{1,q}/p'=x\in\mathrm{V}\,]=l_{1,q}\nu$. <u>Q.e.d. (Claim 12)</u>

<u>Claim 13:</u> $r_{1,q}\nu\mathrel{\leftarrow\!\!\!+\!\!\!+\!\!-}_{\omega}w_1/q$.
<u>Proof of Claim 13:</u> Since $r_{1,q}\mu_{1,q}=w_1/q$, this follows directly from Claim 11. <u>Q.e.d. (Claim 13)</u>

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu\longrightarrow_{\omega+n+1}r_{1,q}\nu$, which again follows from Claim 11, Lemma 13.8 (matching its n_0 to 0 and its n_1 to our n) and our induction hypothesis that R, X is ω-shallow confluent up to n.

$\underline{\underline{\text{Q.e.d. ("The second variable overlap (if any) case")}}}$

"The second critical peak case": There is some $p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q})$ with $l_{1,q}/p \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,qp}, r_{0,qp}), C_{0,qp}))] \cap \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,qp}, r_{0,qp}), C_{0,qp}))] \cup \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) \\ x\xi^{-1}\mu_{0,qp} & \text{else} \end{Bmatrix}$ $(x \in \mathrm{V})$.
By $l_{0,qp}\xi\rho = l_{0,qp}\xi\xi^{-1}\mu_{0,qp} = u/qp = l_{1,q}\mu_{1,q}/p = l_{1,q}\rho/p = (l_{1,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,qp}\xi, l_{1,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\upharpoonleft(\sigma\varphi) = {}_{\mathrm{Y}}\upharpoonleft\rho$.
Define $u' := l_{1,q}\mu_{1,q}[\,p \leftarrow r_{0,qp}\mu_{0,qp}\,]$. We get

$$w_0/q = u/q[\,p' \leftarrow r_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0\,] \leftarrow\!\!\!\!\!\!\!+\!\!\!\!+_{\omega,\Pi'_0\setminus\{p\}} u/q[\,p' \leftarrow l_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0\setminus\{p\}\,][\,p \leftarrow r_{0,qp}\mu_{0,qp}\,] = u'.$$

If $l_{1,q}[\,p \leftarrow r_{0,qp}\xi\,]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q \leftarrow\!\!\!\!\!\!\!+\!\!\!\!+_{\omega,\Pi'_0\setminus\{p\}} u' = l_{1,q}[\,p \leftarrow r_{0,qp}\xi\,]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

Otherwise we have $(\,(l_{1,q}[\,p \leftarrow r_{0,qp}\xi\,]\sigma, C_{0,qp}\xi\sigma, 0), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, p\,) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_{0,qp}\xi\sigma\varphi = C_{0,qp}\mu_{0,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R,X ω-shallow confluent up to n (by our induction hypothesis), due to our assumed ω-shallow [noisy] parallel joinability up to ω (matching the definition's n_0 to 0 and its n_1 to our $n{+}1$) we have $u' = l_{1,q}[\,p \leftarrow r_{0,qp}\xi\,]\sigma\varphi \,{+}\!\!\!\!{+}\!\!\!\!\!\!\rightarrow_{\omega+n+1} v_1 \overset{*}{\longrightarrow}_{\omega[+n]} v_2 \overset{*}{\longleftarrow}_{\omega} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$ for some v_1, v_2. We then have $w_0/q \leftarrow\!\!\!\!\!\!\!+\!\!\!\!+_{\omega,\Pi'_0\setminus\{p\}} u' \,{+}\!\!\!\!{+}\!\!\!\!\!\!\rightarrow_{\omega+n+1,\Pi''} v_1$ for some Π''. Since $\sum_{p'' \in \Omega(\Pi'_0\setminus\{p\},\Pi'')} \lambda(u'/p'') \preceq \sum_{p'' \in \Pi'_0\setminus\{p\}} \lambda(u'/p'') = \sum_{p'' \in \Pi'_0\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p'' \in \Pi'_0} \lambda(u/qp'') = \sum_{p' \in q\Pi'_0} \lambda(u/p') = \sum_{p' \in \Omega(\Pi_0,\{q\})} \lambda(u/p') \preceq \sum_{p' \in \Omega(\Pi_0,\Pi_1)} \lambda(u/p')$ due to our second induction level we get some v'_1 with $w_0/q \,{+}\!\!\!\!{+}\!\!\!\!\!\!\rightarrow_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega[+n]} v'_1 \overset{*}{\longleftarrow}_{\omega} v_1$. From the peak $v'_1 \overset{*}{\longleftarrow}_{\omega} v_1 \overset{*}{\longrightarrow}_{\omega[+n]} v_2$ we finally get $v'_1 \overset{*}{\longrightarrow}_{\omega[+n]} \circ \overset{*}{\longleftarrow}_{\omega} v_2$ by ω-shallow confluence up to $0[+n]$.

Q.e.d. ("The second critical peak case") **Q.e.d. (Lemma A.1)**

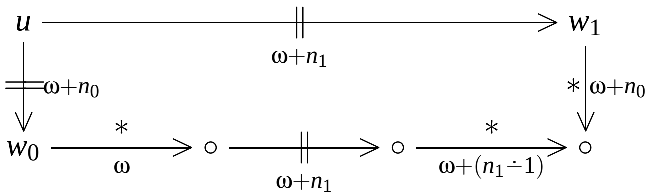
Proof of Lemma A.2

Claim 0: R, X is ω-shallow confluent up to ω.
Proof of Claim 0: Directly by the assumed strong commutation, cf. the proofs of the claims 2 and 3 of the proof of Lemma A.1. Q.e.d. (Claim 0)

Claim 1: If $\xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\Vert\!\!\rightarrow}_{\omega+n_1} \circ \xrightarrow{*}_{\omega+(n_1 \dot{-} 1)}$ strongly commutes over $\xrightarrow{*}_{\omega+n_0}$, then $\longrightarrow_{\omega+n_1}$ and $\longrightarrow_{\omega+n_0}$ are commuting.
Proof of Claim 1: $\xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\Vert\!\!\rightarrow}_{\omega+n_1} \circ \xrightarrow{*}_{\omega+(n_1 \dot{-} 1)}$ and $\xrightarrow{*}_{\omega+n_0}$ are commuting by Lemma 3.3. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{\omega+n_1} \subseteq \xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\Vert\!\!\rightarrow}_{\omega+n_1} \circ \xrightarrow{*}_{\omega+(n_1 \dot{-} 1)} \subseteq \xrightarrow{*}_{\omega+n_1}$, now $\longrightarrow_{\omega+n_1}$ and $\longrightarrow_{\omega+n_0}$ are commuting, too. Q.e.d. (Claim 1)

For $n_0 \preceq n_1 \prec \omega$ we are going to show by induction on $n_0 +_{\omega} n_1$ the following property:

$$w_0 \mathrel{\leftarrow\!\!\Vert\!\!-}_{\omega+n_0} u \mathrel{-\!\!\Vert\!\!\rightarrow}_{\omega+n_1} w_1 \quad \Rightarrow \quad w_0 \xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\Vert\!\!\rightarrow}_{\omega+n_1} \circ \xrightarrow{*}_{\omega+(n_1 \dot{-} 1)} \circ \xleftarrow{*}_{\omega+n_0} w_1 .$$



<u>Claim 2:</u> Let $\delta \prec \omega{+}\omega$. If

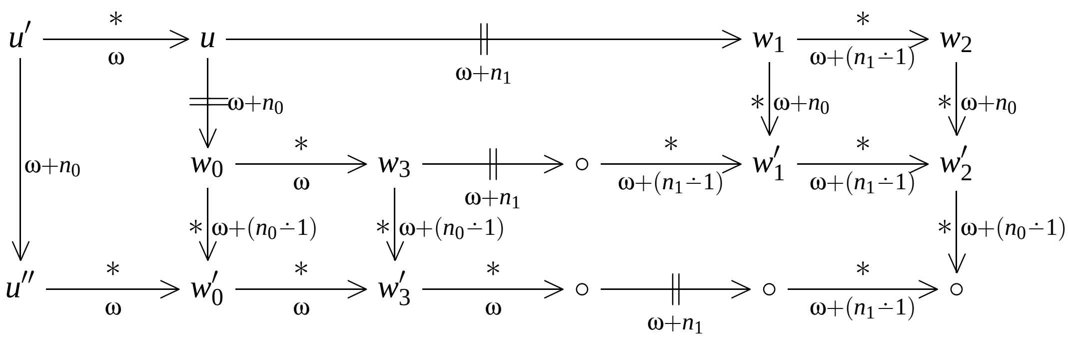
$$\forall n_0,n_1 \prec \omega.\ \left(\begin{array}{l} \left(\begin{array}{l} \quad n_0 \preceq n_1 \\ \wedge\ \ n_0{+_\omega}n_1 \preceq \delta \end{array}\right) \\ \Rightarrow\ \ \forall w_0,w_1,u.\ \left(\begin{array}{l} \quad w_0 \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!-}_{\omega+n_0} u \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} w_1 \\ \Rightarrow\ \ w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \circ \overset{*}{\longleftarrow}_{\omega+n_0} w_1 \end{array}\right) \end{array}\right),$$

then

$$\forall n_0,n_1 \prec \omega.\ \left(\begin{array}{l} \left(\begin{array}{l} \quad n_0 \preceq n_1 \\ \wedge\ \ n_0{+_\omega}n_1 \preceq \delta \end{array}\right) \\ \Rightarrow\ \ \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \text{ strongly commutes over } \overset{*}{\longrightarrow}_{\omega+n_0} \end{array}\right),$$

and R,X is ω-shallow confluent up to δ.

<u>Proof of Claim 2:</u> By induction on δ in $\prec$. First we show the strong commutation. Assume $n_0 \preceq n_1 \prec \omega$ with $n_0{+_\omega}n_1 \preceq \delta$. By Lemma 3.3 it suffices to show that $\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)}$ strongly commutes over $\longrightarrow_{\omega+n_0}$. Assume $u'' \longleftarrow_{\omega+n_0} u' \overset{*}{\longrightarrow}_{\omega} u \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} w_1 \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_2$ (cf. diagram below). By the strong commutation assumed for our lemma and Corollary 2.14, there are w_0 and w_0' with $u'' \overset{*}{\longrightarrow}_{\omega} w_0' \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_0 \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!-}_{\omega+n_0} u$. By the above property there are some w_3, w_1' with $w_0 \overset{*}{\longrightarrow}_{\omega} w_3 \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_1' \overset{*}{\longleftarrow}_{\omega+n_0} w_1$. Next we show that we can close the peak $w_1' \overset{*}{\longleftarrow}_{\omega+n_0} w_1 \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_2$ according to $w_1' \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_2' \overset{*}{\longleftarrow}_{\omega+n_0} w_2$ for some w_2'. In case of $n_1 = 0$ this is possible due to the ω-shallow confluence up to ω given by Claim 0. Otherwise we have $n_0{+_\omega}(n_1\dot{-}1) \prec n_0{+_\omega}n_1 \preceq \delta$ and due to our induction hypothesis (saying that R,X is ω-shallow confluent up to all $\delta' \prec \delta$) this is possible again. By Claim 0 again, we can close the peak $w_0' \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_0 \overset{*}{\longrightarrow}_{\omega} w_3$ according to $w_0' \overset{*}{\longrightarrow}_{\omega} w_3' \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_3$ for some w_3'. To close the whole diagram, we only have to show that we can close the peak $w_3' \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_3 \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_2'$ according to $w_3' \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \circ \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_2'$. In case of $n_0 = 0$ this is possible due to the strong commutation assumed for our lemma. Otherwise we have $n_0\dot{-}1 \prec n_0 \preceq n_1$ and $(n_0\dot{-}1){+_\omega}n_1 \prec n_0{+_\omega}n_1 \preceq \delta$, and then due to our induction hypothesis this is possible again.



Finally we show ω-shallow confluence up to δ. Assume $n_0{+_\omega}n_1 \preceq \delta$ and $w_0 \overset{*}{\longleftarrow}_{\omega+n_0} u \overset{*}{\longrightarrow}_{\omega+n_1} w_1$. Due to symmetry in n_0 and n_1 we may assume $n_0 \preceq n_1$. Above we have shown that $\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n_0}$. By Claim 1 we finally get $w_0 \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0} w_1$ as desired. <u>Q.e.d. (Claim 2)</u>

Note that for $n_0 = 0$ our property follows from $\mathrel{\leftarrow\!\!\!\!\!\parallel}_{\omega} \subseteq \overset{*}{\longleftarrow}_{\omega}$ (by Corollary 2.14) and the assumption of our lemma that for each $n_1 \prec \omega$: $\mathrel{\parallel\!\!\!\!\!\rightarrow}_{\mathrm{R,X},\omega+n_1} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+(n_1 \dot{-} 1)}$ strongly commutes over $\overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}$.

The benefit of Claim 2 is twofold: First, it says that our lemma is valid if the above property holds for all $n_0 \preceq n_1 \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n_i{+}1$ instead of n_i since we may assume $0 \prec n_0 \preceq n_1$) it now suffices to show for $n_0 \preceq n_1 \prec \omega$ that

$$w_0 \mathrel{\leftarrow\!\!\!\!\!\parallel}_{\omega+n_0+1,\Pi_0} u \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\omega+n_1+1,\Pi_1} w_1$$

together with our induction hypotheses that

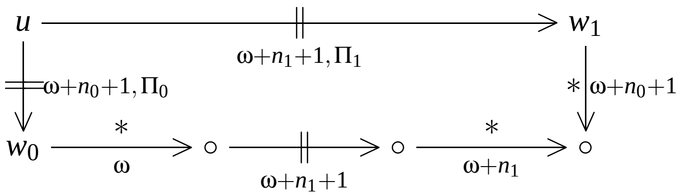
$$\forall\delta \prec (n_0{+}1) +_{\omega} (n_1{+}1).\ \mathrm{R,X} \text{ is } \omega\text{-shallow confluent up to } \delta$$

and (due to $n_0 \preceq n_1{+}1$ and $n_0 +_{\omega} (n_1{+}1) \prec (n_0{+}1) +_{\omega} (n_1{+}1)$)

$$\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \text{ strongly commutes over } \overset{*}{\longrightarrow}_{\omega+n_0}$$

implies

$$w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1 .$$



Note that for the availability of our second induction hypothesis it is important that we have imposed the restriction "$n_0 \preceq n_1$" in opposition to the restriction "$n_0 \succeq n_1$". In the latter case the availability of our second induction hypothesis would require $n_0{+}1 \succeq n_1{+}1 \Rightarrow n_0 \succeq n_1{+}1$ which is not true for $n_0 = n_1$. The additional hypothesis

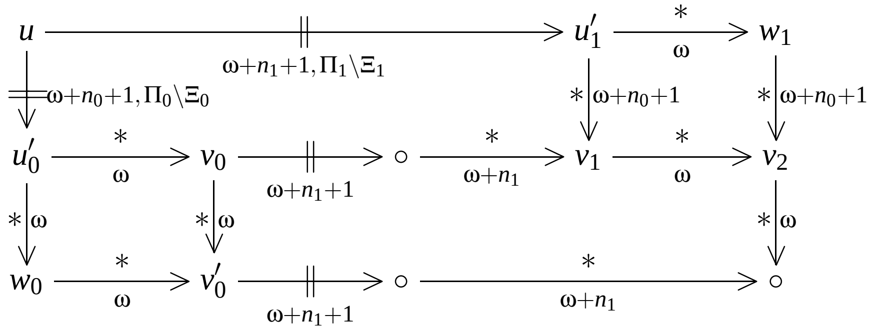
$$\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \text{ strongly commutes over } \overset{*}{\longrightarrow}_{\omega+n_0+1}$$

of the latter restriction is useless for our proof.

W.l.o.g. let the positions of Π_i be maximal in the sense that for any $p \in \Pi_i$ and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $u \mathrel{\parallel\!\!\!\!\!\rightarrow}_{\omega+n_i+1,(\Pi_i \setminus \{p\}) \cup \Xi} w_i$ anymore. Then for each $i \prec 2$ and $p \in \Pi_i$ there are $((l_{i,p}, r_{i,p}), C_{i,p}) \in \mathrm{R}$ and $\mu_{i,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{i,p}\mu_{i,p}$, $r_{i,p}\mu_{i,p} = w_i/p$, $C_{i,p}\mu_{i,p}$ fulfilled w.r.t. $\longrightarrow_{\omega+n_i}$. Finally, for each $i \prec 2$: $w_i = u[\, p \leftarrow r_{i,p}\mu_{i,p} \mid p \in \Pi_i \,]$.

<u>Claim 5:</u> We may assume $\forall i\prec 2.\ \forall p\in\Pi_i.\ l_{i,p}\notin\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)$.

<u>Proof of Claim 5:</u> Define $\Xi_i := \{\, p\in\Pi_i \mid l_{i,p}\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)\,\}$ and $u'_i := u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i\backslash\Xi_i\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $u'_0\stackrel{*}{\longrightarrow}_{\omega}v_0\mathrel{\Vert\!\!\!\longrightarrow}_{\omega+n_1+1}\circ\stackrel{*}{\longrightarrow}_{\omega+n_1}v_1\stackrel{*}{\longleftarrow}_{\omega+n_0+1}u'_1$ for some v_0, v_1 (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{i,p}$) we get $\forall i\prec 2.\ \forall p\in\Xi_i.\ l_{i,p}\mu_{i,p}\longrightarrow_{\omega}r_{i,p}\mu_{i,p}$ and therefore $\forall i\prec 2.\ u'_i\stackrel{*}{\longrightarrow}_{\omega}w_i$. Thus from $v_1\stackrel{*}{\longleftarrow}_{\omega+n_0+1}u'_1\stackrel{*}{\longrightarrow}_{\omega}w_1$ we get $v_1\stackrel{*}{\longrightarrow}_{\omega}v_2\stackrel{*}{\longleftarrow}_{\omega+n_0+1}w_1$ for some v_2 by ω-shallow confluence up to ω (cf. Claim 0). For the same reason we can close the peak $w_0\stackrel{*}{\longleftarrow}_{\omega}u'_0\stackrel{*}{\longrightarrow}_{\omega}v_0$ according to $w_0\stackrel{*}{\longrightarrow}_{\omega}v'_0\stackrel{*}{\longleftarrow}_{\omega}v_0$ for some v'_0. By the assumption of our lemma that $\mathrel{\Vert\!\!\!\longrightarrow}_{\mathrm{R,X},\omega+n_1+1}\circ\stackrel{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{\omega}$, from $v'_0\stackrel{*}{\longleftarrow}_{\omega}v_0\mathrel{\Vert\!\!\!\longrightarrow}_{\omega+n_1+1}\circ\stackrel{*}{\longrightarrow}_{\omega+n_1}v_1\stackrel{*}{\longrightarrow}_{\omega+n_1}v_2$ we can finally conclude $v'_0\mathrel{\Vert\!\!\!\longrightarrow}_{\omega+n_1+1}\circ\stackrel{*}{\longrightarrow}_{\omega+n_1}\circ\stackrel{*}{\longleftarrow}_{\omega}v_2$.



<u>Q.e.d. (Claim 5)</u>

Define the set of inner overlapping positions by

$$\Omega(\Pi_0,\Pi_1) := \bigcup_{i\prec 2}\{\, p\in\Pi_{1-i} \mid \exists q\in\Pi_i.\ \exists q'\in\mathbf{N}^*.\ p=qq'\,\},$$

and the length of a term by $\lambda(f(t_0,\ldots,t_{m-1})) := 1+\sum_{j\prec m}\lambda(t_j)$.

Now we start a second level of induction on $\displaystyle\sum_{p'\in\Omega(\Pi_0,\Pi_1)}\lambda(u/p')$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\, p\in\Pi_0\cup\Pi_1 \mid \neg\exists q\in\Pi_0\cup\Pi_1.\ \exists q'\in\mathbf{N}^+.\ p=qq'\,\}.$$

Since the prefix ordering is wellfounded we have $\forall i\prec 2.\ \forall p\in\Pi_i.\ \exists q\in\Theta.\ \exists q'\in\mathbf{N}^*.\ p=qq'$. Then $\forall i\prec 2.\ w_i=w_i[q\leftarrow w_i/q \mid q\in\Theta]=u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i][q\leftarrow w_i/q \mid q\in\Theta]=u[q\leftarrow w_i/q \mid q\in\Theta]$. Thus, it now suffices to show for all $q\in\Theta$

$$w_0/q\stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{\Vert\!\!\!\longrightarrow}_{\omega+n_1+1}\circ\stackrel{*}{\longrightarrow}_{\omega+n_1}\circ\stackrel{*}{\longleftarrow}_{\omega+n_0+1}w_1/q$$

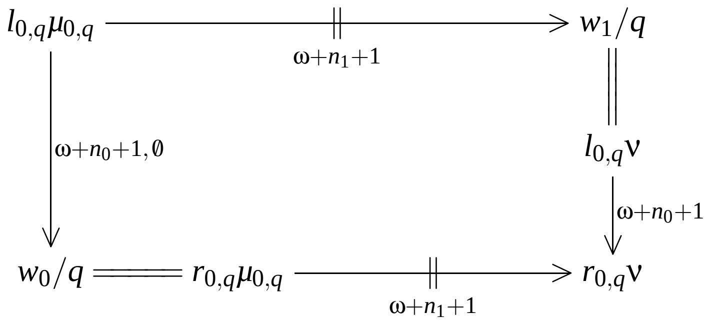
because then we have

$$w_0=u[q\leftarrow w_0/q \mid q\in\Theta]\stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{\Vert\!\!\!\longrightarrow}_{\omega+n_1+1}\circ\stackrel{*}{\longrightarrow}_{\omega+n_1}\circ\stackrel{*}{\longleftarrow}_{\omega+n_0+1}u[q\leftarrow w_1/q \mid q\in\Theta]=w_1.$$

Therefore we are left with the following two cases for $q\in\Theta$:

<u>$q \notin \Pi_1$:</u> Then $q \in \Pi_0$. Define $\Pi'_1 := \{\, p \mid qp \in \Pi_1 \,\}$. We have two cases:

<u>"The variable overlap (if any) case": $\forall p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}$:</u>



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\, (p', p'') \mid l_{0,q}/p' = x \wedge p'p'' \in \Pi'_1 \,\}$.

<u>Claim 7:</u> There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}. \left(\begin{array}{ll} & x\mu_{0,q} \twoheadrightarrow_{\omega+n_1+1} x\nu \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu = x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \end{array} \right).$$

<u>Proof of Claim 7:</u>

In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$\begin{array}{rl} x\mu_{0,q} = x\mu_{0,q}[\, p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \twoheadrightarrow_{\omega+n_1+1} & \\ x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] = x\nu. & \end{array}$$

Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{0,q}/p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ this implies $l_{1,qp'p''}\mu_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. <u>Q.e.d. (Claim 7)</u>

<u>Claim 8:</u> $l_{0,q}\nu = w_1/q$.

<u>Proof of Claim 8:</u>

By Claim 7 we get $w_1/q = u/q[\, p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x) \,] =$
$l_{0,q}[\, p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V} \,][\, p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x) \,] =$
$l_{0,q}[\, p' \leftarrow x\mu_{0,q}[\, p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x) \,] \mid l_{0,q}/p' = x \in \mathrm{V} \,] =$
$l_{0,q}[\, p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V} \,] = l_{0,q}\nu$. <u>Q.e.d. (Claim 8)</u>

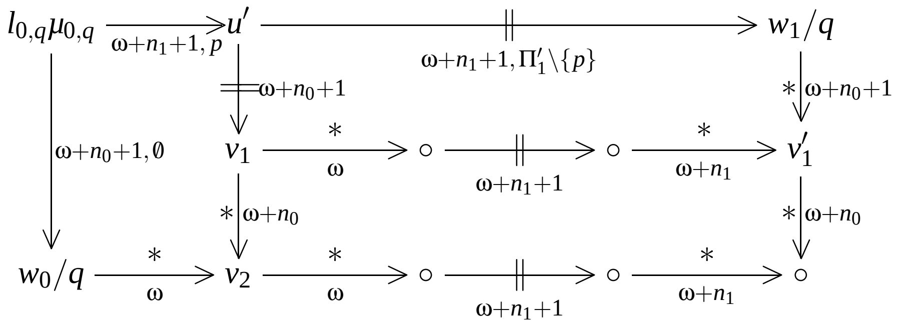
<u>Claim 9:</u> $w_0/q \twoheadrightarrow_{\omega+n_1+1} r_{0,q}\nu$.

<u>Proof of Claim 9:</u> Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. <u>Q.e.d. (Claim 9)</u>

By claims 8 and 9 it now suffices to show $l_{0,q}\nu \longrightarrow_{\omega+n_0+1} r_{0,q}\nu$, which again follows from Lemma 13.8 since R, X is ω-shallow confluent up to $(n_1{+}1) +_{\omega} n_0$ by our induction hypothesis and since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega+n_1+1} x\nu$ by Claim 7 and Corollary 2.14.

<u>Q.e.d. ("The variable overlap (if any) case")</u>

<u>"The critical peak case": There is some $p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:</u>



<u>Claim 10:</u> $p \neq \emptyset$.

<u>Proof of Claim 10:</u> If $p=\emptyset$, then $\emptyset \in \Pi'_1$, then $q \in \Pi_1$, which contradicts our global case assumption. <u>Q.e.d. (Claim 10)</u>

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp},r_{1,qp}),C_{1,qp}))] \cap \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp},r_{1,qp}),C_{1,qp}))] \cup \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{Bmatrix}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1 \setminus \{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \twoheadrightarrow_{\omega+n_1+1,\Pi'_1\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1] = w_1/q.$$

If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \twoheadrightarrow_{\omega+n_1+1,\Pi'_1\setminus\{p\}} w_1/q.$$

Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 1), l_{0,q}\sigma, p) \in \mathrm{CP(R)}$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$. Since $\forall\delta \prec (n_1+1) +_\omega (n_0+1)$. R, X is ω-shallow confluent up to δ (by our induction hypothesis) due to our assumed ω-shallow parallel closedness (matching the definition's n_0 to our n_1+1 and its n_1 to our n_0+1) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \twoheadrightarrow_{\omega+n_0+1} v_1 \overset{*}{\longrightarrow}_{\omega+n_0} v_2 \overset{*}{\longleftarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v_1, v_2. We then have $v_1 \twoheadleftarrow_{\omega+n_0+1,\Pi''} u' \twoheadrightarrow_{\omega+n_1+1,\Pi'_1\setminus\{p\}} w_1/q$ for some Π''. By

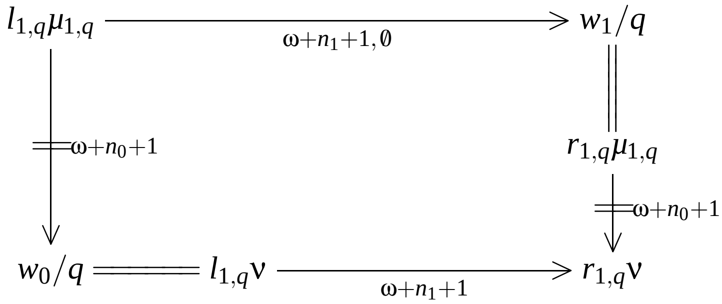
$$\sum_{p''\in\Omega(\Pi'',\Pi'_1\setminus\{p\})} \lambda(u'/p'') \preceq \sum_{p''\in\Pi'_1\setminus\{p\}} \lambda(u'/p'') = \sum_{p''\in\Pi'_1\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p''\in\Pi'_1} \lambda(u/qp'') =$$
$$\sum_{p'\in q\Pi'_1} \lambda(u/p') = \sum_{p'\in\Omega(\{q\},\Pi_1)} \lambda(u/p') \preceq \sum_{p'\in\Omega(\Pi_0,\Pi_1)} \lambda(u/p'),$$

due to our second induction level we get some v'_1 with $v_1 \overset{*}{\longrightarrow}_{\omega} \circ \twoheadrightarrow_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} v'_1 \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1/q$. Finally by our induction hypothesis that $\overset{*}{\longrightarrow}_{\omega} \circ \twoheadrightarrow_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n_0}$ the peak at v_1 can be closed according to $v_2 \overset{*}{\longrightarrow}_{\omega} \circ \twoheadrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0} v'_1$.

<u>Q.e.d. ("The critical peak case")</u> <u>Q.e.d. ("$q \notin \Pi_1$")</u>

$\underline{\underline{q \in \Pi_1}}$: Define $\Pi'_0 := \{\ p \mid qp \in \Pi_0\ \}$. We have two cases:

"The second variable overlap (if any) case": $\forall p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q}).\ l_{1,q}/p \in \mathrm{V}$:



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\ (p', p'') \mid l_{1,q}/p' = x \ \wedge\ p'p'' \in \Pi'_0\ \}$.

<u>Claim 11:</u> There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}. \left(\begin{array}{ll} & x\nu \leftarrow\!\!\!+\!\!\!\!-_{\omega+n_0+1} x\mu_{1,q} \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu \end{array} \right).$$

<u>Proof of Claim 11:</u>
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{1,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{1,q}/p'' = l_{1,q}\mu_{1,q}/p'p'' = u/qp'p'' = l_{0,qp'p''}\mu_{0,qp'p''}$ we have

$$x\mu_{1,q} = x\mu_{1,q}[\,p'' \leftarrow l_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \rightarrow\!\!\!\!+\!\!\!\!\!\rightarrow_{\omega+n_0+1}$$
$$x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu.$$

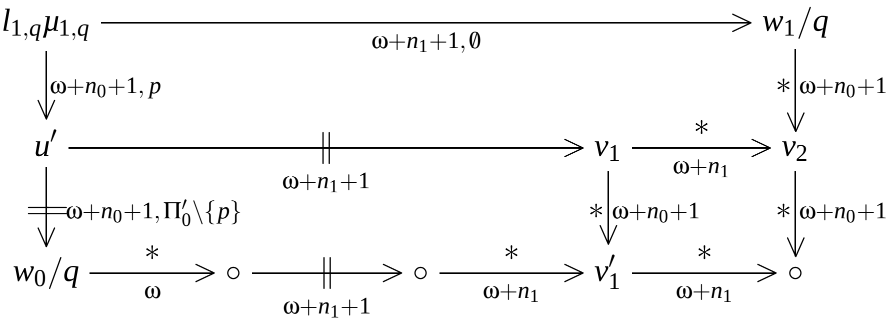
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{1,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{1,q}/p'' = l_{0,qp'p''}\mu_{0,qp'p''}$ this implies $l_{0,qp'p''}\mu_{0,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{0,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. <u>Q.e.d. (Claim 11)</u>

<u>Claim 12:</u> $w_0/q = l_{1,q}\nu$.
<u>Proof of Claim 12:</u>
By Claim 11 we get $w_0/q = u/q[\,p'p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{1,q}[\,p' \leftarrow x\mu_{1,q} \mid l_{1,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{1,q}[\,p' \leftarrow x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{1,q}/p' = x \in \mathrm{V}\,] =$
$l_{1,q}[\,p' \leftarrow x\nu \mid l_{1,q}/p' = x \in \mathrm{V}\,] = l_{1,q}\nu$. <u>Q.e.d. (Claim 12)</u>

<u>Claim 13:</u> $r_{1,q}\nu \leftarrow\!\!\!+\!\!\!\!-_{\omega+n_0+1} w_1/q$.
<u>Proof of Claim 13:</u> Since $r_{1,q}\mu_{1,q} = w_1/q$, this follows directly from Claim 11. <u>Q.e.d. (Claim 13)</u>

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{\omega+n_1+1} r_{1,q}\nu$, which again follows from Claim 11, Corollary 2.14, Lemma 13.8 (matching its n_0 to our n_0+1 and its n_1 to our n_1), and our induction hypothesis that R, X is ω-shallow confluent up to $(n_0+1) +_\omega n_1$.
Q.e.d. ("The second variable overlap (if any) case")

“The second critical peak case”: There is some $p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q})$ with $l_{1,q}/p \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,qp}, r_{0,qp}), C_{0,qp}))] \cap \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,qp}, r_{0,qp}), C_{0,qp}))] \cup \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) \\ x\xi^{-1}\mu_{0,qp} & \text{else} \end{Bmatrix}$ $(x \in \mathrm{V})$.
By $l_{0,qp}\xi\rho = l_{0,qp}\xi\xi^{-1}\mu_{0,qp} = u/qp = l_{1,q}\mu_{1,q}/p = l_{1,q}\rho/p = (l_{1,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,qp}\xi, l_{1,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{1,q}\mu_{1,q}[p \leftarrow r_{0,qp}\mu_{0,qp}]$. We get

$$w_0/q = u/q[p' \leftarrow r_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0] \leftarrow\!\!\!\!\!+\!\!\!+\!\!\!-_{\omega+n_0+1,\Pi'_0\setminus\{p\}}$$
$$u/q[p' \leftarrow l_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0\setminus\{p\}][p \leftarrow r_{0,qp}\mu_{0,qp}] = u'.$$

If $l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q \leftarrow\!\!\!\!\!+\!\!\!+\!\!\!-_{\omega+n_0+1,\Pi'_0\setminus\{p\}} u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

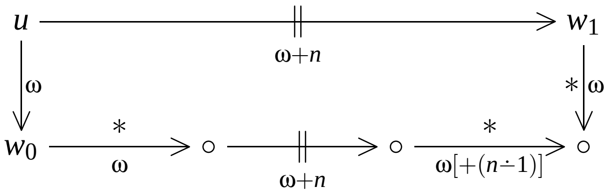
Otherwise we have $((l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma, C_{0,qp}\xi\sigma, 1), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_{0,qp}\xi\sigma\varphi = C_{0,qp}\mu_{0,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$. Since $\forall\delta \prec (n_0{+}1) +_\omega (n_1{+}1)$. R, X is ω-shallow confluent up to δ (by our induction hypothesis) due to our assumed ω-shallow noisy parallel joinability (matching the definition's n_0 to our $n_0{+}1$ and its n_1 to our $n_1{+}1$) we have $u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi \,-\!\!\!+\!\!\!+\!\!\!\!\!\rightarrow_{\omega+n_1+1} v_1 \overset{*}{\longrightarrow}_{\omega+n_1} v_2 \overset{*}{\longleftarrow}_{\omega+n_0+1} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$ for some v_1, v_2. We then have $w_0/q \leftarrow\!\!\!\!\!+\!\!\!+\!\!\!-_{\omega+n_0+1,\Pi'_0\setminus\{p\}} u' -\!\!\!+\!\!\!+\!\!\!\!\!\rightarrow_{\omega+n_1+1,\Pi''} v_1$ for some Π''. Since $\sum_{p'' \in \Omega(\Pi'_0\setminus\{p\},\Pi'')} \lambda(u'/p'') \preceq \sum_{p'' \in \Pi'_0\setminus\{p\}} \lambda(u'/p'') = \sum_{p'' \in \Pi'_0\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p'' \in \Pi'_0} \lambda(u/qp'') = \sum_{p' \in q\Pi'_0} \lambda(u/p') = \sum_{p' \in \Omega(\Pi_0,\{q\})} \lambda(u/p') \preceq \sum_{p' \in \Omega(\Pi_0,\Pi_1)} \lambda(u/p')$ due to our second induction level we get some v'_1 with $w_0/q \overset{*}{\longrightarrow}_\omega \circ -\!\!\!+\!\!\!+\!\!\!\!\!\rightarrow_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} v'_1 \overset{*}{\longleftarrow}_{\omega+n_0+1} v_1$. Finally the peak at v_1 can be closed according to $v'_1 \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} v_2$ by our induction hypothesis saying that R, X is ω-shallow confluent up to $(n_0{+}1) +_\omega n_1$.

Q.e.d. (“The second critical peak case”) **Q.e.d. (Lemma A.2)**

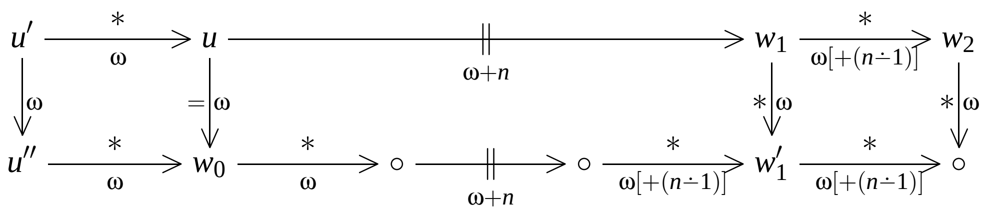
Proof of Lemma A.3

For $n \prec \omega$ we are going to show by induction on n the following property:

$$w_0 \longleftarrow_{\omega} u \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} w_1 \quad \Rightarrow \quad w_0 \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \circ \stackrel{*}{\longleftarrow}_{\omega} w_1 .$$



Claim 1: If the above property holds for a fixed $n \prec \omega$, and
$\forall k \prec n.\ (\mathrm{R},\mathrm{X}$ is ω-shallow confluent up to $k)$, then $\stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{\omega}$.
Proof of Claim 1: By Lemma 3.3 it suffices to show that $\stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ strongly commutes over $\longrightarrow_{\omega}$. Assume $u'' \longleftarrow_{\omega} u' \stackrel{*}{\longrightarrow}_{\omega} u \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} w_1 \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w_2$ (cf. diagram below). By the strong confluence of $\longrightarrow_{\omega}$ assumed for our lemma we can close the peak $u'' \longleftarrow_{\omega} u' \stackrel{*}{\longrightarrow}_{\omega} u$ according to $u'' \stackrel{*}{\longrightarrow}_{\omega} w_0 \stackrel{=}{\longleftarrow}_{\omega} u$ for some w_0. By the above property there is some w_1' with $w_0 \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w_1' \stackrel{*}{\longleftarrow}_{\omega} w_1$. We only have to show that we can close the peak $w_1' \stackrel{*}{\longleftarrow}_{\omega} w_1 \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w_2$ according to $w_1' \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \circ \stackrel{*}{\longleftarrow}_{\omega} w_2$. [In case of $n=0$:] This is possible due to confluence of $\longrightarrow_{\omega}$. [Otherwise we have $n\dot{-}1 \prec n$ and due to the assumed ω-shallow confluence up to $n\dot{-}1$ this is possible again.]



Q.e.d. (Claim 1)

Claim 2: If the above property holds for a fixed $n \prec \omega$, and
$\forall k \prec n.\ (\mathrm{R},\mathrm{X}$ is ω-shallow confluent up to $k)$, then $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega}$ are commuting.
Proof of Claim 2: $\stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ and $\stackrel{*}{\longrightarrow}_{\omega}$ are commuting by Lemma 3.3 and Claim 1. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{\omega+n} \subseteq \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n} \circ \stackrel{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \subseteq \stackrel{*}{\longrightarrow}_{\omega+n}$, now $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega}$ are commuting, too. Q.e.d. (Claim 2)

Claim 3: If the above property holds for all $n \preceq m$ for some $m \prec \omega$, then R, X is ω-shallow confluent up to m.
Proof of Claim 3: By induction on m in $\prec$. Assume $i +_{\omega} n \preceq m$ and $w_0 \stackrel{*}{\longleftarrow}_{\omega+i} u \stackrel{*}{\longrightarrow}_{\omega+n} w_1$. By definition of '$+_{\omega}$' and $i +_{\omega} n \prec \omega$ w.l.o.g. we have $i=0$ and $n \preceq m$. By Claim 2 and our induction hypothesis we finally get $w_0 \stackrel{*}{\longrightarrow}_{\omega+n} \circ \stackrel{*}{\longleftarrow}_{\omega} w_1$ as desired. Q.e.d. (Claim 3)

Note that our property for is trivial for $n=0$ since then by Corollary 2.14 we have $\mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n} = \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega} \subseteq \overset{*}{\longrightarrow}_{\omega}$ and $\longrightarrow_{\omega}$ is confluent.

The benefit of claims 1 and 3 is twofold: First, they say that our lemma is valid if the above property holds for all $n \prec \omega$. Second, they strengthen the property when used as induction hypothesis. Thus (writing $n+1$ instead of n since we may assume $0 \prec n$) it now suffices to show for $n \prec \omega$ that

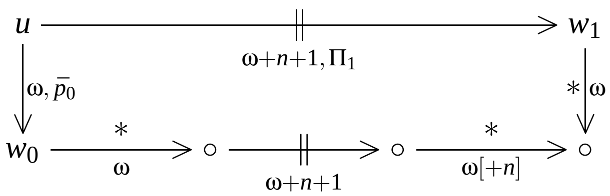
$$w_0 \mathrel{\longleftarrow\!\!\!\!\!+}_{\omega,\bar{p}_0} u \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1,\Pi_1} w_1$$

together with our induction hypothesis that

$$\mathrm{R}, \mathrm{X} \text{ is } \omega\text{-shallow confluent up to } n$$

implies

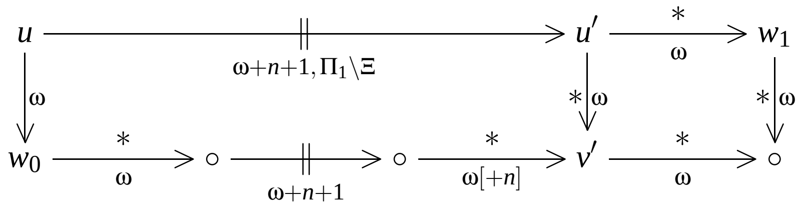
$$w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega[+n]} \circ \overset{*}{\longleftarrow}_{\omega} w_1 .$$



There are $((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}) \in \mathrm{R}$ and $\mu_{0,\bar{p}_0} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ such that $l_{0,\bar{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, $u/\bar{p}_0 = l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$, $C_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ is fulfilled w.r.t. $\longrightarrow_{\omega}$, and $w_0 = u[\,\bar{p}_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}\,]$.

W.l.o.g. let the positions of Π_1 be maximal in the sense that for any $p \in \Pi_1$ and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $u \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1,(\Pi_1 \setminus \{p\}) \cup \Xi} w_1$ anymore. Then for each $p \in \Pi_1$ there are $((l_{1,p}, r_{1,p}), C_{1,p}) \in \mathrm{R}$ and $\mu_{1,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ such that $u/p = l_{1,p}\mu_{1,p}$, $r_{1,p}\mu_{1,p} = w_1/p$, $C_{1,p}\mu_{1,p}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$, and $w_1 = u[\,p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1\,]$.

<u>Claim 5:</u> We may assume $\forall p\in\Pi_1.\ l_{1,p}\notin\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$.
<u>Proof of Claim 5:</u> Define $\Xi := \{\ p\in\Pi_1 \mid l_{1,p}\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\ \}$ and $u' := u[\,p\leftarrow r_{1,p}\mu_{1,p} \mid p\in\Pi_1\backslash\Xi\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $w_0\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n+1}\circ\overset{*}{\longrightarrow}_{\omega[+n]}v'\overset{*}{\longleftarrow}_{\omega}u'$ for some v' (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{1,p}$) we get $\forall p\in\Xi.\ l_{1,p}\mu_{1,p}\longrightarrow_{\omega}r_{1,p}\mu_{1,p}$. Thus from $v'\overset{*}{\longleftarrow}_{\omega}u'\overset{*}{\longrightarrow}_{\omega}w_1$ we get $v'\overset{*}{\longrightarrow}_{\omega}\circ\overset{*}{\longleftarrow}_{\omega}w_1$ by confluence of $\longrightarrow_{\omega}$.



<u>Q.e.d. (Claim 5)</u>

Now we start a second level of induction on $|\Pi_1|$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\ p\in\{\bar{p}_0\}\cup\Pi_1 \mid \neg\exists q\in\{\bar{p}_0\}\cup\Pi_1.\ \exists q'\in\mathbf{N}^{+}.\ p=qq'\ \}.$$

Since the prefix ordering is wellfounded we have $\forall p\in\{\bar{p}_0\}\cup\Pi_1.\ \exists q\in\Theta.\ \exists q'\in\mathbf{N}^{*}.\ p=qq'$. It now suffices to show for all $q\in\Theta$

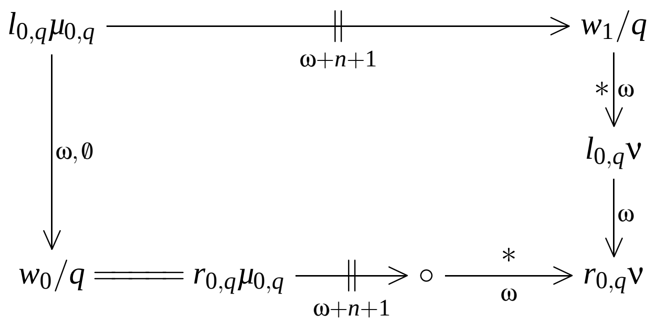
$$w_0/q\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n+1}\circ\overset{*}{\longrightarrow}_{\omega[+n]}\circ\overset{*}{\longleftarrow}_{\omega}w_1/q$$

because then we have $w_0=w_0[q\leftarrow w_0/q \mid q\in\Theta]=u[\,\bar{p}_0\leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}\,][q\leftarrow w_0/q \mid q\in\Theta]=$
$u[q\leftarrow w_0/q \mid q\in\Theta]\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n+1}\circ\overset{*}{\longrightarrow}_{\omega[+n]}\circ\overset{*}{\longleftarrow}_{\omega}u[q\leftarrow w_1/q \mid q\in\Theta]=$
$u[\,p\leftarrow r_{1,p}\mu_{1,p} \mid p\in\Pi_1\,][q\leftarrow w_1/q \mid q\in\Theta]=w_1[q\leftarrow w_1/q \mid q\in\Theta]=w_1$.

Therefore we are left with the following two cases for $q\in\Theta$:

$\underline{q \notin \Pi_1}$: Then $q = \bar{p}_0$. Define $\Pi_1' := \{\, p \mid qp \in \Pi_1 \,\}$. We have two cases:

"The variable overlap (if any) case": $\forall p \in \Pi_1' \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}$:



Define a function Γ on V by ($x \in$ V): $\Gamma(x) := \{\, (p', p'') \mid l_{0,q}/p' = x \wedge p'p'' \in \Pi_1' \,\}$.

Claim 7: There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}.\ \left(\begin{array}{l} x\mu_{0,q} \twoheadrightarrow_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} x\nu \\ \wedge\ \ \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu \overset{*}{\longleftarrow}_{\omega} x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \end{array} \right).$$

Proof of Claim 7:
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$x\mu_{0,q} = x\mu_{0,q}[\,p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \twoheadrightarrow_{\omega+n+1} x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu.$$

Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q} = l_{0,\bar{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ is not linear in x. By the conditions of our lemma, this implies $x \in \mathrm{V}_{\mathcal{C}}$. Therefore $x\mu_{0,q} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$. Together with
$\forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega+n+1} x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$ this implies
$\forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega} x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$
by Lemma 2.10. By confluence of $\longrightarrow_{\omega}$ and Lemma 2.10 again, there is some $t \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ with
$\forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \overset{*}{\longrightarrow}_{\omega} t$. Therefore we can define $x\nu := t$ in this case. This is appropriate since by $\exists p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega} x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \overset{*}{\longrightarrow}_{\omega} x\nu$ we have $x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega} x\nu$. Q.e.d. (Claim 7)

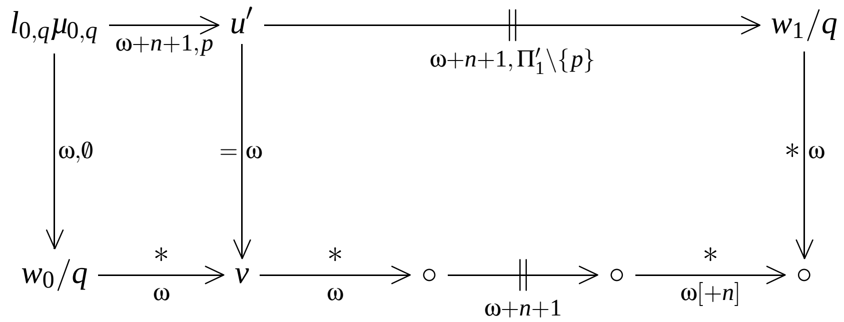
Claim 8: $l_{0,q}\nu \overset{*}{\longleftarrow}_{\omega} w_1/q$.
Proof of Claim 8:
By Claim 7 we get $w_1/q = u/q[\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{0,q}/p' = x \in \mathrm{V}\,] \overset{*}{\longrightarrow}_{\omega}$
$l_{0,q}[\,p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V}\,] = l_{0,q}\nu$. Q.e.d. (Claim 8)

Claim 9: $w_0/q \twoheadrightarrow_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} r_{0,q}\nu$.
Proof of Claim 9: Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows from Claim 7. Q.e.d. (Claim 9)

By claims 8 and 9 it now suffices to show $r_{0,q}\nu \longleftarrow_{\omega} l_{0,q}\nu$, which again follows from Lemma 13.2 since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega+n+1} x\nu$ by Claim 7 and Corollary 2.14.

Q.e.d. ("The variable overlap (if any) case")

"The critical peak case": There is some $p \in \Pi_1' \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:



Claim 10: $p \neq \emptyset$.
Proof of Claim 10: If $p = \emptyset$, then $\emptyset \in \Pi_1'$, then $q \in \Pi_1$, which contradicts our global case assumption. Q.e.d. (Claim 10)

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cap \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cup \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{ \begin{matrix} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{matrix} \right\}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\upharpoonright(\sigma\varphi) = {}_{\mathrm{Y}}\upharpoonright\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi_1' \setminus \{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \mathrel{+\!\!\!\Vert\!\!\!\rightarrow}_{\omega+n+1, \Pi_1'\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi_1'] = w_1/q.$$

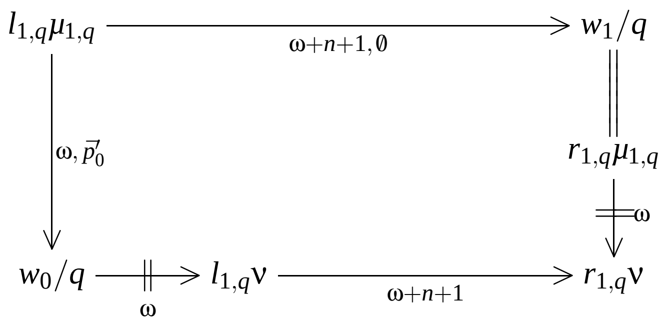
If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \mathrel{+\!\!\!\Vert\!\!\!\rightarrow}_{\omega+n+1, \Pi_1'\setminus\{p\}} w_1/q.$$

Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 0), l_{0,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega}$. Since R, X is ω-shallow confluent up to n (by our induction hypothesis), due to our assumed ω-shallow closedness up to ω (matching the definition's n_0 to our $n+1$ and its n_1 to 0) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \overset{=}{\longrightarrow}_{\omega} v \overset{*}{\longleftarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v. We then have $v \overset{=}{\longleftarrow}_{\omega} u' \mathrel{+\!\!\!\Vert\!\!\!\rightarrow}_{\omega+n+1, \Pi_1'\setminus\{p\}} w_1/q$. We can finish the proof in this case due to our second induction level since $|\Pi_1' \setminus \{p\}| \prec |\Pi_1'| \preceq |\Pi_1|$.
Q.e.d. ("The critical peak case") Q.e.d. ("$q \notin \Pi_1$")

$q\in\Pi_1$: If there is no $\bar{p}'_0$ with $q\bar{p}'_0=\bar{p}_0$, then the proof is finished due to $w_0/q = u/q \longrightarrow_{\omega+n+1} w_1/q$. Otherwise, we can define $\bar{p}'_0$ by $q\bar{p}'_0=\bar{p}_0$. We have two cases:

"The second variable overlap case":
There are $x\in\mathrm{V}$ and p', p'' such that $l_{1,q}/p'=x$ and $p'p''=\bar{p}'_0$:



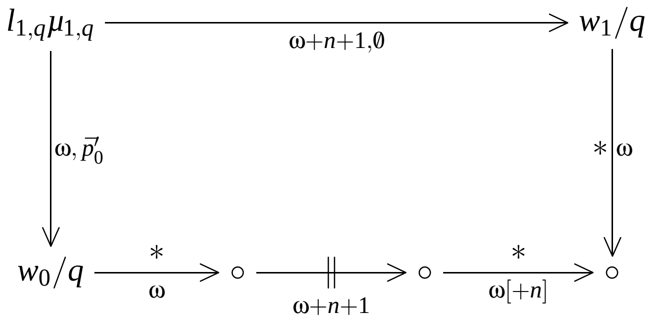
Claim 11: For $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ defined by $x\nu = x\mu_{1,q}[p''\leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}]$ and $\forall y\in\mathrm{V}\setminus\{x\}.\ y\nu = y\mu_{1,q}$ we get $\forall y\in\mathrm{V}.\ y\mu_{1,q} \mathrel{+\!\!\!\Vert\!\!\!\rightarrow}_\omega y\nu$.
Proof of Claim 11:
Due to $x\mu_{1,q}/p'' = l_{1,q}\mu_{1,q}/p'p'' = u/qp'p'' = u/q\bar{p}'_0 = u/\bar{p}_0 = l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ we have $x\mu_{1,q} = x\mu_{1,q}[p''\leftarrow l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] \longrightarrow_\omega x\mu_{1,q}[p''\leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] = x\nu$. Q.e.d. (Claim 11)

Claim 12: $w_0/q \mathrel{+\!\!\!\Vert\!\!\!\rightarrow}_\omega l_{1,q}\nu$.
Proof of Claim 12:
By Claim 11 we get $w_0/q = u/q[p'p''\leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] =$
$l_{1,q}[p'''\leftarrow y\mu_{1,q} \mid l_{1,q}/p'''=y\in\mathrm{V}][p'p''\leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] =$
$l_{1,q}[p'''\leftarrow y\mu_{1,q} \mid l_{1,q}/p'''=y\in\mathrm{V}\wedge x\neq y][p'''\leftarrow x\mu_{1,q} \mid l_{1,q}/p'''=x\wedge p'''\neq p']$
$[p'\leftarrow x\mu_{1,q}[p''\leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}]] =$
$l_{1,q}[p'''\leftarrow y\nu \mid l_{1,q}/p'''=y\in\mathrm{V}\wedge x\neq y][p'''\leftarrow x\mu_{1,q} \mid l_{1,q}/p'''=x\wedge p'''\neq p'][p'\leftarrow x\nu] \mathrel{+\!\!\!\Vert\!\!\!\rightarrow}_\omega$
$l_{1,q}[p'''\leftarrow y\nu \mid l_{1,q}/p'''=y\in\mathrm{V}\wedge x\neq y][p'''\leftarrow x\nu \mid l_{1,q}/p'''=x\wedge p'''\neq p'][p'\leftarrow x\nu] =$
$l_{1,q}[p'''\leftarrow y\nu \mid l_{1,q}/p'''=y\in\mathrm{V}] = l_{1,q}\nu$. Q.e.d. (Claim 12)

Claim 13: $r_{1,q}\nu \mathrel{\leftarrow\!\!\!\Vert\!\!\!+}_\omega w_1/q$.
Proof of Claim 13: Since $r_{1,q}\mu_{1,q} = w_1/q$, this follows directly from Claim 11. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{\omega+n+1} r_{1,q}\nu$, which again follows from Claim 11, Lemma 13.8 (matching its n_0 to 0 and its n_1 to our n) and our induction hypothesis that R, X is ω-shallow confluent up to n. Q.e.d. ("The second variable overlap case")

"The second critical peak case": $\overline{p}'_0 \in \mathcal{POS}(l_{1,q})$ with $l_{1,q}/\overline{p}'_0 \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}))] \cap \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}))] \cup \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{ \begin{array}{ll} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) \\ x\xi^{-1}\mu_{0,\bar{p}_0} & \text{else} \end{array} \right\}$ $(x \in \mathrm{V})$.
By $l_{0,\bar{p}_0}\xi\rho = l_{0,\bar{p}_0}\xi\xi^{-1}\mu_{0,\bar{p}_0} = u/\overline{p}_0 = l_{1,q}\mu_{1,q}/\overline{p}'_0 = l_{1,q}\rho/\overline{p}'_0 = (l_{1,q}/\overline{p}'_0)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,\bar{p}_0}\xi, l_{1,q}/\overline{p}'_0)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
If $l_{1,q}[\overline{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q = l_{1,q}\mu_{1,q}[\overline{p}'_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] = l_{1,q}[\overline{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

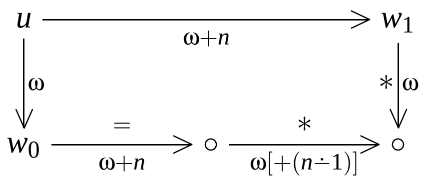
Otherwise we have $((l_{1,q}[\overline{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma, C_{0,\bar{p}_0}\xi\sigma, 0), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, \overline{p}'_0) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_{0,\bar{p}_0}\xi\sigma\varphi = C_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ is fulfilled w.r.t. $\longrightarrow_{\omega}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R, X ω-shallow confluent up to n (by our induction hypothesis), due to our assumed ω-shallow [noisy] weak parallel joinability up to ω (matching the definition's n_0 to 0 and its n_1 to our $n{+}1$) we have $w_0/q = l_{1,q}[\overline{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma\varphi \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega[+n]} \circ \overset{*}{\longleftarrow}_{\omega} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$.

Q.e.d. ("The second critical peak case") **Q.e.d. (Lemma A.3)**

Proof of Lemma A.4

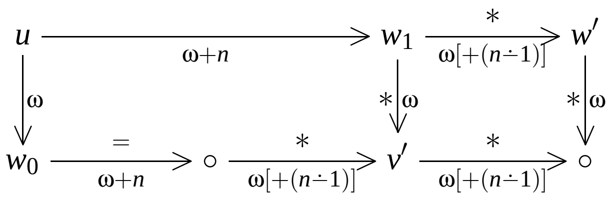
For $n \prec \omega$ we are going to show by induction on n the following property:

$$w_0 \,{}_{\omega}\!\longleftarrow u \longrightarrow_{\omega+n} w_1 \quad\Rightarrow\quad w_0 \overset{=}{\longrightarrow}_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \circ \,{}_{\omega}\!\overset{*}{\longleftarrow} w_1 .$$



Claim 1: If the above property holds for a fixed $n \prec \omega$, and
$\forall k \prec n.\ (\mathrm{R,X}$ is ω-shallow confluent up to $k)$, then $\longrightarrow_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega}$.

Proof of Claim 1: By Lemma 3.3 it suffices to show that $\longrightarrow_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ strongly commutes over $\longrightarrow_{\omega}$. Assume $w_0 \,{}_{\omega}\!\longleftarrow u \longrightarrow_{\omega+n} w_1 \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w'$ (cf. diagram below). By the above property there is some v' with $w_0 \overset{=}{\longrightarrow}_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} v' \,{}_{\omega}\!\overset{*}{\longleftarrow} w_1$. We only have to show that we can close the peak $v' \,{}_{\omega}\!\overset{*}{\longleftarrow} w_1 \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} w'$ according to $v' \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \circ \,{}_{\omega}\!\overset{*}{\longleftarrow} w'$. [In case of $n = 0$:] This is possible due to confluence of $\longrightarrow_{\omega}$. [Otherwise we have $n \dot{-} 1 \prec n$ and due to the assumed ω-shallow confluence up to $n \dot{-} 1$ this is possible again.]



Q.e.d. (Claim 1)

Claim 2: If the above property holds for a fixed $n \prec \omega$, and
$\forall k \prec n.\ (\mathrm{R,X}$ is ω-shallow confluent up to $k)$, then $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega}$ are commuting.

Proof of Claim 2: $\longrightarrow_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]}$ and $\overset{*}{\longrightarrow}_{\omega}$ are commuting by Lemma 3.3 and Claim 1. Since by Lemma 2.12 we have $\longrightarrow_{\omega+n} \subseteq \longrightarrow_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega[+(n\dot{-}1)]} \subseteq \overset{*}{\longrightarrow}_{\omega+n}$, now $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega}$ are commuting, too. Q.e.d. (Claim 2)

Claim 3: If the above property holds for all $n \preceq m$ for some $m \prec \omega$, then R, X is ω-shallow confluent up to m.

Proof of Claim 3: By induction on m in $\prec$. Assume $i +_{\omega} n \preceq m$ and $w_0 \,{}_{\omega+i}\!\overset{*}{\longleftarrow} u \overset{*}{\longrightarrow}_{\omega+n} w_1$. By definition of '$+_{\omega}$' and $i +_{\omega} n \prec \omega$ w.l.o.g. we have $i = 0$ and $n \preceq m$. By Claim 2 and our induction hypothesis we finally get $w_0 \overset{*}{\longrightarrow}_{\omega+n} \circ \,{}_{\omega}\!\overset{*}{\longleftarrow} w_1$ as desired. Q.e.d. (Claim 3)

Note that our property for is trivial for $n = 0$ since $\longrightarrow_{\omega}$ is confluent.

The benefit of claims 1 and 3 is twofold: First, they say that our lemma is valid if the above property holds for all $n \prec \omega$. Second, they strengthen the property when used as induction hypothesis. Thus (writing $n+1$ instead of n since we may assume $0 \prec n$) it now suffices to show for $n \prec \omega$ that

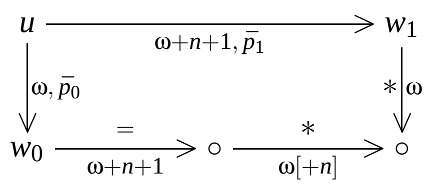
$$w_0 \longleftarrow_{\omega, \bar{p}_0} u \longrightarrow_{\omega+n+1, \bar{p}_1} w_1$$

together with our induction hypothesis that

$$\mathrm{R}, \mathrm{X} \text{ is } \omega\text{-shallow confluent up to } n$$

implies

$$w_0 \overset{=}{\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega[+n]} \circ \overset{*}{\longleftarrow}_{\omega} w_1.$$



Now for each $i \prec 2$ there are $((l_i, r_i), C_i) \in \mathrm{R}$ and $\mu_i \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/\bar{p}_i = l_i\mu_i$, $w_i = u[\bar{p}_i \leftarrow r_i\mu_i]$, $l_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, $C_0\mu_0$ fulfilled w.r.t. $\longrightarrow_{\omega}$, $C_1\mu_1$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$.

<u>Claim 5:</u> We may assume $l_1 \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$.
<u>Proof of Claim 5:</u> In case of $l_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ by Lemma 13.2 (matching both its μ and ν to our μ_1) we get $l_1\mu_1 \longrightarrow_{\omega} r_1\mu_1$. Then the proof is finished by confluence of $\longrightarrow_{\omega}$. <u>Q.e.d. (Claim 5)</u>

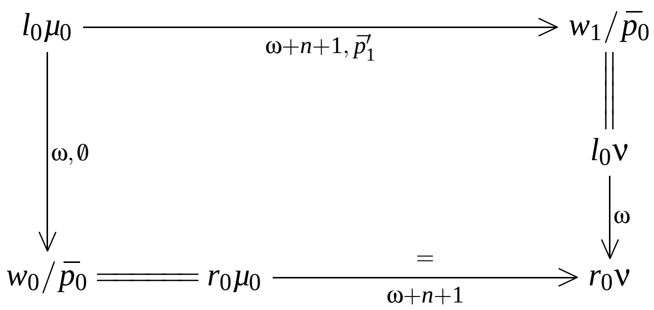
In case of $\bar{p}_0 \parallel \bar{p}_1$ we have $w_i/\bar{p}_{1-i} = u[\bar{p}_i \leftarrow r_i\mu_i]/\bar{p}_{1-i} = u/\bar{p}_{1-i} = l_{1-i}\mu_{1-i}$ and therefore $w_0 \longrightarrow_{\omega+n+1} u[\bar{p}_k \leftarrow r_k\mu_k \mid k \prec 2] \longleftarrow_{\omega} w_1$, i.e. our proof is finished. Thus, according to whether $\bar{p}_0$ is a prefix of $\bar{p}_1$ or vice versa, we have the following two cases left:

There is some $\overline{p}'_1$ with $\overline{p}_0\overline{p}'_1 = \overline{p}_1$ and $\overline{p}'_1 \neq \emptyset$:

We have two cases:

"The variable overlap case":

There are $x \in \mathrm{V}$ and p', p'' such that $l_0/p' = x \wedge p'p'' = \overline{p}'_1$:



Claim 6: We have $x\mu_0/p'' = l_1\mu_1$ and may assume $x \in \mathrm{V}_{\mathrm{SIG}}$.
Proof of Claim 6: We have $x\mu_0/p'' = l_0\mu_0/p'p'' = u/\overline{p}_0p'p'' = u/\overline{p}_0\overline{p}'_1 = u/\overline{p}_1 = l_1\mu_1$.
If $x \in \mathrm{V}_{\mathcal{C}}$, then $x\mu_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $x\mu_0/p'' \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $l_1\mu_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, and then $l_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which we may assume not to be the case by Claim 5. Q.e.d. (Claim 6)

Claim 7: We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_0[p'' \leftarrow r_1\mu_1]$ and $\forall y \in \mathrm{V}\setminus\{x\}.\, y\nu = y\mu_0$.
Then we have $x\mu_0 \longrightarrow_{\omega+n+1} x\nu$.
Proof of Claim 7: This follows directly from Claim 6. Q.e.d. (Claim 7)

Claim 8: $l_0\nu = w_1/\overline{p}_0$.
Proof of Claim 8: By the left-linearity assumption of our lemma and Claim 6 we may assume $\{\, p''' \mid l_0/p''' = x \,\} = \{p'\}$. Thus, by Claim 7 we get $w_1/\overline{p}_0 = u/\overline{p}_0[\overline{p}'_1 \leftarrow r_1\mu_1] =$
$l_0[p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V}][\overline{p}'_1 \leftarrow r_1\mu_1] =$
$l_0[p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_0][p'p'' \leftarrow r_1\mu_1] =$
$l_0[p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_0[p'' \leftarrow r_1\mu_1]] =$
$l_0[p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V}] = l_0\nu$. Q.e.d. (Claim 8)

Claim 9: $w_0/\overline{p}_0 \overset{=}{\longrightarrow}_{\omega+n+1} r_0\nu$.
Proof of Claim 9: By the right-linearity assumption of our lemma and Claim 6 we may assume $|\{\, p''' \mid r_0/p''' = x \,\}| \preceq 1$. Thus by Claim 7 we get: $w_0/\overline{p}_0 = r_0\mu_0 =$
$r_0[p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}][p''' \leftarrow x\mu_0 \mid r_0/p''' = x] \overset{=}{\longrightarrow}_{\omega+n+1}$
$r_0[p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}][p''' \leftarrow x\nu \mid r_0/p''' = x] =$
$r_0[p''' \leftarrow y\nu \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}][p''' \leftarrow x\nu \mid r_0/p''' = x] = r_0\nu$. Q.e.d. (Claim 9)

By claims 8 and 9 it now suffices to show $r_{0,q}\nu \longleftarrow_{\omega} l_{0,q}\nu$, which again follows from Lemma 13.2 since $\forall y \in \mathrm{V}.\, y\mu_{0,q} \overset{*}{\longrightarrow}_{\omega+n+1} y\nu$ by Claim 7. Q.e.d. ("The variable overlap case")

<u>“The critical peak case”: $\overline{p}'_1 \in \mathcal{POS}(l_0) \wedge l_0/\overline{p}'_1 \notin \mathrm{V}$:</u>

$$\begin{array}{ccccc}
l_0\mu_0 & & \xrightarrow[\omega+n+1,\,\overline{p}'_1]{} & & w_1/\overline{p}_0 \\
\downarrow{\scriptstyle \omega,\emptyset} & & & & {\scriptstyle *}\downarrow{\scriptstyle \omega} \\
w_0/\overline{p}_0 & \xrightarrow[\omega+n+1]{=} & \circ & \xrightarrow[\omega[+n]]{*} & \circ
\end{array}$$

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_1,r_1),C_1))] \cap \mathcal{V}(((l_0,r_0),C_0)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_1,r_1),C_1))] \cup \mathcal{V}(((l_0,r_0),C_0))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{matrix} x\mu_0 & \text{if } x \in \mathcal{V}(((l_0,r_0),C_0)) \\ x\xi^{-1}\mu_1 & \text{else} \end{matrix}\right\}$ $(x \in \mathrm{V})$.
By $l_1\xi\rho = l_1\xi\xi^{-1}\mu_1 = u/\overline{p}_1 = u/\overline{p}_0\overline{p}'_1 = l_0\mu_0/\overline{p}'_1 = l_0\rho/\overline{p}'_1 = (l_0/\overline{p}'_1)\rho$
let $\sigma := \mathrm{mgu}(\{(l_1\xi, l_0/\overline{p}'_1)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
If $l_0[\overline{p}'_1 \leftarrow r_1\xi]\sigma = r_0\sigma$, then the proof is finished due to

$$w_0/\overline{p}_0 = r_0\mu_0 = r_0\sigma\varphi = l_0[\overline{p}'_1 \leftarrow r_1\xi]\sigma\varphi = l_0\mu_0[\overline{p}'_1 \leftarrow r_1\mu_1] = w_1/\overline{p}_0.$$

Otherwise we have $((l_0[\overline{p}'_1 \leftarrow r_1\xi], C_1\xi, 1), (r_0, C_0, 0), l_0, \sigma, \overline{p}'_1) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $\overline{p}'_1 \neq \emptyset$ (due the global case assumption); $C_1\xi\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_0\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\omega}$. Since R,X is ω-shallow confluent up to n (by our induction hypothesis), due to our assumed ω-shallow [noisy] anti-closedness up to ω (matching the definition’s n_0 to our $n+1$ and its n_1 to 0) we have $w_1/q = l_0\mu_0[\overline{p}'_1 \leftarrow r_1\mu_1] = l_0[\overline{p}'_1 \leftarrow r_1\xi]\sigma\varphi \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega[+n]} \circ \overset{=}{\longleftarrow}_{\omega+n+1} r_0\sigma\varphi = r_0\mu_0 = w_0/q$.

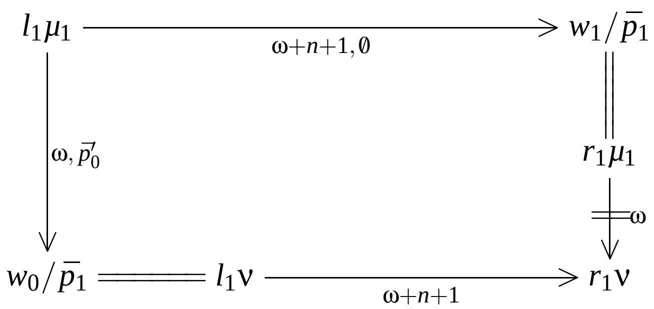
<u>Q.e.d. (“The critical peak case”)</u> <u>Q.e.d. (“There is some $\overline{p}'_1$ with $\overline{p}_0\overline{p}'_1 = \overline{p}_1$ and $\overline{p}'_1 \neq \emptyset$”)</u>

There is some $\overline{p_0'}$ with $\overline{p_1}\,\overline{p_0'} = \overline{p_0}$:

We have two cases:

"The second variable overlap case":

There are $x \in V$ and p', p'' such that $l_1/p' = x \wedge p'p'' = \overline{p_0'}$:



We have $x\mu_1/p'' = l_1\mu_1/p'p'' = u/\overline{p_1}p'p'' = u/\overline{p_1}\,\overline{p_0'} = u/\overline{p_0} = l_0\mu_0$.

Claim 11: We can define $\nu \in \mathcal{SUB}(V, \mathcal{T}(X))$ by $x\nu = x\mu_1[\,p'' \leftarrow r_0\mu_0\,]$ and $\forall y \in V\setminus\{x\}.\ y\nu = y\mu_1$. Then we have $x\mu_1 \longrightarrow_{\omega} x\nu$.
Proof of Claim 11: This follows directly from the above equality and Lemma 2.10.

Q.e.d. (Claim 11)

Claim 12: $w_0/\overline{p_1} = l_1\nu$.
Proof of Claim 12:
By the left-linearity assumption of our lemma and Claim 5 we may assume $\{\,p''' \mid l_1/p''' = x\,\} = \{p'\}$. Thus, by Claim 11 we get $w_0/\overline{p_1} = u/\overline{p_1}[\,\overline{p_0'} \leftarrow r_0\mu_0\,] =$
$l_1[\,p''' \leftarrow y\mu_1 \mid l_1/p''' = y \in V\,][\,\overline{p_0'} \leftarrow r_0\mu_0\,] =$
$l_1[\,p''' \leftarrow y\mu_1 \mid l_1/p''' = y \in V \wedge y \neq x\,][\,p' \leftarrow x\mu_1\,][\,p'p'' \leftarrow r_0\mu_0\,] =$
$l_1[\,p''' \leftarrow y\nu \mid l_1/p''' = y \in V \wedge y \neq x\,][\,p' \leftarrow x\mu_1[\,p'' \leftarrow r_0\mu_0\,]\,] =$
$l_1[\,p''' \leftarrow y\nu \mid l_1/p''' = y \in V\,] = l_1\nu$.

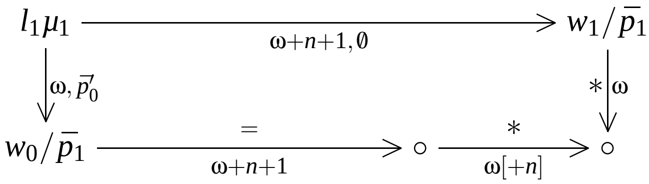
Q.e.d. (Claim 12)

Claim 13: $r_1\nu \mathrel{\leftarrow\!\!\!\!+\!\!\!\!+\!\!-}_{\omega} w_1/\overline{p_1}$.
Proof of Claim 13: Since $r_1\mu_1 = w_1/\overline{p_1}$, this follows directly from Claim 11. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{\omega+n+1} r_{1,q}\nu$, which again follows from Claim 11, Lemma 13.8 (matching its n_0 to 0 and its n_1 to our n) and our induction hypothesis that R, X is ω-shallow confluent up to n.

Q.e.d. ("The second variable overlap (if any) case")

"The second critical peak case": $\overline{p'_0} \in \mathcal{POS}(l_1) \wedge l_1/\overline{p'_0} \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_0,r_0),C_0))] \cap \mathcal{V}(((l_1,r_1),C_1)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_0,r_0),C_0))] \cup \mathcal{V}(((l_1,r_1),C_1))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{matrix} x\mu_1 & \text{if } x \in \mathcal{V}(((l_1,r_1),C_1)) \\ x\xi^{-1}\mu_0 & \text{else} \end{matrix}\right\}$ $(x \in \mathrm{V})$.
By $l_0\xi\rho = l_0\xi\xi^{-1}\mu_0 = u/\overline{p_0} = u/\overline{p_1}\overline{p'_0} = l_1\mu_1/\overline{p'_0} = l_1\rho/\overline{p'_0} = (l_1/\overline{p'_0})\rho$
let $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/\overline{p'_0})\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
If $l_1[\overline{p'_0} \leftarrow r_0\xi]\sigma = r_1\sigma$, then the proof is finished due to

$$w_0/\overline{p_1} = l_1\mu_1[\overline{p'_0} \leftarrow r_0\mu_0] = l_1[\overline{p'_0} \leftarrow r_0\xi]\sigma\varphi = r_1\sigma\varphi = r_1\mu_1 = w_1/\overline{p_1}.$$

Otherwise we have $((l_1[\overline{p'_0} \leftarrow r_0\xi], C_0\xi, 0), (r_1, C_1, 1), l_1, \sigma, \overline{p'_0}) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_0\xi\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\omega}$; $C_1\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R,X ω-shallow confluent up to n (by our induction hypothesis), due to our assumed ω-shallow [noisy] strong joinability up to ω (matching the definition's n_0 to 0 and its n_1 to our $n{+}1$) we have $w_0/\overline{p_1} = l_1\mu_1[\overline{p'_0} \leftarrow r_0\mu_0] = l_1[\overline{p'_0} \leftarrow r_0\xi]\sigma\varphi \overset{=}{\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega[+n]} \circ \overset{*}{\longleftarrow}_{\omega} r_1\sigma\varphi = r_1\mu_1 = w_1/\overline{p_1}$.
Q.e.d. ("The second critical peak case") **Q.e.d. (Lemma A.4)**

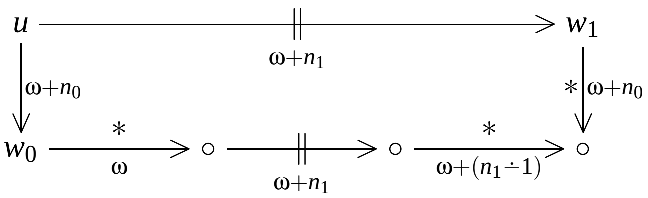
Proof of Lemma A.5

Claim 0: R,X is ω-shallow confluent up to ω.
Proof of Claim 0: Directly by the assumed strong commutation of $\longrightarrow_{\mathrm{R},\mathrm{X},\omega+n} \circ \overset{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+(n\dot{-}1)}$ over $\overset{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega}$, cf. the proofs of the claims 2 and 3 of the proof of Lemma A.4. Q.e.d. (Claim 0)

Claim 1: If $\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n_0}$, then $\longrightarrow_{\omega+n_1}$ and $\longrightarrow_{\omega+n_0}$ are commuting.
Proof of Claim 1: $\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)}$ and $\overset{*}{\longrightarrow}_{\omega+n_0}$ are commuting by Lemma 3.3. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{\omega+n_1} \subseteq \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \subseteq \overset{*}{\longrightarrow}_{\omega+n_1}$, now $\longrightarrow_{\omega+n_1}$ and $\longrightarrow_{\omega+n_0}$ are commuting, too. Q.e.d. (Claim 1)

For $n_0 \preceq n_1 \prec \omega$ we are going to show by induction on $n_0 +_{\omega} n_1$ the following property:

$$w_0 \longleftarrow_{\omega+n_0} u \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n_1} w_1 \quad \Rightarrow \quad w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \circ \overset{*}{\longleftarrow}_{\omega+n_0} w_1$$



<u>Claim 2:</u> Let $\delta \prec \omega{+}\omega$. If

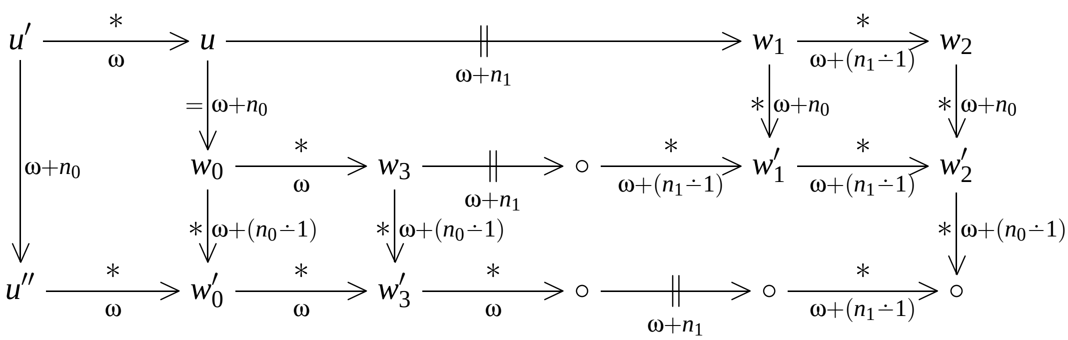
$$\forall n_0, n_1 \prec \omega.\ \left(\begin{array}{l} \left(\begin{array}{ll} & n_0 \preceq n_1 \\ \wedge & n_0 +_\omega n_1 \preceq \delta \end{array} \right) \\ \Rightarrow\ \forall w_0, w_1, u.\ \left(\begin{array}{ll} & w_0 \longleftarrow_{\omega+n_0} u \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} w_1 \\ \Rightarrow & w_0 \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \circ \stackrel{*}{\longleftarrow}_{\omega+n_0} w_1 \end{array} \right) \end{array} \right),$$

then

$$\forall n_0, n_1 \prec \omega.\ \left(\begin{array}{l} \left(\begin{array}{ll} & n_0 \preceq n_1 \\ \wedge & n_0 +_\omega n_1 \preceq \delta \end{array} \right) \\ \Rightarrow\ \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \text{ strongly commutes over } \stackrel{*}{\longrightarrow}_{\omega+n_0} \end{array} \right),$$

and R, X is ω-shallow confluent up to δ.

<u>Proof of Claim 2:</u> By induction on δ in $\prec$. First we show the strong commutation. Assume $n_0 \preceq n_1 \prec \omega$ with $n_0 +_\omega n_1 \preceq \delta$. By Lemma 3.3 it suffices to show that $\stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)}$ strongly commutes over $\longrightarrow_{\omega+n_0}$. Assume $u'' \longleftarrow_{\omega+n_0} u' \stackrel{*}{\longrightarrow}_{\omega} u \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} w_1 \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} w_2$ (cf. diagram below). By the strong commutation assumption of our lemma there are w_0 and w_0' with $u'' \stackrel{*}{\longrightarrow}_{\omega} w_0' \stackrel{*}{\longleftarrow}_{\omega+(n_0 \dot{-} 1)} w_0 \stackrel{=}{\longleftarrow}_{\omega+n_0} u$. By the above property there are some w_3, w_1' with $w_0 \stackrel{*}{\longrightarrow}_{\omega} w_3 \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} w_1' \stackrel{*}{\longleftarrow}_{\omega+n_0} w_1$. Next we show that we can close the peak $w_1' \stackrel{*}{\longleftarrow}_{\omega+n_0} w_1 \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} w_2$ according to $w_1' \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} w_2' \stackrel{*}{\longleftarrow}_{\omega+n_0} w_2$ for some w_2'. In case of $n_1 = 0$ this is possible due to the ω-shallow confluence up to ω given by Claim 0. Otherwise we have $n_0 +_\omega (n_1 \dot{-} 1) \prec n_0 +_\omega n_1 \preceq \delta$ and due to our induction hypothesis (saying that R, X is ω-shallow confluent up to all $\delta' \prec \delta$) this is possible again. By Claim 0 again, we can close the peak $w_0' \stackrel{*}{\longleftarrow}_{\omega+(n_0 \dot{-} 1)} w_0 \stackrel{*}{\longrightarrow}_{\omega} w_3$ according to $w_0' \stackrel{*}{\longrightarrow}_{\omega} w_3' \stackrel{*}{\longleftarrow}_{\omega+(n_0 \dot{-} 1)} w_3$ for some w_3'. To close the whole diagram, we only have to show that we can close the peak $w_3' \stackrel{*}{\longleftarrow}_{\omega+(n_0 \dot{-} 1)} w_3 \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} w_2'$ according to $w_3' \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \circ \stackrel{*}{\longleftarrow}_{\omega+(n_0 \dot{-} 1)} w_2'$. In case of $n_0 = 0$ this is possible since it is assumed for our lemma (below the strong commutation assumption). Otherwise we have $n_0 \dot{-} 1 \prec n_0 \preceq n_1$ and $(n_0 \dot{-} 1) +_\omega n_1 \prec n_0 +_\omega n_1 \preceq \delta$, and then due to our induction hypothesis this is possible again.



Finally we show ω-shallow confluence up to δ. Assume $n_0 +_\omega n_1 \preceq \delta$ and $w_0 \stackrel{*}{\longleftarrow}_{\omega+n_0} u \stackrel{*}{\longrightarrow}_{\omega+n_1} w_1$. Due to symmetry in n_0 and n_1 we may assume $n_0 \preceq n_1$. Above we have shown that $\stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\!+\!\!\!\!+\!\!\!\rightarrow}_{\omega+n_1} \circ \stackrel{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{\omega+n_0}$. By Claim 1 we finally get $w_0 \stackrel{*}{\longrightarrow}_{\omega+n_1} \circ \stackrel{*}{\longleftarrow}_{\omega+n_0} w_1$ as desired. <u>Q.e.d. (Claim 2)</u>

Note that for $n_0 = 0$ our property follows from the assumption of our lemma (below the strong commutation assumption).

The benefit of Claim 2 is twofold: First, it says that our lemma is valid if the above property holds for all $n_0 \preceq n_1 \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n_i{+}1$ instead of n_i since we may assume $0 \prec n_0 \preceq n_1$) it now suffices to show for $n_0 \preceq n_1 \prec \omega$ that

$$w_0 \longleftarrow_{\omega+n_0+1,\bar{p}_0} u \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n_1+1,\Pi_1} w_1$$

together with our induction hypotheses that

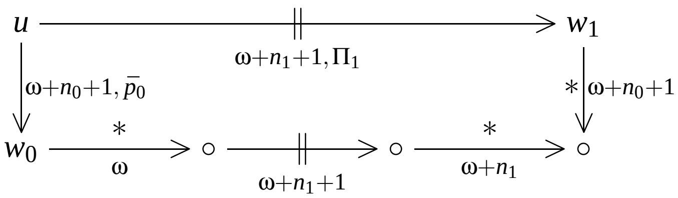
$$\forall \delta \prec (n_0{+}1) +_{\omega} (n_1{+}1).\ \mathrm{R},\mathrm{X} \text{ is } \omega\text{-shallow confluent up to } \delta$$

and (due to $n_0 \preceq n_1{+}1$ and $n_0 +_{\omega} (n_1{+}1) \prec (n_0{+}1) +_{\omega} (n_1{+}1)$)

$$\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \text{ strongly commutes over } \overset{*}{\longrightarrow}_{\omega+n_0}$$

implies

$$w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1.$$



Note that for the availability of our second induction hypothesis it is important that we have imposed the restriction "$n_0 \preceq n_1$" in opposition to the restriction "$n_0 \succeq n_1$". In the latter case the availability of our second induction hypothesis would require $n_0{+}1 \succeq n_1{+}1 \Rightarrow n_0 \succeq n_1{+}1$ which is not true for $n_0 = n_1$. The additional hypothesis

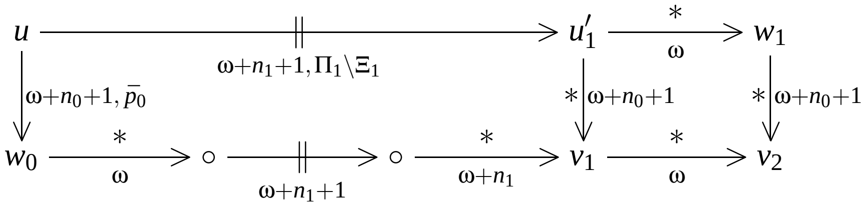
$$\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \text{ strongly commutes over } \overset{*}{\longrightarrow}_{\omega+n_0+1}$$

of the latter restriction is useless for our proof.

There are $((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}) \in \mathrm{R}$ and $\mu_{0,\bar{p}_0} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$, $C_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$, and $w_0 = u[\,p \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}\,]$.

W.l.o.g. let the positions of Π_1 be maximal in the sense that for any $p \in \Pi_1$ and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $u \mathrel{+\!\!\!\!\longrightarrow}_{\omega+n_1+1,(\Pi_1 \setminus \{p\}) \cup \Xi} w_1$ anymore. Then for each $p \in \Pi_1$ there are $((l_{1,p}, r_{1,p}), C_{1,p}) \in \mathrm{R}$ and $\mu_{1,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{1,p}\mu_{1,p}$, $r_{1,p}\mu_{1,p} = w_1/p$, $C_{1,p}\mu_{1,p}$ fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$. Finally, $w_1 = u[\,p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1\,]$.

<u>Claim 5:</u>
We may assume $l_{0,\bar{p}_0} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_C)$ and $\forall p \in \Pi_1.\ l_{1,p} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_C)$.
<u>Proof of Claim 5:</u> In case of $l_{0,\bar{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_C)$ we get $w_0 \longleftarrow_{\omega} u$ by Lemma 13.2 (matching both its μ and ν to our $\mu_{0,\bar{p}_0}$) and then our property follows from the assumption of our lemma (below the strong commutation assumption). For the second restriction define $\Xi_1 := \{\ p \in \Pi_1 \mid l_{1,p} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_C)\ \}$ and $u'_1 := u[\,p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1 \setminus \Xi_1\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{\|\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} v_1 \overset{*}{\longleftarrow}_{\omega+n_0+1} u'_1$ for some v_1 (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{1,p}$) we get $\forall p \in \Xi_1.\ l_{1,p}\mu_{1,p} \longrightarrow_{\omega} r_{1,p}\mu_{1,p}$ and therefore $u'_1 \overset{*}{\longrightarrow}_{\omega} w_1$. Thus from $v_1 \overset{*}{\longleftarrow}_{\omega+n_0+1} u'_1 \overset{*}{\longrightarrow}_{\omega} w_1$ we get $v_1 \overset{*}{\longrightarrow}_{\omega} v_2 \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1$ for some v_2 by ω-shallow confluence up to ω (cf. Claim 0).



<u>Q.e.d. (Claim 5)</u>

Now we start a second level of induction on $|\Pi_1|$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\ p \in \{\bar{p}_0\} \cup \Pi_1 \mid \neg\exists q \in \{\bar{p}_0\} \cup \Pi_1.\ \exists q' \in \mathbf{N}^+.\ p = qq'\ \}.$$

Since the prefix ordering is wellfounded we have $\forall p \in \{\bar{p}_0\} \cup \Pi_1.\ \exists q \in \Theta.\ \exists q' \in \mathbf{N}^*.\ p = qq'$. It now suffices to show for all $q \in \Theta$

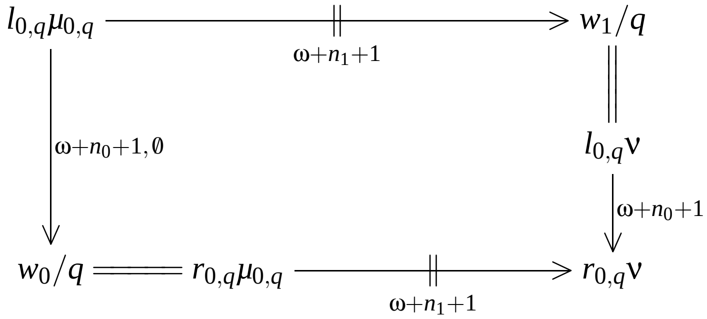
$$w_0/q \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{\|\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1/q$$

because then we have $w_0 = w_0[\,q \leftarrow w_0/q \mid q \in \Theta\,] = u[\,\bar{p}_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}\,][\,q \leftarrow w_0/q \mid q \in \Theta\,] =$
$u[\,q \leftarrow w_0/q \mid q \in \Theta\,] \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{\|\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} u[\,q \leftarrow w_1/q \mid q \in \Theta\,] =$
$u[\,p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1\,][\,q \leftarrow w_1/q \mid q \in \Theta\,] = w_1[\,q \leftarrow w_1/q \mid q \in \Theta\,] = w_1$.

Therefore we are left with the following two cases for $q \in \Theta$:

$q \notin \Pi_1$: Then $q = \bar{p}_0$. Define $\Pi'_1 := \{\ p \mid qp \in \Pi_1\ \}$. We have two cases:

“The variable overlap (if any) case”: $\forall p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}$:



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\ (p', p'') \mid l_{0,q}/p' = x\ \wedge\ p'p'' \in \Pi'_1\ \}$.

Claim 7: There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}.\ \left(\begin{array}{ll} & x\mu_{0,q} \mathrel{\|\!\!\!\longrightarrow}_{\omega+n_1+1} x\nu \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu = x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \end{array}\right).$$

Proof of Claim 7:
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$\begin{array}{rl} x\mu_{0,q} = x\mu_{0,q}[\,p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] & \mathrel{\|\!\!\!\longrightarrow}_{\omega+n_1+1} \\ x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] & = x\nu. \end{array}$$

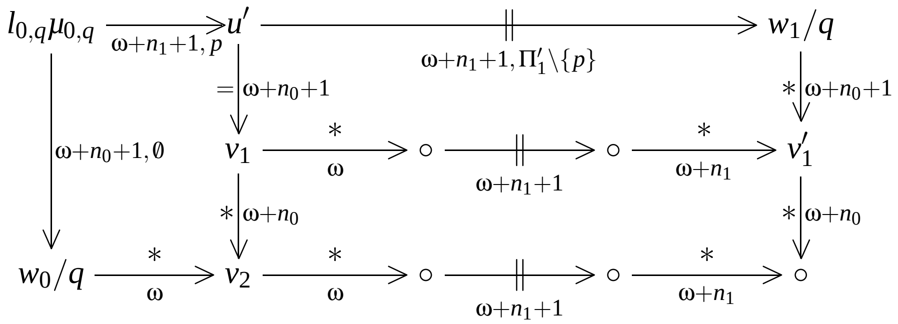
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{0,q}/p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ this implies $l_{1,qp'p''}\mu_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. Q.e.d. (Claim 7)

Claim 8: $l_{0,q}\nu = w_1/q$.
Proof of Claim 8:
By Claim 7 we get $w_1/q = u/q[\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{0,q}/p' = x \in \mathrm{V}\,] =$
$l_{0,q}[\,p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V}\,] = l_{0,q}\nu$. Q.e.d. (Claim 8)

Claim 9: $w_0/q \mathrel{\|\!\!\!\longrightarrow}_{\omega+n_1+1} r_{0,q}\nu$.
Proof of Claim 9: Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. Q.e.d. (Claim 9)

By claims 8 and 9 it now suffices to show $l_{0,q}\nu \longrightarrow_{\omega+n_0+1} r_{0,q}\nu$, which again follows from Lemma 13.8 since R, X is ω-shallow confluent up to $(n_1+1) +_\omega n_0$ by our induction hypothesis and since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega+n_1+1} x\nu$ by Claim 7 and Corollary 2.14.
Q.e.d. (“The variable overlap (if any) case”)

“The critical peak case”: There is some $p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:



Claim 10: $p \neq \emptyset$.

Proof of Claim 10: If $p = \emptyset$, then $\emptyset \in \Pi'_1$, then $q \in \Pi_1$, which contradicts our global case assumption. Q.e.d. (Claim 10)

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cap \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cup \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{cases} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{cases}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1\setminus\{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \mathrel{+\!\!\!\longrightarrow}_{\omega+n_1+1,\Pi'_1\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1] = w_1/q.$$

If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \mathrel{+\!\!\!\longrightarrow}_{\omega+n_1+1,\Pi'_1\setminus\{p\}} w_1/q.$$

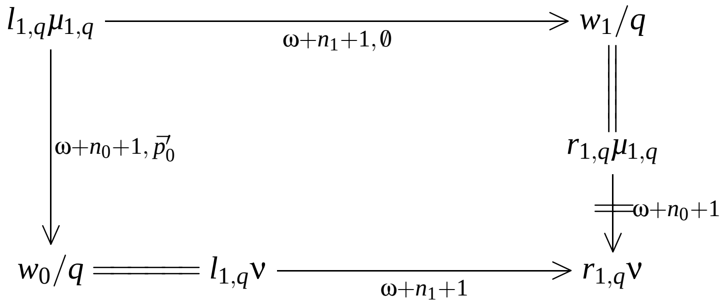
Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 1), l_{0,q}\sigma, p) \in \mathrm{CP(R)}$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$. Since $\forall\delta \prec (n_1{+}1) +_\omega (n_0{+}1)$. R, X is ω-shallow confluent up to δ (by our induction hypothesis) due to our assumed ω-shallow closedness (matching the definition's n_0 to our $n_1{+}1$ and its n_1 to our $n_0{+}1$) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \overset{=}{\longrightarrow}_{\omega+n_0+1} v_1 \overset{*}{\longrightarrow}_{\omega+n_0} v_2 \overset{*}{\longleftarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v_1, v_2. We then have $v_1 \overset{=}{\longleftarrow}_{\omega+n_0+1} u' \mathrel{+\!\!\!\longrightarrow}_{\omega+n_1+1,\Pi'_1\setminus\{p\}} w_1/q$. By $|\Pi'_1\setminus\{p\}| \prec |\Pi'_1| \preceq |\Pi_1|$, due to our second induction level we get some v'_1 with $v_1 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} v'_1 \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1/q$. Finally by our induction hypothesis that $\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n_0}$ the peak at v_1 can be closed according to $v_2 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{+\!\!\!\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0} v'_1$.

Q.e.d. (“The critical peak case”) Q.e.d. (“$q \notin \Pi_1$”)

$\underline{\underline{q \in \Pi_1}}$: If there is no $\bar{p}'_0$ with $q\bar{p}'_0 = \bar{p}_0$, then the proof is finished due to $w_0/q = u/q = l_{1,q}\mu_{1,q} \longrightarrow_{\omega+n_1+1} r_{1,q}\mu_{1,q} = w_1/q$. Otherwise, we can define $\bar{p}'_0$ by $q\bar{p}'_0 = \bar{p}_0$. We have two cases:

“The second variable overlap case”:

There are $x \in \mathrm{V}$ and p', p'' such that $l_{1,q}/p' = x \wedge p'p'' = \bar{p}'_0$:



Claim 11a: We have $x\mu_{1,q}/p'' = l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ and may assume $x \in \mathrm{V_{SIG}}$.
Proof of Claim 11a: We have $x\mu_{1,q}/p'' = l_{1,q}\mu_{1,q}/p'p'' = u/qp'p'' = u/q\bar{p}'_0 = u/\bar{p}_0 = l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$. If $x \in \mathrm{V}_\mathcal{C}$, then $x\mu_{1,q} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_\mathcal{C})$, then $x\mu_{1,q}/p'' \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_\mathcal{C})$, then $l_{0,\bar{p}_0}\mu_{0,\bar{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_\mathcal{C})$, and then $l_{0,\bar{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ which we may assume not to be the case by Claim 5.

Q.e.d. (Claim 11a)

Claim 11b: We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_{1,q}[p'' \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}]$ and $\forall y \in \mathrm{V}\setminus\{x\}.\ y\nu = y\mu_{1,q}$. Then we have $x\mu_{1,q} \longrightarrow_{\omega+n_0+1} x\nu$.
Proof of Claim 11b: This follows directly from Claim 11a. Q.e.d. (Claim 11b)

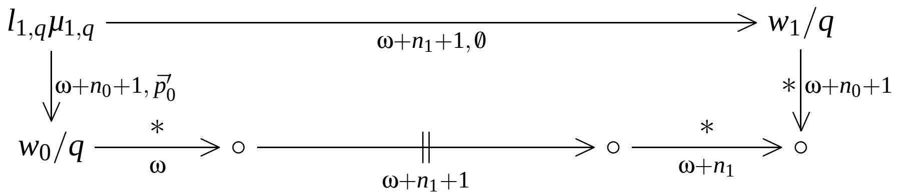
Claim 12: $w_0/q = l_{1,q}\nu$.
Proof of Claim 12: By the left-linearity assumption of our lemma, Claim 5, and Claim 11a we may assume $\{\, p''' \mid l_{1,q}/p''' = x \,\} = \{p'\}$. Thus, by Claim 11b we get $w_0/q =$
$u/q[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] =$
$l_{1,q}[p''' \leftarrow y\mu_{1,q} \mid l_{1,q}/p''' = y \in \mathrm{V}][\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] =$
$l_{1,q}[p''' \leftarrow y\mu_{1,q} \mid l_{1,q}/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_{1,q}][p'p'' \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] =$
$l_{1,q}[p''' \leftarrow y\nu \mid l_{1,q}/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_{1,q}[p'' \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}]] =$
$l_{1,q}[p''' \leftarrow y\nu \mid l_{1,q}/p''' = y \in \mathrm{V}] = l_{1,q}\nu$. Q.e.d. (Claim 12)

Claim 13: $r_{1,q}\nu \leftarrow\!\!\!+\!\!\!\!\!\!+\!\!\!-_{\omega+n_0+1} w_1/q$.
Proof of Claim 13: Since $r_{1,q}\mu_{1,q} = w_1/q$, this follows directly from Claim 11b.

Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{\omega+n_1+1} r_{1,q}\nu$, which again follows from Claim 11b, Lemma 13.8 (matching its n_0 to our n_0+1 and its n_1 to our n_1), and our induction hypothesis that R, X is ω-shallow confluent up to $(n_0+1) +_\omega n_1$.
Q.e.d. (“The second variable overlap case”)

"The second critical peak case": $\bar{p}'_0 \in \mathcal{POS}(l_{1,q}) \wedge l_{1,q}/\bar{p}'_0 \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}))] \cap \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}))] \cup \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) \\ x\xi^{-1}\mu_{0,\bar{p}_0} & \text{else} \end{Bmatrix} \; (x \in \mathrm{V})$.
By $l_{0,\bar{p}_0}\xi\rho = l_{0,\bar{p}_0}\xi\xi^{-1}\mu_{0,\bar{p}_0} = u/\bar{p}_0 = u/q\bar{p}'_0 = l_{1,q}\mu_{1,q}/\bar{p}'_0 = l_{1,q}\rho/\bar{p}'_0 = (l_{1,q}/\bar{p}'_0)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,\bar{p}_0}\xi, l_{1,q}/\bar{p}'_0)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}{\upharpoonright}(\sigma\varphi) = {}_{\mathrm{Y}}{\upharpoonright}\rho$.
If $l_{1,q}[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q = l_{1,q}\mu_{1,q}[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] = l_{1,q}[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

Otherwise we have $((l_{1,q}[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma, C_{0,\bar{p}_0}\xi\sigma, 1), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, \bar{p}'_0) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_{0,\bar{p}_0}\xi\sigma\varphi = C_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$. Since $\forall\delta \prec (n_0{+}1) +_\omega (n_1{+}1)$. R, X is ω-shallow confluent up to δ (by our induction hypothesis) due to our assumed ω-shallow noisy weak parallel joinability (matching the definition's n_0 to our $n_0{+}1$ and its n_1 to our $n_1{+}1$) we have $w_0/q = l_{1,q}\mu_{1,q}[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] = l_{1,q}[\bar{p}'_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma\varphi \overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow\!\!\!\!\!\!\shortparallel\;\;_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$.
Q.e.d. ("The second critical peak case") **Q.e.d. (Lemma A.5)**

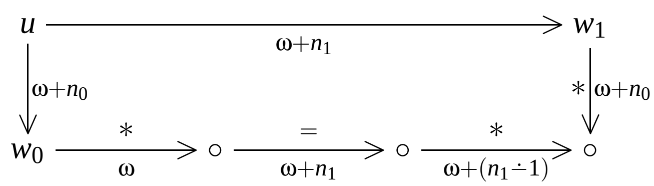
Proof of Lemma A.6

Claim 0: R, X is ω-shallow confluent up to ω.
Proof of Claim 0: Directly by the assumed strong commutation, cf. the proofs of the claims 2 and 3 of the proof of Lemma A.1. Q.e.d. (Claim 0)

Claim 1: If $\overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n_0}$, then $\longrightarrow_{\omega+n_1}$ and $\longrightarrow_{\omega+n_0}$ are commuting.
Proof of Claim 1: $\overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)}$ and $\overset{*}{\longrightarrow}_{\omega+n_0}$ are commuting by Lemma 3.3. Since by Lemma 2.12 we have $\longrightarrow_{\omega+n_1} \subseteq \overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \subseteq \overset{*}{\longrightarrow}_{\omega+n_1}$, now $\longrightarrow_{\omega+n_1}$ and $\longrightarrow_{\omega+n_0}$ are commuting, too. Q.e.d. (Claim 1)

For $n_0 \preceq n_1 \prec \omega$ we are going to show by induction on $n_0 +_\omega n_1$ the following property:

$$w_0 \longleftarrow_{\omega+n_0} u \longrightarrow_{\omega+n_1} w_1 \quad \Rightarrow \quad w_0 \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1 \dot{-} 1)} \circ \overset{*}{\longleftarrow}_{\omega+n_0} w_1.$$



<u>Claim 2:</u> Let $\delta \prec \omega{+}\omega$. If

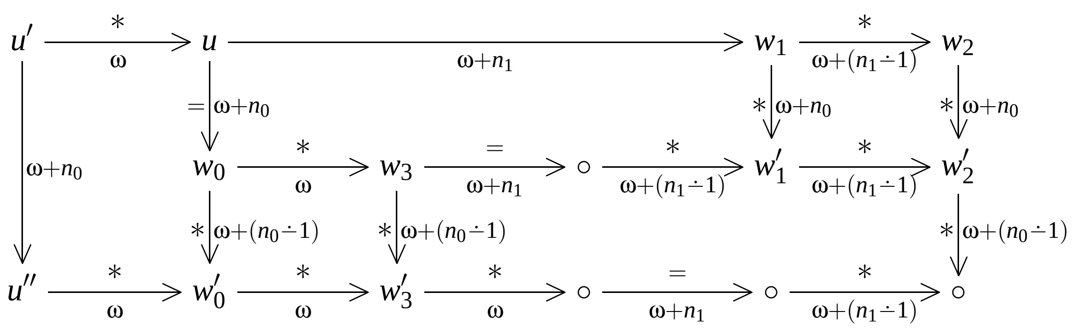
$$\forall n_0,n_1 \prec \omega.\ \left(\begin{array}{l} \left(\begin{array}{rl} & n_0 \preceq n_1 \\ \wedge & n_0 +_\omega n_1 \preceq \delta \end{array}\right) \\ \Rightarrow\ \forall w_0,w_1,u.\ \left(\begin{array}{rl} & w_0 \longleftarrow_{\omega+n_0} u \longrightarrow_{\omega+n_1} w_1 \\ \Rightarrow & w_0 \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \circ \overset{*}{\longleftarrow}_{\omega+n_0} w_1 \end{array}\right)\end{array}\right),$$

then

$$\forall n_0,n_1 \prec \omega.\ \left(\begin{array}{l} \left(\begin{array}{rl} & n_0 \preceq n_1 \\ \wedge & n_0 +_\omega n_1 \preceq \delta \end{array}\right) \\ \Rightarrow\ \overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \text{ strongly commutes over } \overset{*}{\longrightarrow}_{\omega+n_0} \end{array}\right),$$

and R,X is ω-shallow confluent up to δ.

<u>Proof of Claim 2:</u> By induction on δ in $\prec$. First we show the strong commutation. Assume $n_0 \preceq n_1 \prec \omega$ with $n_0 +_\omega n_1 \preceq \delta$. By Lemma 3.3 it suffices to show that $\overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)}$ strongly commutes over $\longrightarrow_{\omega+n_0}$. Assume $u'' \longleftarrow_{\omega+n_0} u' \overset{*}{\longrightarrow}_{\omega} u \longrightarrow_{\omega+n_1} w_1 \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_2$ (cf. diagram below). By the strong commutation assumed for our lemma, there are w_0 and w'_0 with $u'' \overset{*}{\longrightarrow}_{\omega} w'_0 \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_0 \overset{=}{\longleftarrow}_{\omega+n_0} u$. By the above property there are some w_3, w'_1 with $w_0 \overset{*}{\longrightarrow}_{\omega} w_3 \overset{=}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w'_1 \overset{*}{\longleftarrow}_{\omega+n_0} w_1$. Next we show that we can close the peak $w'_1 \overset{*}{\longleftarrow}_{\omega+n_0} w_1 \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w_2$ according to $w'_1 \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w'_2 \overset{*}{\longleftarrow}_{\omega+n_0} w_2$ for some w'_2. In case of $n_1 = 0$ this is possible due to the ω-shallow confluence up to ω given by Claim 0. Otherwise we have $n_0 +_\omega (n_1 \dot{-} 1) \prec n_0 +_\omega n_1 \preceq \delta$ and due to our induction hypothesis (saying that R,X is ω-shallow confluent up to all $\delta' \prec \delta$) this is possible again. By Claim 0 again, we can close the peak $w'_0 \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_0 \overset{*}{\longrightarrow}_{\omega} w_3$ according to $w'_0 \overset{*}{\longrightarrow}_{\omega} w'_3 \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_3$ for some w'_3. To close the whole diagram, we only have to show that we can close the peak $w'_3 \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w_3 \overset{=}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} w'_2$ according to $w'_3 \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)} \circ \overset{*}{\longleftarrow}_{\omega+(n_0\dot{-}1)} w'_2$. In case of $n_0 = 0$ this is possible due to the strong commutation assumed for our lemma. Otherwise we have $n_0 \dot{-} 1 \prec n_0 \preceq n_1$ and $(n_0 \dot{-} 1) +_\omega n_1 \prec n_0 +_\omega n_1 \preceq \delta$, and then due to our induction hypothesis this is possible again.



Finally we show ω-shallow confluence up to δ. Assume $n_0 +_\omega n_1 \preceq \delta$ and $w_0 \overset{*}{\longleftarrow}_{\omega+n_0} u \overset{*}{\longrightarrow}_{\omega+n_1} w_1$. Due to symmetry in n_0 and n_1 we may assume $n_0 \preceq n_1$. Above we have shown that $\overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+n_1} \circ \overset{*}{\longrightarrow}_{\omega+(n_1\dot{-}1)}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n_0}$. By Claim 1 we finally get $w_0 \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0} w_1$ as desired. <u>Q.e.d. (Claim 2)</u>

Note that for $n_0 = 0$ our property follows from the strong commutation assumption of our lemma.

The benefit of Claim 2 is twofold: First, it says that our lemma is valid if the above property holds for all $n_0 \preceq n_1 \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n_i{+}1$ instead of n_i since we may assume $0 \prec n_0 \preceq n_1$) it now suffices to show for $n_0 \preceq n_1 \prec \omega$ that

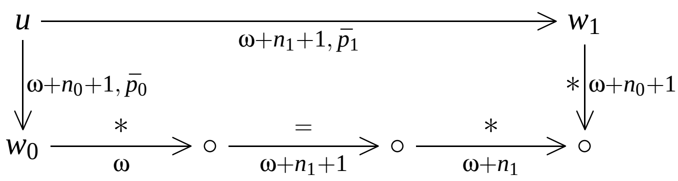
$$w_0 \longleftarrow_{\omega+n_0+1,\bar{p}_0} u \longrightarrow_{\omega+n_1+1,\bar{p}_1} w_1$$

together with our induction hypotheses that

$$\forall\delta \prec (n_0{+}1) +_\omega (n_1{+}1).\ \mathrm{R},\mathrm{X} \text{ is } \omega\text{-shallow confluent up to } \delta$$

implies

$$w_0 \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1.$$



Now for each $i \prec 2$ there are $((l_i,r_i),C_i) \in \mathrm{R}$ and $\mu_i \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $u/\bar{p}_i = l_i\mu_i$, $w_i = u[\,\bar{p}_i \leftarrow r_i\mu_i\,]$, and $C_i\mu_i$ fulfilled w.r.t. $\longrightarrow_{\omega+n_i}$.

<u>Claim 5:</u> We may assume $\forall i \prec 2.\ l_i \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$.
<u>Proof of Claim 5:</u> In case of $l_i \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ we get $u \longrightarrow_\omega w_i$ by Lemma 13.2 (matching both its μ and ν to our μ_i). In case of “$i = 0$” our property follows from the strong commutation assumption of our lemma. In case of “$i = 1$” our property follows from Claim 0. <u>Q.e.d. (Claim 5)</u>

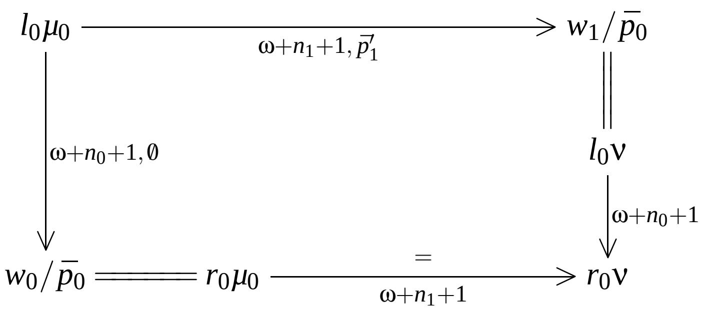
In case of $\bar{p}_0 \parallel \bar{p}_1$ we have $w_i/\bar{p}_{1-i} = u[\,\bar{p}_i \leftarrow r_i\mu_i\,]/\bar{p}_{1-i} = u/\bar{p}_{1-i} = l_{1-i}\mu_{1-i}$ and therefore $w_i \longrightarrow_{\omega+n_{i}+1} u[\,\bar{p}_k \leftarrow r_k\mu_k \mid k \prec 2\,]$, i.e. our proof is finished. Thus, according to whether $\bar{p}_0$ is a prefix of $\bar{p}_1$ or vice versa, we have the following two cases left:

<u>There is some $\bar{p}'_1$ with $\bar{p}_0\bar{p}'_1 = \bar{p}_1$ and $\bar{p}'_1 \neq \emptyset$:</u>

We have two cases:

<u>"The variable overlap case":</u>

<u>There are $x \in \mathrm{V}$ and p', p'' such that $l_0/p' = x \wedge p'p'' = \bar{p}'_1$:</u>



<u>Claim 6:</u> We have $x\mu_0/p'' = l_1\mu_1$ and may assume $x \in \mathrm{V}_{\mathrm{SIG}}$.
<u>Proof of Claim 6:</u> We have $x\mu_0/p'' = l_0\mu_0/p'p'' = u/\bar{p}_0 p'p'' = u/\bar{p}_0\bar{p}'_1 = u/\bar{p}_1 = l_1\mu_1$.
If $x \in \mathrm{V}_{\mathcal{C}}$, then $x\mu_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $x\mu_0/p'' \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $l_1\mu_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, and then $l_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which we may assume not to be the case by Claim 5. <u>Q.e.d. (Claim 6)</u>

<u>Claim 7:</u> We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_0[\, p'' \leftarrow r_1\mu_1\,]$ and $\forall y \in \mathrm{V}\setminus\{x\}.\ y\nu = y\mu_0$.
Then we have $x\mu_0 \longrightarrow_{\omega+n_1+1} x\nu$.
<u>Proof of Claim 7:</u> This follows directly from Claim 6. <u>Q.e.d. (Claim 7)</u>

<u>Claim 8:</u> $l_0\nu = w_1/\bar{p}_0$.
<u>Proof of Claim 8:</u> By the left-linearity assumption of our lemma and claims 5 and 6 we may assume $\{\, p''' \mid l_0/p''' = x \,\} = \{p'\}$. Thus, by Claim 7 we get $w_1/\bar{p}_0 = u/\bar{p}_0[\,\bar{p}'_1 \leftarrow r_1\mu_1\,] =$
$l_0[\, p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V}\,][\,\bar{p}'_1 \leftarrow r_1\mu_1\,] =$
$l_0[\, p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x\,][\, p' \leftarrow x\mu_0\,][\, p'p'' \leftarrow r_1\mu_1\,] =$
$l_0[\, p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x\,][\, p' \leftarrow x\mu_0[\, p'' \leftarrow r_1\mu_1\,]\,] =$
$l_0[\, p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V}\,] = l_0\nu$. <u>Q.e.d. (Claim 8)</u>

<u>Claim 9:</u> $w_0/\bar{p}_0 \overset{=}{\longrightarrow}_{\omega+n_1+1} r_0\nu$.
<u>Proof of Claim 9:</u> By the right-linearity assumption of our lemma and claims 5 and 6 we may assume $|\{\, p''' \mid r_0/p''' = x \,\}| \preceq 1$. Thus by Claim 7 we get: $w_0/\bar{p}_0 = r_0\mu_0 =$
$r_0[\, p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}\,][\, p''' \leftarrow x\mu_0 \mid r_0/p''' = x\,] \overset{=}{\longrightarrow}_{\omega+n_1+1}$
$r_0[\, p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}\,][\, p''' \leftarrow x\nu \mid r_0/p''' = x\,] =$
$r_0[\, p''' \leftarrow y\nu \mid r_0/p''' = y \in \mathrm{V}\setminus\{x\}\,][\, p''' \leftarrow x\nu \mid r_0/p''' = x\,] = r_0\nu$. <u>Q.e.d. (Claim 9)</u>

By claims 8 and 9 it now suffices to show $l_0\nu \longrightarrow_{\omega+n_0+1} r_0\nu$, which again follows from Lemma 13.8 (matching its n_0 to our $n_1{+}1$ and its n_1 to our n_0) since R, X is quasi-normal and ω-shallow confluent up to $(n_1{+}1) +_\omega n_0$ by our induction hypothesis, and since $\forall y \in \mathrm{V}.$ $y\mu_0 \overset{*}{\longrightarrow}_{\omega+n_1+1} y\nu$ by Claim 7. <u>Q.e.d. ("The variable overlap case")</u>

"The critical peak case": $\overline{p'_1} \in \mathcal{POS}(l_0) \wedge l_0/\overline{p'_1} \notin \mathrm{V}$:

$$\begin{array}{ccccccc}
l_0\mu_0 & & \xrightarrow[\omega+n_1+1,\,\overline{p'_1}]{} & & & & w_1/\overline{p_0} \\
\downarrow{\scriptstyle \omega+n_0+1,\,\emptyset} & & & & & & {\scriptstyle *}\downarrow{\scriptstyle \omega+n_0+1} \\
w_0/\overline{p_0} & \xrightarrow[\omega]{*} & \circ & \xrightarrow[\omega+n_1+1]{=} & \circ & \xrightarrow[\omega+n_1]{*} & \circ
\end{array}$$

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_1,r_1),C_1))] \cap \mathcal{V}(((l_0,r_0),C_0)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_1,r_1),C_1))] \cup \mathcal{V}(((l_0,r_0),C_0))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_0 & \text{if } x \in \mathcal{V}(((l_0,r_0),C_0)) \\ x\xi^{-1}\mu_1 & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_1\xi\rho = l_1\xi\xi^{-1}\mu_1 = u/\overline{p_1} = u/\overline{p_0}\,\overline{p'_1} = l_0\mu_0/\overline{p'_1} = l_0\rho/\overline{p'_1} = (l_0/\overline{p'_1})\rho$
let $\sigma := \mathrm{mgu}(\{(l_1\xi, l_0/\overline{p'_1})\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}{\upharpoonleft}(\sigma\varphi) = {}_{\mathrm{Y}}{\upharpoonleft}\rho$.
If $l_0[\overline{p'_1} \leftarrow r_1\xi]\sigma = r_0\sigma$, then the proof is finished due to

$$w_0/\overline{p_0} = r_0\mu_0 = r_0\sigma\varphi = l_0[\overline{p'_1} \leftarrow r_1\xi]\sigma\varphi = l_0\mu_0[\overline{p'_1} \leftarrow r_1\mu_1] = w_1/\overline{p_0}.$$

Otherwise we have $((l_0[\overline{p'_1} \leftarrow r_1\xi], C_1\xi, 1), (r_0, C_0, 1), l_0, \sigma, \overline{p'_1}) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $\overline{p'_1} \neq \emptyset$ (due the global case assumption); $C_1\xi\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$; $C_0\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$. Since $\forall\delta \prec (n_1+1)+_{\omega}(n_0+1)$. R, X is ω-shallow confluent up to δ (by our induction hypothesis), due to our assumed ω-shallow noisy anti-closedness (matching the definition's n_0 to our n_1+1 and its n_1 to n_0+1) we have $w_1/\overline{p_0} = l_0\mu_0[\overline{p'_1} \leftarrow r_1\mu_1] = l_0[\overline{p'_1} \leftarrow r_1\xi]\sigma\varphi \xrightarrow{*}_{\omega+n_0+1} \circ \xleftarrow{*}_{\omega+n_1} \circ \xleftarrow{=}_{\omega+n_1+1} \circ \xleftarrow{*}_{\omega} r_0\sigma\varphi = r_0\mu_0 = w_0/\overline{p_0}$.

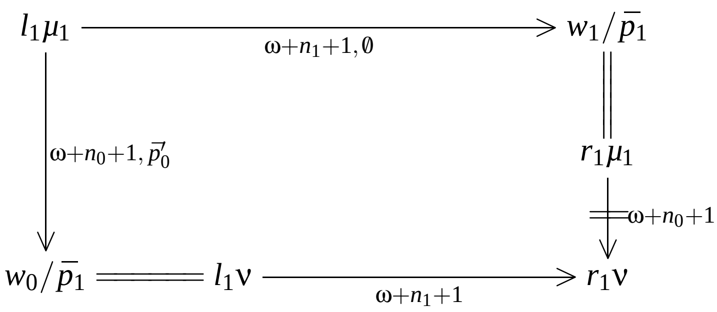
Q.e.d. ("The critical peak case") Q.e.d. ("There is some $\overline{p'_1}$ with $\overline{p_0}\,\overline{p'_1} = \overline{p_1}$ and $\overline{p'_1} \neq \emptyset$")

There is some $\overline{p'_0}$ with $\overline{p_1}\,\overline{p'_0} = \overline{p_0}$:

We have two cases:

"The second variable overlap case":

There are $x\in V$ and p', p'' such that $l_1/p' = x \wedge p'p'' = \overline{p'_0}$:



Claim 11a: We have $x\mu_1/p'' = l_0\mu_0$ and may assume $x\in V_{SIG}$.
Proof of Claim 11a: We have $x\mu_1/p'' = l_1\mu_1/p'p'' = u/\overline{p_1}p'p'' = u/\overline{p_1}\,\overline{p'_0} = u/\overline{p_0} = l_0\mu_0$.
If $x\in V_{\mathcal{C}}$, then $x\mu_1\in\mathcal{T}(\mathrm{cons}, V_{\mathcal{C}})$, then $x\mu_1/p''\in\mathcal{T}(\mathrm{cons}, V_{\mathcal{C}})$, then $l_0\mu_0\in\mathcal{T}(\mathrm{cons}, V_{\mathcal{C}})$, and then $l_0\in\mathcal{T}(\mathrm{cons}, V_{SIG}\uplus V_{\mathcal{C}})$ which we may assume not to be the case by Claim 5. Q.e.d. (Claim 11a)

Claim 11b: We can define $\nu\in\mathcal{SUB}(V,\mathcal{T}(X))$ by $x\nu = x\mu_1[p''\leftarrow r_0\mu_0]$ and $\forall y\in V\setminus\{x\}.\ y\nu = y\mu_1$. Then we have $x\mu_1 \longrightarrow_{\omega+n_0+1} x\nu$.
Proof of Claim 11b: This follows directly from Claim 11a. Q.e.d. (Claim 11b)

Claim 12: $w_0/\overline{p_1} = l_1\nu$.
Proof of Claim 12:
By the left-linearity assumption of our lemma and claims 5 and 11a we may assume $\{\, p''' \mid l_1/p''' = x \,\} = \{p'\}$. Thus, by Claim 11b we get $w_0/\overline{p_1} = u/\overline{p_1}[\overline{p'_0} \leftarrow r_0\mu_0] =$
$l_1[p''' \leftarrow y\mu_1 \mid l_1/p''' = y\in V][\overline{p'_0} \leftarrow r_0\mu_0] =$
$l_1[p''' \leftarrow y\mu_1 \mid l_1/p''' = y\in V \wedge y\neq x][p' \leftarrow x\mu_1][p'p'' \leftarrow r_0\mu_0] =$
$l_1[p''' \leftarrow y\nu \mid l_1/p''' = y\in V \wedge y\neq x][p' \leftarrow x\mu_1[p'' \leftarrow r_0\mu_0]] =$
$l_1[p''' \leftarrow y\nu \mid l_1/p''' = y\in V] = l_1\nu$. Q.e.d. (Claim 12)

Claim 13: $r_1\nu \leftarrow\!\!\!\Vert\!\!-_{\omega+n_0+1} w_1/\overline{p_1}$.
Proof of Claim 13: Since $r_1\mu_1 = w_1/\overline{p_1}$, this follows directly from Claim 11b. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_1\nu \longrightarrow_{\omega+n_1+1} r_1\nu$, which again follows from Claim 11b, Lemma 13.8 (matching its n_0 to our n_0+1 and its n_1 to our n_1), and our induction hypothesis that R, X is ω-shallow confluent up to $(n_0+1) +_\omega n_1$.
Q.e.d. ("The second variable overlap case")

<u>“The second critical peak case”: $\bar{p}'_0 \in \mathcal{POS}(l_1) \wedge l_1/\bar{p}'_0 \notin \mathrm{V}$:</u>

$$\begin{array}{ccccccc}
l_1\mu_1 & & \xrightarrow[\omega+n_1+1,\,\emptyset]{} & & & & w_1/\bar{p}_1 \\
\Big\downarrow{\scriptstyle \omega+n_0+1,\,\bar{p}'_0} & & & & & & {\scriptstyle *}\Big\downarrow{\scriptstyle \omega+n_0+1} \\
w_0/\bar{p}_1 & \xrightarrow[\omega]{*} & \circ & \xrightarrow[\omega+n_1+1]{=} & \circ & \xrightarrow[\omega+n_1]{*} & \circ
\end{array}$$

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_0,r_0),C_0))] \cap \mathcal{V}(((l_1,r_1),C_1)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_0,r_0),C_0))] \cup \mathcal{V}(((l_1,r_1),C_1))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_1 & \text{if } x \in \mathcal{V}(((l_1,r_1),C_1)) \\ x\xi^{-1}\mu_0 & \text{else} \end{array}\right\}$ $(x\in\mathrm{V})$.
By $l_0\xi\rho = l_0\xi\xi^{-1}\mu_0 = u/\bar{p}_0 = u/\bar{p}_1\bar{p}'_0 = l_1\mu_1/\bar{p}'_0 = l_1\rho/\bar{p}'_0 = (l_1/\bar{p}'_0)\rho$
let $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/\bar{p}'_0)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}{\upharpoonright}(\sigma\varphi) = {}_{\mathrm{Y}}{\upharpoonright}\rho$.
If $l_1[\bar{p}'_0 \leftarrow r_0\xi]\sigma = r_1\sigma$, then the proof is finished due to

$$w_0/\bar{p}_1 = l_1\mu_1[\bar{p}'_0 \leftarrow r_0\mu_0] = l_1[\bar{p}'_0 \leftarrow r_0\xi]\sigma\varphi = r_1\sigma\varphi = r_1\mu_1 = w_1/\bar{p}_1.$$

Otherwise we have $((l_1[\bar{p}'_0 \leftarrow r_0\xi], C_0\xi, 1), (r_1, C_1, 1), l_1, \sigma, \bar{p}'_0) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_0\xi\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_0}$; $C_1\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$. Since $\forall\delta \prec (n_0{+}1) +_{\omega} (n_1{+}1)$. R, X is ω-shallow confluent up to δ (by our induction hypothesis) due to our assumed ω-shallow noisy strong joinability (matching the definition’s n_0 to our $n_0{+}1$ and its n_1 to our $n_1{+}1$) we have $w_0/\bar{p}_1 = l_1\mu_1[\bar{p}'_0 \leftarrow r_0\mu_0] = l_1[\bar{p}'_0 \leftarrow r_0\xi]\sigma\varphi \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega+n_1} \circ \overset{*}{\longleftarrow}_{\omega+n_0+1} r_1\sigma\varphi = r_1\mu_1 = w_1/\bar{p}_1$.
<u>Q.e.d. (“The second critical peak case”)</u> **Q.e.d. (Lemma A.6)**

Proof of Lemma A.7 For each literal L in C we have to show that $L\nu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+n_1}$. Note that we already know that $L\mu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X},\omega+n_1}$. If $\mathcal{V}(C) \subseteq \mathrm{V}_{\mathcal{C}}$, then for all x in $\mathcal{V}(C)$ we have $x\mu \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then by Lemma 2.10 $x\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+0} y\mu$. Thus, by the disjunctive assumption of our lemma we may assume $n_0 \preceq n_1$.
<u>$L = (s_0{=}s_1)$:</u> We have $s_0\nu \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_0} s_0\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} s_1\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_0} s_1\nu$ for some t_0. By our ω-level confluence up to n_1 and $n_0 \preceq n_1$, we get some v with $s_0\nu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} v \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_0$ and then (due to $v \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} s_1\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_0} s_1\nu$) $v \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} \circ \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} s_1\nu$.
<u>$L = (\mathrm{Def}\, s)$:</u> We know the existence of $t \in \mathcal{GT}(\mathrm{cons})$ with $s\nu \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_0} s\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t$. By our ω-level confluence up to n_1 and $n_0 \preceq n_1$, there is some t' with $s\nu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t' \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t$. By Lemma 2.10 we get $t' \in \mathcal{GT}(\mathrm{cons})$.
<u>$L = (s_0{\neq}s_1)$:</u> There exist some $t_0, t_1 \in \mathcal{GT}(\mathrm{cons})$ with $\forall i \prec 2.\ s_i\nu \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_0} s_i\mu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t_i$ and $t_0 \not\downarrow_{\mathrm{R,X},\omega+n_1} t_1$. Just like above we get $t'_0, t'_1 \in \mathcal{GT}(\mathrm{cons})$ with $\forall i \prec 2.\ s_i\nu \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t'_i \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_i$. Finally $t'_0 \overset{*}{\longleftarrow}_{\mathrm{R,X},\omega+n_1} t_0 \not\downarrow_{\mathrm{R,X},\omega+n_1} t_1 \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega+n_1} t'_1$ implies $t'_0 \not\downarrow_{\mathrm{R,X},\omega+n_1} t'_1$. **Q.e.d. (Lemma A.7)**

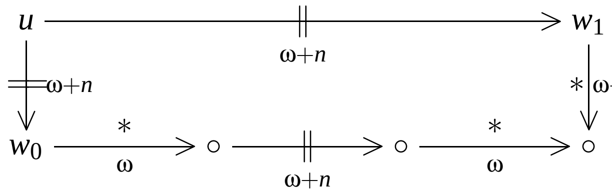
Proof of Lemma A.8

Claim 0: R,X is ω-shallow confluent up to ω.
Proof of Claim 0: Directly by the assumed strong commutation, cf. the proofs of the claims 2 and 3 of the proof of Lemma A.1. Q.e.d. (Claim 0)

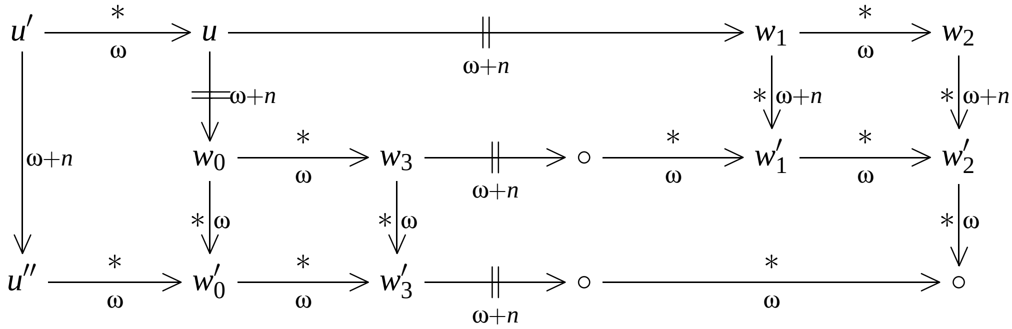
Claim 1: If $\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}$ strongly commutes over $\xrightarrow{*}_{\omega+n}$, then $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega+n}$ are commuting.
Proof of Claim 1: $\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}$ and $\xrightarrow{*}_{\omega+n}$ are commuting by Lemma 3.3. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{\omega+n}\subseteq\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}\subseteq\xrightarrow{*}_{\omega+n}$, now $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega+n}$ are commuting, too. Q.e.d. (Claim 1)

For $n\prec\omega$ we are going to show by induction on n the following property:

$$w_0\mathrel{\leftarrow\!\!\!\Vert\!\!-}_{\omega+n}u\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}w_1\quad\Rightarrow\quad w_0\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}\circ\xleftarrow{*}_{\omega+n}w_1.$$



Claim 2: Let $\delta\prec\omega$. If $\forall n\preceq\delta.\,\forall w_0,w_1,u.\left(\begin{array}{l} w_0\mathrel{\leftarrow\!\!\!\Vert\!\!-}_{\omega+n}u\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}w_1\\ \Rightarrow\ w_0\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}\circ\xleftarrow{*}_{\omega+n}w_1\end{array}\right)$, then $\forall n\preceq\delta.\left(\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}\text{ strongly commutes over }\xrightarrow{*}_{\omega+n}\right)$, and R,X is ω-level confluent up to δ.
Proof of Claim 2: First we show the strong commutation. Assume $n\preceq\delta$. By Lemma 3.3 it suffices to show that $\xrightarrow{*}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}$ strongly commutes over $\longrightarrow_{\omega+n}$. Assume $u''\longleftarrow_{\omega+n}u'\xrightarrow{*}_{\omega}u\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}w_1\xrightarrow{*}_{\omega}w_2$ (cf. diagram below). By the strong commutation assumed for our lemma and Corollary 2.14, there are w_0 and w_0' with $u''\xrightarrow{*}_{\omega}w_0'\xleftarrow{*}_{\omega}w_0\mathrel{\leftarrow\!\!\!\Vert\!\!-}_{\omega+n}u$. By the above property there are some w_3, w_1' with $w_0\xrightarrow{*}_{\omega}w_3\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}w_1'\xleftarrow{*}_{\omega+n}w_1$. By Claim 0 we can close the peak $w_1'\xleftarrow{*}_{\omega+n}w_1\xrightarrow{*}_{\omega}w_2$ according to $w_1'\xrightarrow{*}_{\omega}w_2'\xleftarrow{*}_{\omega+n}w_2$ for some w_2'. By Claim 0 again, we can close the peak $w_0'\xleftarrow{*}_{\omega}w_0\xrightarrow{*}_{\omega}w_3$ according to $w_0'\xrightarrow{*}_{\omega}w_3'\xleftarrow{*}_{\omega}w_3$ for some w_3'. To close the whole diagram, we only have to show that we can close the peak $w_3'\xleftarrow{*}_{\omega}w_3\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}w_2'$ according to $w_3'\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+n}\circ\xrightarrow{*}_{\omega}\circ\xleftarrow{*}_{\omega}w_2'$, which is possible due to the strong commutation assumed for our lemma.



Finally we show ω-level confluence up to δ. Assume $n_0, n_1 \prec \omega$ with $\max\{n_0,n_1\} \preceq \delta$ and $w_0 \xleftarrow{*}_{\omega+n_0} u \xrightarrow{*}_{\omega+n_1} w_1$. By Lemma 2.12 we get $w_0 \xleftarrow{*}_{\omega+\max\{n_0,n_1\}} u \xrightarrow{*}_{\omega+\max\{n_0,n_1\}} w_1$. Since $\max\{n_0,n_1\} \preceq \delta$, above we have shown that $\xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+\max\{n_0,n_1\}} \circ \xrightarrow{*}_{\omega}$ strongly commutes over $\xrightarrow{*}_{\omega+\max\{n_0,n_1\}}$. By Claim 1 we finally get $w_0 \xrightarrow{*}_{\omega+\max\{n_0,n_1\}} \circ \xleftarrow{*}_{\omega+\max\{n_0,n_1\}} w_1$ as desired.

Q.e.d. (Claim 2)

Note that for $n=0$ our property follows from $\mathrel{\leftarrow\!\!\parallel\!\!-}_{\omega} \subseteq \xleftarrow{*}_{\omega}$ (by Corollary 2.14) and Claim 0.

The benefit of Claim 2 is twofold: First, it says that our lemma is valid if the above property holds for all $n \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n+1$ instead of n since we may assume $0 \prec n$) it now suffices to show for $n \prec \omega$ that

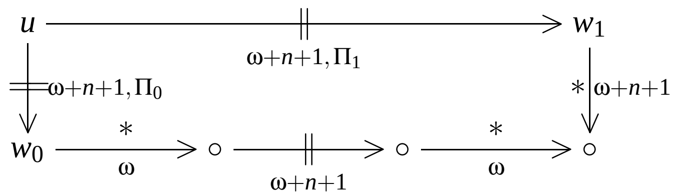
$$w_0 \mathrel{\leftarrow\!\!\parallel\!\!-}_{\omega+n+1,\Pi_0} u \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n+1,\Pi_1} w_1$$

together with our induction hypotheses that

$$\mathrm{R}, \mathrm{X} \text{ is } \omega\text{-level confluent up to } n$$

implies

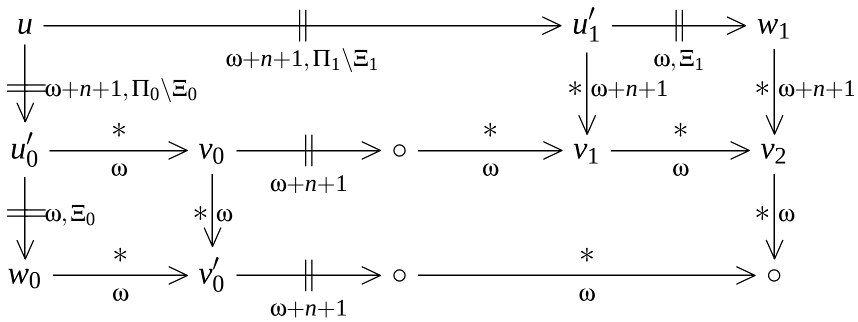
$$w_0 \xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n+1} \circ \xrightarrow{*}_{\omega} \circ \xleftarrow{*}_{\omega+n_1+1} w_1.$$



W.l.o.g. let the positions of Π_i be maximal in the sense that for any $p \in \Pi_i$ and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $u \mathrel{-\!\!\parallel\!\!\rightarrow}_{\omega+n+1,(\Pi_i \setminus \{p\}) \cup \Xi} w_i$ anymore. Then for each $i \prec 2$ and $p \in \Pi_i$ there are $((l_{i,p}, r_{i,p}), C_{i,p}) \in \mathrm{R}$ and $\mu_{i,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{i,p}\mu_{i,p}$, $r_{i,p}\mu_{i,p} = w_i/p$, $C_{i,p}\mu_{i,p}$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Finally, for each $i \prec 2$: $w_i = u[\, p \leftarrow r_{i,p}\mu_{i,p} \mid p \in \Pi_i \,]$.

<u>Claim 5:</u> We may assume $\forall i\prec 2.\ \forall p\in\Pi_i.\ l_{i,p}\notin\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{C})$.

<u>Proof of Claim 5:</u> Define $\Xi_i := \{\, p\in\Pi_i \mid l_{i,p}\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{C})\,\}$ and $u'_i := u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i\setminus\Xi_i\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $u'_0 \overset{*}{\longrightarrow}_{\omega} v_0 \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} v_1 \overset{*}{\longleftarrow}_{\omega+n+1} u'_1$ for some v_0, v_1 (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{i,p}$) we get $\forall i\prec 2.\ \forall p\in\Xi_i.\ l_{i,p}\mu_{i,p} \longrightarrow_{\omega} r_{i,p}\mu_{i,p}$ and therefore $\forall i\prec 2.\ u'_i \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\omega,\Xi_i} w_i$. Thus from $v_1 \overset{*}{\longleftarrow}_{\omega+n_0+1} u'_1 \overset{*}{\longrightarrow}_{\omega} w_1$ we get $v_1 \overset{*}{\longrightarrow}_{\omega} v_2 \overset{*}{\longleftarrow}_{\omega+n_0+1} w_1$ for some v_2 by ω-shallow confluence up to ω (cf. Claim 0). For the same reason we can close the peak $w_0 \overset{*}{\longleftarrow}_{\omega} u'_0 \overset{*}{\longrightarrow}_{\omega} v_0$ according to $w_0 \overset{*}{\longrightarrow}_{\omega} v'_0 \overset{*}{\longleftarrow}_{\omega} v_0$ for some v'_0. By the assumption of our lemma that $\mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\mathrm{R,X},\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X},\omega}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega}$, from $v'_0 \overset{*}{\longleftarrow}_{\omega} v_0 \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega} v_1 \overset{*}{\longrightarrow}_{\omega} v_2$ we can finally conclude $v'_0 \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega} v_2$.



<u>Q.e.d. (Claim 5)</u>

Define the set of inner overlapping positions by

$$\Omega(\Pi_0,\Pi_1) := \bigcup_{i\prec 2}\{\, p\in\Pi_{1-i} \mid \exists q\in\Pi_i.\ \exists q'\in\mathbf{N}^*.\ p=qq' \,\},$$

and the length of a term by $\lambda(f(t_0,\ldots,t_{m-1})) := 1+\sum_{j\prec m}\lambda(t_j)$.

Now we start a second level of induction on $\displaystyle\sum_{p'\in\Omega(\Pi_0,\Pi_1)}\lambda(u/p')$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\, p\in\Pi_0\cup\Pi_1 \mid \neg\exists q\in\Pi_0\cup\Pi_1.\ \exists q'\in\mathbf{N}^+.\ p=qq' \,\}.$$

Since the prefix ordering is wellfounded we have $\forall i\prec 2.\ \forall p\in\Pi_i.\ \exists q\in\Theta.\ \exists q'\in\mathbf{N}^*.\ p=qq'$. Then $\forall i\prec 2.\ w_i = w_i[q\leftarrow w_i/q \mid q\in\Theta] = u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i\,][q\leftarrow w_i/q \mid q\in\Theta] = u[q\leftarrow w_i/q \mid q\in\Theta]$. Thus, it now suffices to show for all $q\in\Theta$

$$w_0/q \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} w_1/q$$

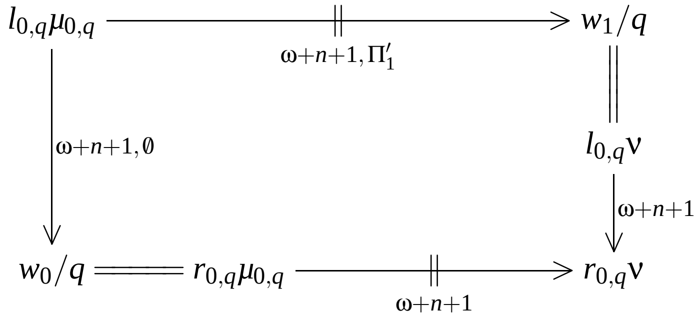
because then we have

$$w_0 = u[q\leftarrow w_0/q \mid q\in\Theta] \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\shortparallel\!\!\rightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} u[q\leftarrow w_1/q \mid q\in\Theta] = w_1.$$

Therefore we are left with the following two cases for $q\in\Theta$:

$\underline{\underline{q \notin \Pi_1}}$: Then $q \in \Pi_0$. Define $\Pi'_1 := \{\, p \mid qp \in \Pi_1 \,\}$. We have two cases:

<u>“The variable overlap (if any) case”: $\forall p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}$:</u>



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\, (p', p'') \mid l_{0,q}/p' = x \ \wedge\ p'p'' \in \Pi'_1 \,\}$.

<u>Claim 7:</u> There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}.\ \left(\begin{array}{ll} & x\mu_{0,q} \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\!\!\!+\ \ \ _{\omega+n+1} x\nu \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu = x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \end{array} \right).$$

<u>Proof of Claim 7:</u>
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$\begin{array}{rl} x\mu_{0,q} = & x\mu_{0,q}[\,p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\!\!\!+\ \ \ _{\omega+n+1} \\ & x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu. \end{array}$$

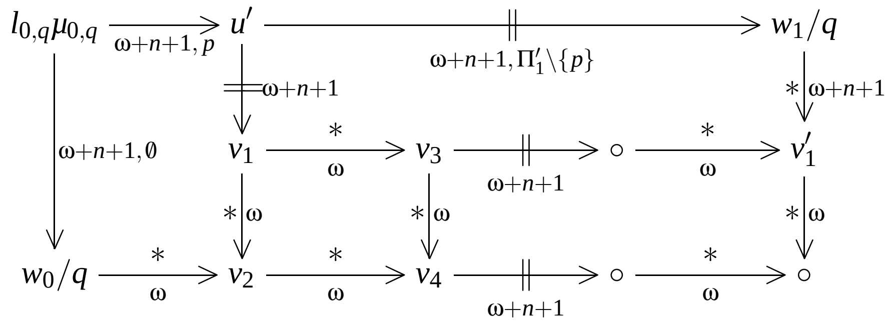
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{0,q}/p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ this implies $l_{1,qp'p''}\mu_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. <u>Q.e.d. (Claim 7)</u>

<u>Claim 8:</u> $l_{0,q}\nu = w_1/q$.
<u>Proof of Claim 8:</u>
By Claim 7 we get $w_1/q = u/q[\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{0,q}/p' = x \in \mathrm{V}\,] =$
$l_{0,q}[\,p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V}\,] = l_{0,q}\nu$. <u>Q.e.d. (Claim 8)</u>

<u>Claim 9:</u> $w_0/q \longrightarrow\!\!\!\!\!\!\!\!\!\!\!+\!\!\!+\ \ \ _{\omega+n+1} r_{0,q}\nu$.
<u>Proof of Claim 9:</u> Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. <u>Q.e.d. (Claim 9)</u>

By claims 8 and 9 it now suffices to show $l_{0,q}\nu \longrightarrow_{\omega+n+1} r_{0,q}\nu$, which again follows from Lemma A.7 (matching its n_0 to our $n{+}1$ and its n_1 to our n) since R, X is ω-level confluent up to n by our induction hypothesis and since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \stackrel{*}{\longrightarrow}_{\omega+n+1} x\nu$ by Claim 7 and Corollary 2.14.
<u>Q.e.d. (“The variable overlap (if any) case”)</u>

“The critical peak case”: There is some $p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:



Claim 10: $p \neq \emptyset$.
Proof of Claim 10: If $p = \emptyset$, then $\emptyset \in \Pi'_1$, then $q \in \Pi_1$, which contradicts our global case assumption. Q.e.d. (Claim 10)

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cap \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cup \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{matrix} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{matrix}\right\}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1\setminus\{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1, \Pi'_1\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1] = w_1/q.$$

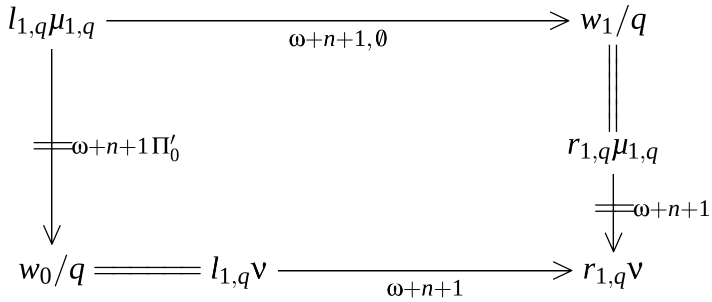
If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1, \Pi'_1\setminus\{p\}} w_1/q.$$

Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 1), l_{0,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R, X is ω-level confluent up to n (by our induction hypothesis) and ω-shallow confluent up to ω (by Claim 0) due to our assumed ω-level parallel closedness (matching the definition's n to our $n{+}1$) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1} v_1 \overset{*}{\longrightarrow}_{\omega} v_2 \overset{*}{\longleftarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v_1, v_2. We then have $v_1 \mathrel{\longleftarrow\!\!\!\!\!+}_{\omega+n+1, \Pi''} u' \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1, \Pi'_1\setminus\{p\}} w_1/q$ for some Π''. By $\sum_{p'' \in \Omega(\Pi'', \Pi'_1\setminus\{p\})} \lambda(u'/p'') \preceq$
$\sum_{p'' \in \Pi'_1\setminus\{p\}} \lambda(u'/p'') = \sum_{p'' \in \Pi'_1\setminus\{p\}} \lambda(u/qp'') \prec$
$\sum_{p'' \in \Pi'_1} \lambda(u/qp'') = \sum_{p' \in q\Pi'_1} \lambda(u/p') = \sum_{p' \in \Omega(\{q\}, \Pi_1)} \lambda(u/p') \preceq \sum_{p' \in \Omega(\Pi_0, \Pi_1)} \lambda(u/p')$, due to our second induction level we get some v'_1, v_3 with $v_1 \overset{*}{\longrightarrow}_{\omega} v_3 \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} v'_1 \overset{*}{\longleftarrow}_{\omega+n+1} w_1/q$. By Claim 0 we can close the peak at v_1 according to $v_2 \overset{*}{\longrightarrow}_{\omega} v_4 \overset{*}{\longleftarrow}_{\omega} v_3$ for some v_4. Finally by the assumption of our lemma that $\mathrel{+\!\!\!\!\!\longrightarrow}_{\mathrm{R},\mathrm{X},\omega+n_1+1} \circ \overset{*}{\longrightarrow}_{\mathrm{R},\mathrm{X},\omega}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega}$, the peak at v_3 can be closed according to $v_4 \mathrel{+\!\!\!\!\!\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega} v'_1$.
Q.e.d. (“The critical peak case”) Q.e.d. (“$q \notin \Pi_1$”)

<u>$q \in \Pi_1$:</u> Define $\Pi'_0 := \{\ p \mid qp \in \Pi_0\ \}$. We have two cases:

<u>"The second variable overlap (if any) case": $\forall p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q}).\ l_{1,q}/p \in \mathrm{V}$:</u>



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\ (p', p'') \mid l_{1,q}/p' = x \ \wedge\ p'p'' \in \Pi'_0\ \}$.

<u>Claim 11:</u> There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}.\ \left(\begin{array}{ll} & x\nu \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!-}_{\omega+n+1} x\mu_{1,q} \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu \end{array}\right).$$

<u>Proof of Claim 11:</u>
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{1,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{1,q}/p'' = l_{1,q}\mu_{1,q}/p'p'' = u/qp'p'' = l_{0,qp'p''}\mu_{0,qp'p''}$ we have

$$x\mu_{1,q} = x\mu_{1,q}[\,p'' \leftarrow l_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mathrel{-\!\!\!+\!\!\!+\!\!\!\rightarrow}_{\omega+n+1}$$
$$x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu.$$

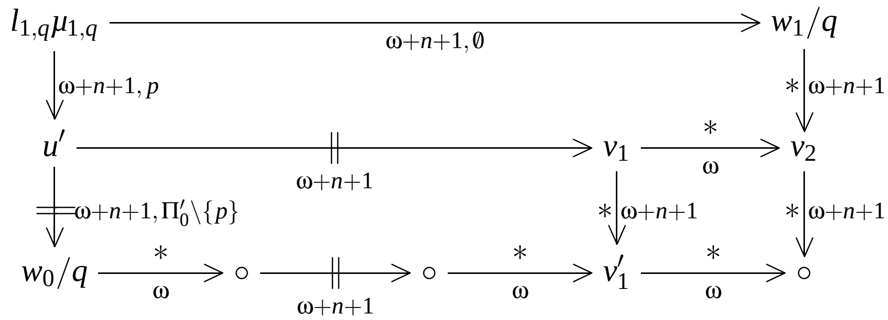
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{1,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{1,q}/p'' = l_{0,qp'p''}\mu_{0,qp'p''}$ this implies $l_{0,qp'p''}\mu_{0,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{0,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. <u>Q.e.d. (Claim 11)</u>

<u>Claim 12:</u> $w_0/q = l_{1,q}\nu$.
<u>Proof of Claim 12:</u>
By Claim 11 we get $w_0/q = u/q[\,p'p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{1,q}[\,p' \leftarrow x\mu_{1,q} \mid l_{1,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{1,q}[\,p' \leftarrow x\mu_{1,q}[\,p'' \leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{1,q}/p' = x \in \mathrm{V}\,] =$
$l_{1,q}[\,p' \leftarrow x\nu \mid l_{1,q}/p' = x \in \mathrm{V}\,] = l_{1,q}\nu$. <u>Q.e.d. (Claim 12)</u>

<u>Claim 13:</u> $r_{1,q}\nu \mathrel{\leftarrow\!\!\!+\!\!\!+\!\!\!-}_{\omega+n+1} w_1/q$.
<u>Proof of Claim 13:</u> Since $r_{1,q}\mu_{1,q} = w_1/q$, this follows directly from Claim 11. <u>Q.e.d. (Claim 13)</u>

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{\omega+n+1} r_{1,q}\nu$, which again follows from Lemma A.7 (matching its n_0 to our $n{+}1$ and its n_1 to our n) since R, X is ω-level confluent up to n by our induction hypothesis and since $\forall x \in \mathrm{V}.\ x\mu_{1,q} \overset{*}{\longrightarrow}_{\omega+n+1} x\nu$ by Claim 11 and Corollary 2.14.
<u>Q.e.d. ("The second variable overlap (if any) case")</u>

"The second critical peak case": There is some $p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q})$ with $l_{1,q}/p \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,qp},r_{0,qp}),C_{0,qp}))] \cap \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,qp},r_{0,qp}),C_{0,qp}))] \cup \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q})) \\ x\xi^{-1}\mu_{0,qp} & \text{else} \end{Bmatrix}$ $(x{\in}\mathrm{V})$.
By $l_{0,qp}\xi\rho = l_{0,qp}\xi\xi^{-1}\mu_{0,qp} = u/qp = l_{1,q}\mu_{1,q}/p = l_{1,q}\rho/p = (l_{1,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,qp}\xi, l_{1,q}/p)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}{\upharpoonright}(\sigma\varphi) = {}_{\mathrm{Y}}{\upharpoonright}\rho$.
Define $u' := l_{1,q}\mu_{1,q}[p \leftarrow r_{0,qp}\mu_{0,qp}]$. We get

$$w_0/q = u/q[p' \leftarrow r_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0] \leftarrow\!\!\Vert\!\!-_{\omega+n+1,\Pi'_0\setminus\{p\}}$$
$$u/q[p' \leftarrow l_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0\setminus\{p\}][p \leftarrow r_{0,qp}\mu_{0,qp}] = u'.$$

If $l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q \leftarrow\!\!\Vert\!\!-_{\omega+n+1,\Pi'_0\setminus\{p\}} u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

Otherwise we have $((l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma, C_{0,qp}\xi\sigma, 1), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, p) \in \mathrm{CP(R)}$ (due to Claim 5); $C_{0,qp}\xi\sigma\varphi = C_{0,qp}\mu_{0,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R,X is ω-level confluent up to n (by our induction hypothesis) and ω-shallow confluent up to ω (by Claim 0) due to our assumed ω-level parallel joinability (matching the definition's n to our $n{+}1$) we have $u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi \;{-\!\!\Vert\!\!\rightarrow}_{\omega+n+1} v_1 \overset{*}{\longrightarrow}_{\omega} v_2 \overset{*}{\longleftarrow}_{\omega+n+1} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$ for some v_1, v_2. We then have $w_0/q \leftarrow\!\!\Vert\!\!-_{\omega+n+1,\Pi'_0\setminus\{p\}} u' \;{-\!\!\Vert\!\!\rightarrow}_{\omega+n+1,\Pi''} v_1$ for some Π''. Since $\sum_{p''\in\Omega(\Pi'_0\setminus\{p\},\Pi'')} \lambda(u'/p'') \preceq \sum_{p''\in\Pi'_0\setminus\{p\}} \lambda(u'/p'') = \sum_{p''\in\Pi'_0\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p''\in\Pi'_0} \lambda(u/qp'') = \sum_{p'\in q\Pi'_0} \lambda(u/p') = \sum_{p'\in\Omega(\Pi_0,\{q\})} \lambda(u/p') \preceq \sum_{p'\in\Omega(\Pi_0,\Pi_1)} \lambda(u/p')$ due to our second induction level we get some v'_1 with $w_0/q \overset{*}{\longrightarrow}_{\omega} \circ \;{-\!\!\Vert\!\!\rightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} v'_1 \overset{*}{\longleftarrow}_{\omega+n+1} v_1$. Finally the peak at v_1 can be closed according to $v'_1 \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} v_2$ by Claim 0.

Q.e.d. ("The second critical peak case") **Q.e.d. (Lemma A.8)**

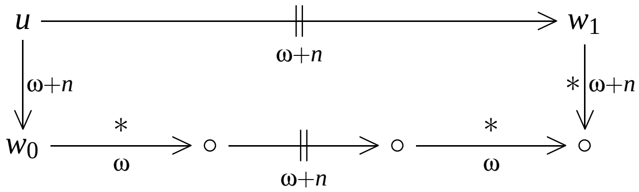
Proof of Lemma A.9

<u>Claim 0:</u> R, X is ω-shallow confluent up to ω.
<u>Proof of Claim 0:</u> Directly by the assumed strong commutation, cf. the proofs of the claims 2 and 3 of the proof of Lemma A.1. <u>Q.e.d. (Claim 0)</u>

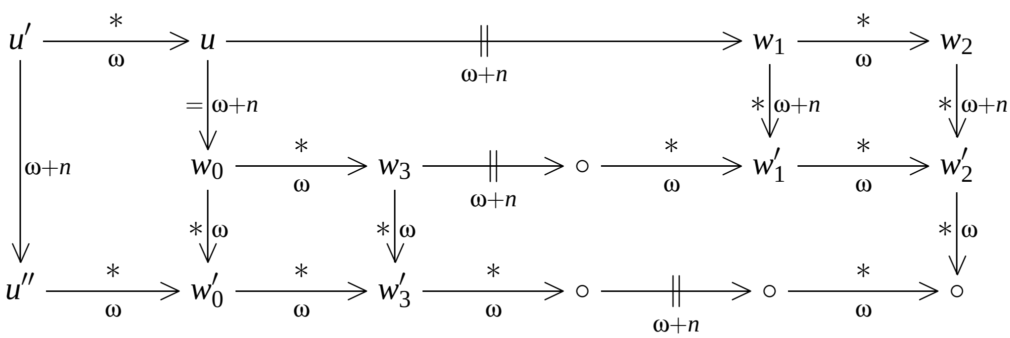
<u>Claim 1:</u> If $\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+n}$, then $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega+n}$ are commuting.
<u>Proof of Claim 1:</u> $\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}$ and $\overset{*}{\longrightarrow}_{\omega+n}$ are commuting by Lemma 3.3. Since by Corollary 2.14 and Lemma 2.12 we have $\longrightarrow_{\omega+n}\subseteq\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}\subseteq\overset{*}{\longrightarrow}_{\omega+n}$, now $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega+n}$ are commuting, too. <u>Q.e.d. (Claim 1)</u>

For $n\prec\omega$ we are going to show by induction on n the following property:

$$w_0\longleftarrow_{\omega+n}u\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}w_1\quad\Rightarrow\quad w_0\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}\circ\overset{*}{\longleftarrow}_{\omega+n}w_1.$$



<u>Claim 2:</u> Let $\delta\prec\omega$. If $\forall n\preceq\delta.\ \forall w_0,w_1,u.\left(\begin{array}{l} w_0\longleftarrow_{\omega+n}u\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}w_1\\ \Rightarrow\ w_0\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}\circ\overset{*}{\longleftarrow}_{\omega+n}w_1\end{array}\right)$, then $\forall n\preceq\delta.\left(\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}\text{ strongly commutes over }\overset{*}{\longrightarrow}_{\omega+n}\right)$, and R, X is ω-level confluent up to δ.
<u>Proof of Claim 2:</u> First we show the strong commutation. Assume $n\preceq\delta$. By Lemma 3.3 it suffices to show that $\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}$ strongly commutes over $\longrightarrow_{\omega+n}$. Assume $u''\longleftarrow_{\omega+n}u'\overset{*}{\longrightarrow}_{\omega}u\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}w_1\overset{*}{\longrightarrow}_{\omega}w_2$ (cf. diagram below). By the strong commutation assumed for our lemma, there are w_0 and w_0' with $u''\overset{*}{\longrightarrow}_{\omega}w_0'\overset{*}{\longleftarrow}_{\omega}w_0\overset{=}{\longleftarrow}_{\omega+n}u$. By the above property there are some w_3, w_1' with $w_0\overset{*}{\longrightarrow}_{\omega}w_3\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}w_1'\overset{*}{\longleftarrow}_{\omega+n}w_1$. By Claim 0 we can close the peak $w_1'\overset{*}{\longleftarrow}_{\omega+n}w_1\overset{*}{\longrightarrow}_{\omega}w_2$ according to $w_1'\overset{*}{\longrightarrow}_{\omega}w_2'\overset{*}{\longleftarrow}_{\omega+n}w_2$ for some w_2'. By Claim 0 again, we can close the peak $w_0'\overset{*}{\longleftarrow}_{\omega}w_0\overset{*}{\longrightarrow}_{\omega}w_3$ according to $w_0'\overset{*}{\longrightarrow}_{\omega}w_3'\overset{*}{\longleftarrow}_{\omega}w_3$ for some w_3'. To close the whole diagram, we only have to show that we can close the peak $w_3'\overset{*}{\longleftarrow}_{\omega}w_3\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}w_2'$ according to $w_3'\overset{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\!\|\!\!\!\rightarrow}_{\omega+n}\circ\overset{*}{\longrightarrow}_{\omega}\circ\overset{*}{\longleftarrow}_{\omega}w_2'$, which is possible since it is assumed for our lemma (below the strong commutation assumption).



Finally we show ω-level confluence up to δ. Assume $n_0, n_1 \prec \omega$ with $\max\{n_0, n_1\} \preceq \delta$ and $w_0 \overset{*}{\longleftarrow}_{\omega+n_0} u \overset{*}{\longrightarrow}_{\omega+n_1} w_1$. By Lemma 2.12 we get $w_0 \overset{*}{\longleftarrow}_{\omega+\max\{n_0,n_1\}} u \overset{*}{\longrightarrow}_{\omega+\max\{n_0,n_1\}} w_1$. Since $\max\{n_0, n_1\} \preceq \delta$, above we have shown that $\overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\Vert\!\!\!\longrightarrow}_{\omega+\max\{n_0,n_1\}} \circ \overset{*}{\longrightarrow}_{\omega}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+\max\{n_0,n_1\}}$. By Claim 1 we finally get $w_0 \overset{*}{\longrightarrow}_{\omega+\max\{n_0,n_1\}} \circ \overset{*}{\longleftarrow}_{\omega+\max\{n_0,n_1\}} w_1$ as desired.

Q.e.d. (Claim 2)

Note that for $n = 0$ our property follows from Corollary 2.14 and Claim 0.

The benefit of Claim 2 is twofold: First, it says that our lemma is valid if the above property holds for all $n \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n+1$ instead of n since we may assume $0 \prec n$) it now suffices to show for $n \prec \omega$ that

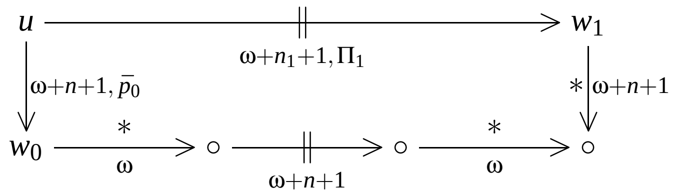
$$w_0 \longleftarrow_{\omega+n+1,\bar{p}_0} u \mathrel{-\!\!\Vert\!\!\!\longrightarrow}_{\omega+n+1,\Pi_1} w_1$$

together with our induction hypotheses that

$$\mathrm{R}, \mathrm{X} \text{ is } \omega\text{-level confluent up to } n$$

implies

$$w_0 \overset{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\Vert\!\!\!\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} w_1 .$$

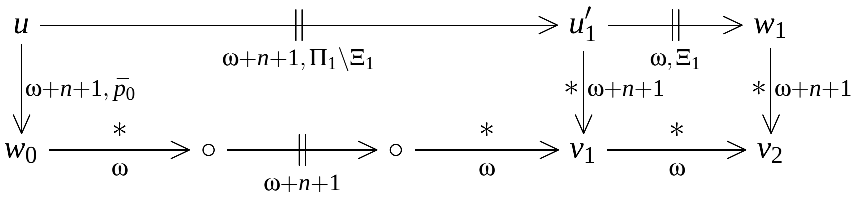


There are $((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}) \in \mathrm{R}$ and $\mu_{0,\bar{p}_0} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$, $C_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$, and $w_0 = u[p \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}]$.

W.l.o.g. let the positions of Π_1 be maximal in the sense that for any $p \in \Pi_1$ and $\Xi \subseteq \mathcal{POS}(u) \cap (p\mathbf{N}^+)$ we do not have $u \mathrel{-\!\!\Vert\!\!\!\longrightarrow}_{\omega+n+1,(\Pi_1\setminus\{p\})\cup\Xi} w_1$ anymore. Then for each $p \in \Pi_1$ there are $((l_{1,p}, r_{1,p}), C_{1,p}) \in \mathrm{R}$ and $\mu_{1,p} \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $u/p = l_{1,p}\mu_{1,p}$, $r_{1,p}\mu_{1,p} = w_1/p$, $C_{1,p}\mu_{1,p}$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Finally, $w_1 = u[p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1]$.

<u>Claim 5:</u>
We may assume $l_{0,\bar{p}_0} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ and $\forall p \in \Pi_1.\ l_{1,p} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$.

<u>Proof of Claim 5:</u> In case of $l_{0,\bar{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ we get $w_0 \longleftarrow_{\omega} u$ by Lemma 13.2 (matching both its μ and ν to our $\mu_{0,\bar{p}_0}$) and then our property follows from the assumption of our lemma (below the strong commutation assumption). For the second restriction define $\Xi_1 := \{\ p \in \Pi_1 \mid l_{1,p} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})\ \}$ and $u_1' := u[\,p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1 \setminus \Xi_1\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $w_0 \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n+1} \circ \stackrel{*}{\longrightarrow}_{\omega} v_1 \stackrel{*}{\longleftarrow}_{\omega+n+1} u_1'$ for some v_1 (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{1,p}$) we get $\forall p \in \Xi_1.\ l_{1,p}\mu_{1,p} \longrightarrow_{\omega} r_{1,p}\mu_{1,p}$ and therefore $u_1' \mathrel{-\!\!\|\!\!\rightarrow}_{\omega,\Xi_1} w_1$. Thus from $v_1 \stackrel{*}{\longleftarrow}_{\omega+n_0+1} u_1' \stackrel{*}{\longrightarrow}_{\omega} w_1$ we get $v_1 \stackrel{*}{\longrightarrow}_{\omega} v_2 \stackrel{*}{\longleftarrow}_{\omega+n_0+1} w_1$ for some v_2 by ω-shallow confluence up to ω (cf. Claim 0).



<u>Q.e.d. (Claim 5)</u>

Now we start a second level of induction on $|\Pi_1|$ in $\prec$.

Define the set of top positions by

$$\Theta := \{\ p \in \{\bar{p}_0\} \cup \Pi_1 \mid \neg\exists q \in \{\bar{p}_0\} \cup \Pi_1.\ \exists q' \in \mathbf{N}^+.\ p = qq'\ \}.$$

Since the prefix ordering is wellfounded we have $\forall p \in \{\bar{p}_0\} \cup \Pi_1.\ \exists q \in \Theta.\ \exists q' \in \mathbf{N}^*.\ p = qq'$. It now suffices to show for all $q \in \Theta$

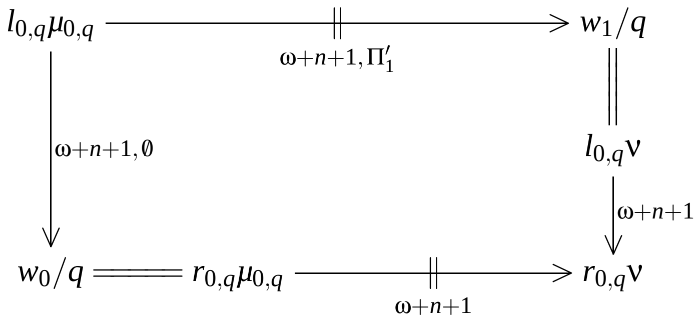
$$w_0/q \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n+1} \circ \stackrel{*}{\longrightarrow}_{\omega} \circ \stackrel{*}{\longleftarrow}_{\omega+n+1} w_1/q$$

because then we have $w_0 = w_0[\,q \leftarrow w_0/q \mid q \in \Theta\,] = u[\,\bar{p}_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}\,][\,q \leftarrow w_0/q \mid q \in \Theta\,] =$
$u[\,q \leftarrow w_0/q \mid q \in \Theta\,] \stackrel{*}{\longrightarrow}_{\omega} \circ \mathrel{-\!\!\|\!\!\rightarrow}_{\omega+n+1} \circ \stackrel{*}{\longrightarrow}_{\omega} \circ \stackrel{*}{\longleftarrow}_{\omega+n+1} u[\,q \leftarrow w_1/q \mid q \in \Theta\,] =$
$u[\,p \leftarrow r_{1,p}\mu_{1,p} \mid p \in \Pi_1\,][\,q \leftarrow w_1/q \mid q \in \Theta\,] = w_1[\,q \leftarrow w_1/q \mid q \in \Theta\,] = w_1$.

Therefore we are left with the following two cases for $q \in \Theta$:

$\underline{\underline{q \notin \Pi_1}}$: Then $q = \bar{p}_0$. Define $\Pi_1' := \{ p \mid qp \in \Pi_1 \}$. We have two cases:

$\underline{\underline{\text{“The variable overlap (if any) case”: } \forall p \in \Pi_1' \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}:}}$



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{ (p', p'') \mid l_{0,q}/p' = x \wedge p'p'' \in \Pi_1' \}$.

Claim 7: There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}. \left(\begin{array}{ll} & x\mu_{0,q} \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n+1} x\nu \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu = x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \end{array} \right).$$

Proof of Claim 7:
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$x\mu_{0,q} = x\mu_{0,q}[\,p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n+1}$$
$$x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu.$$

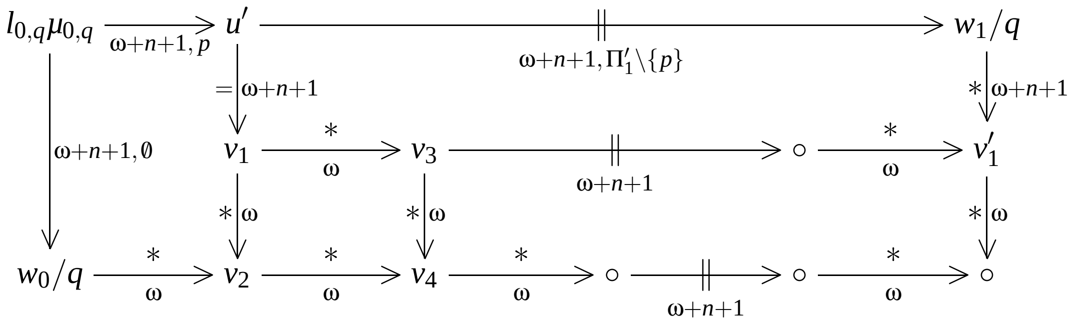
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{0,q}/p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ this implies $l_{1,qp'p''}\mu_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. Q.e.d. (Claim 7)

Claim 8: $l_{0,q}\nu = w_1/q$.
Proof of Claim 8:
By Claim 7 we get $w_1/q = u/q[\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{0,q}/p' = x \in \mathrm{V}\,] =$
$l_{0,q}[\,p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V}\,] = l_{0,q}\nu$. Q.e.d. (Claim 8)

Claim 9: $w_0/q \mathrel{+\!\!\!\!\!\!\longrightarrow}_{\omega+n+1} r_{0,q}\nu$.
Proof of Claim 9: Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. Q.e.d. (Claim 9)

By claims 8 and 9 it now suffices to show $l_{0,q}\nu \longrightarrow_{\omega+n+1} r_{0,q}\nu$, which again follows from Lemma A.7 (matching its n_0 to our $n{+}1$ and its n_1 to our n) since R, X is ω-level confluent up to n by our induction hypothesis and since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \overset{*}{\longrightarrow}_{\omega+n+1} x\nu$ by Claim 7 and Corollary 2.14.
Q.e.d. (“The variable overlap (if any) case”)

<u>“The critical peak case”: There is some $p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:</u>



<u>Claim 10:</u> $p \neq \emptyset$.
<u>Proof of Claim 10:</u> If $p = \emptyset$, then $\emptyset \in \Pi'_1$, then $q \in \Pi_1$, which contradicts our global case assumption. <u>Q.e.d. (Claim 10)</u>

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cap \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp}, r_{1,qp}), C_{1,qp}))] \cup \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{ \begin{array}{ll} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q}, r_{0,q}), C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{array} \right\} \ (x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1 \setminus \{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \longrightarrow\!\!\!\!\!\!\!\!\!\!\|\!\|\;\;_{\omega+n+1, \Pi'_1\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi'_1] = w_1/q.$$

If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \longrightarrow\!\!\!\!\!\!\!\!\!\!\|\!\|\;\;_{\omega+n+1, \Pi'_1\setminus\{p\}} w_1/q.$$

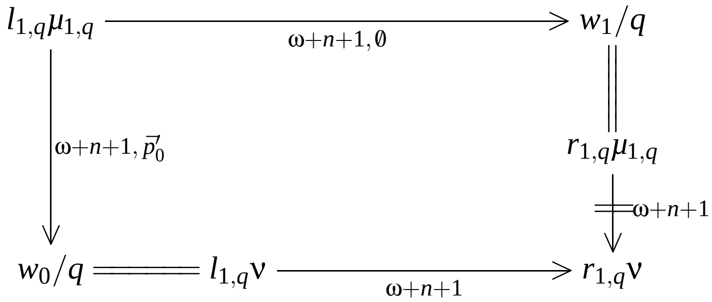
Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 1), l_{0,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R, X is ω-level confluent up to n (by our induction hypothesis) and ω-shallow confluent up to ω (by Claim 0) due to our assumed ω-level closedness (matching the definition's n to our $n+1$) we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \overset{=}{\longrightarrow}_{\omega+n+1} v_1 \overset{*}{\longrightarrow}_{\omega} v_2 \overset{*}{\longleftarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v_1, v_2. We then have $v_1 \overset{=}{\longleftarrow}_{\omega+n+1} u' \longrightarrow\!\!\!\!\!\!\!\!\!\!\|\!\|\;\;_{\omega+n+1, \Pi'_1\setminus\{p\}} w_1/q$. By $|\Pi'_1 \setminus \{p\}| \prec |\Pi'_1| \preceq |\Pi_1|$, due to our second induction level we get some v'_1 with $v_1 \overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow\!\!\!\!\!\!\!\!\!\!\|\!\|\;\;_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} v'_1 \overset{*}{\longleftarrow}_{\omega+n+1} w_1/q$. By Claim 0 we can close the peak at v_1 according to $v_2 \overset{*}{\longrightarrow}_{\omega} v_4 \overset{*}{\longleftarrow}_{\omega} v_3$ for some v_4. Finally by the assumption of our lemma (below the strong commutation assumption) the peak at v_3 can be closed according to $v_4 \overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow\!\!\!\!\!\!\!\!\!\!\|\!\|\;\;_{\omega+n} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n} v'_1$.

<u>Q.e.d. (“The critical peak case”)</u> <u>Q.e.d. (“$q \notin \Pi_1$”)</u>

$q \in \Pi_1$: If there is no $\overline{p}'_0$ with $q\overline{p}'_0 = \overline{p}_0$, then the proof is finished due to $w_0/q = u/q = l_{1,q}\mu_{1,q} \longrightarrow_{\omega+n+1} r_{1,q}\mu_{1,q} = w_1/q$. Otherwise, we can define $\overline{p}'_0$ by $q\overline{p}'_0 = \overline{p}_0$. We have two cases:

"The second variable overlap case":

There are $x \in \mathrm{V}$ and p', p'' such that $l_{1,q}/p' = x \wedge p'p'' = \overline{p}'_0$:



Claim 11a: We have $x\mu_{1,q}/p'' = l_{0,\overline{p}_0}\mu_{0,\overline{p}_0}$ and may assume $x \in \mathrm{V}_{\mathrm{SIG}}$.
Proof of Claim 11a: We have $x\mu_{1,q}/p'' = l_{1,q}\mu_{1,q}/p'p'' = u/qp'p'' = u/q\overline{p}'_0 = u/\overline{p}_0 = l_{0,\overline{p}_0}\mu_{0,\overline{p}_0}$. If $x \in \mathrm{V}_{\mathcal{C}}$, then $x\mu_{1,q} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $x\mu_{1,q}/p'' \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $l_{0,\overline{p}_0}\mu_{0,\overline{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, and then $l_{0,\overline{p}_0} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which we may assume not to be the case by Claim 5.
Q.e.d. (Claim 11a)

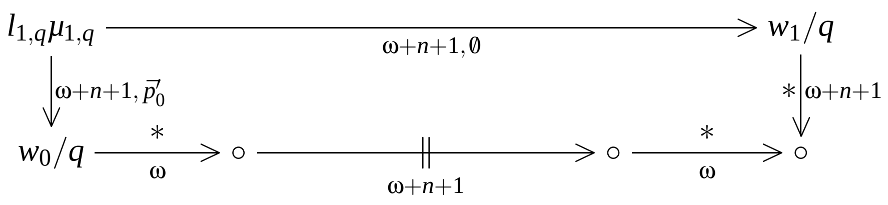
Claim 11b: We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_{1,q}[p'' \leftarrow r_{0,\overline{p}_0}\mu_{0,\overline{p}_0}]$ and $\forall y \in \mathrm{V} \setminus \{x\}.\ y\nu = y\mu_{1,q}$. Then we have $x\mu_{1,q} \longrightarrow_{\omega+n+1} x\nu$.
Proof of Claim 11b: This follows directly from Claim 11a. Q.e.d. (Claim 11b)

Claim 12: $w_0/q = l_{1,q}\nu$.
Proof of Claim 12: By the left-linearity assumption of our lemma, Claim 5, and Claim 11a we may assume $\{\, p''' \mid l_{1,q}/p''' = x \,\} = \{p'\}$. Thus, by Claim 11b we get $w_0/q =$
$u/q[\overline{p}'_0 \leftarrow r_{0,\overline{p}_0}\mu_{0,\overline{p}_0}] =$
$l_{1,q}[p''' \leftarrow y\mu_{1,q} \mid l_{1,q}/p''' = y \in \mathrm{V}][\overline{p}'_0 \leftarrow r_{0,\overline{p}_0}\mu_{0,\overline{p}_0}] =$
$l_{1,q}[p''' \leftarrow y\mu_{1,q} \mid l_{1,q}/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_{1,q}][p'p'' \leftarrow r_{0,\overline{p}_0}\mu_{0,\overline{p}_0}] =$
$l_{1,q}[p''' \leftarrow y\nu \mid l_{1,q}/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_{1,q}[p'' \leftarrow r_{0,\overline{p}_0}\mu_{0,\overline{p}_0}]] =$
$l_{1,q}[p''' \leftarrow y\nu \mid l_{1,q}/p''' = y \in \mathrm{V}] = l_{1,q}\nu$. Q.e.d. (Claim 12)

Claim 13: $r_{1,q}\nu \leftarrow\!\!\!\!+\!\!\!\!\!+\!\!\!- _{\omega+n+1} w_1/q$.
Proof of Claim 13: Since $r_{1,q}\mu_{1,q} = w_1/q$, this follows directly from Claim 11b.
Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu \longrightarrow_{\omega+n+1} r_{1,q}\nu$, which again follows from Lemma A.7 (matching its n_0 to our $n+1$ and its n_1 to our n) since R, X is ω-level confluent up to n by our induction hypothesis and since $\forall x \in \mathrm{V}.\ x\mu_{1,q} \xrightarrow{*}_{\omega+n+1} x\nu$ by Claim 11b.
Q.e.d. ("The second variable overlap case")

"The second critical peak case": $\vec{p}_0 \in \mathcal{POS}(l_{1,q}) \wedge l_{1,q}/\vec{p}_0 \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}))] \cap \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,\bar{p}_0}, r_{0,\bar{p}_0}), C_{0,\bar{p}_0}))] \cup \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{ \begin{array}{ll} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q}, r_{1,q}), C_{1,q})) \\ x\xi^{-1}\mu_{0,\bar{p}_0} & \text{else} \end{array} \right\}$ $(x \in \mathrm{V})$.
By $l_{0,\bar{p}_0}\xi\rho = l_{0,\bar{p}_0}\xi\xi^{-1}\mu_{0,\bar{p}_0} = u/\bar{p}_0 = u/q\vec{p}_0 = l_{1,q}\mu_{1,q}/\vec{p}_0 = l_{1,q}\rho/\vec{p}_0 = (l_{1,q}/\vec{p}_0)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,\bar{p}_0}\xi, l_{1,q}/\vec{p}_0)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
If $l_{1,q}[\vec{p}_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q = l_{1,q}\mu_{1,q}[\vec{p}_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] = l_{1,q}[\vec{p}_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

Otherwise we have $((l_{1,q}[\vec{p}_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma, C_{0,\bar{p}_0}\xi\sigma, 1), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, \vec{p}_0) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_{0,\bar{p}_0}\xi\sigma\varphi = C_{0,\bar{p}_0}\mu_{0,\bar{p}_0}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R, X is ω-level confluent up to n (by our induction hypothesis) and ω-shallow confluent up to ω (by Claim 0) due to our assumed ω-level weak parallel joinability (matching the definition's n to our $n{+}1$) we have $w_0/q = l_{1,q}\mu_{1,q}[\vec{p}_0 \leftarrow r_{0,\bar{p}_0}\mu_{0,\bar{p}_0}] = l_{1,q}[\vec{p}_0 \leftarrow r_{0,\bar{p}_0}\xi]\sigma\varphi \overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow\!\!\!\!\!\!\!+\!\!\!+_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$.

Q.e.d. ("The second critical peak case") **Q.e.d. (Lemma A.9)**

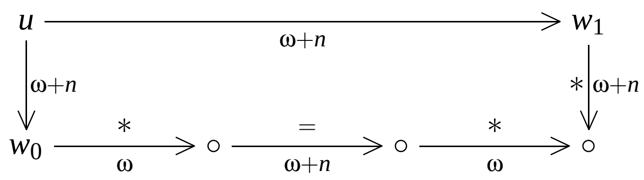
Proof of Lemma A.10

Claim 0: R,X is ω-shallow confluent up to ω.
Proof of Claim 0: Directly by the assumed strong commutation, cf. the proofs of the claims 2 and 3 of the proof of Lemma A.1. Q.e.d. (Claim 0)

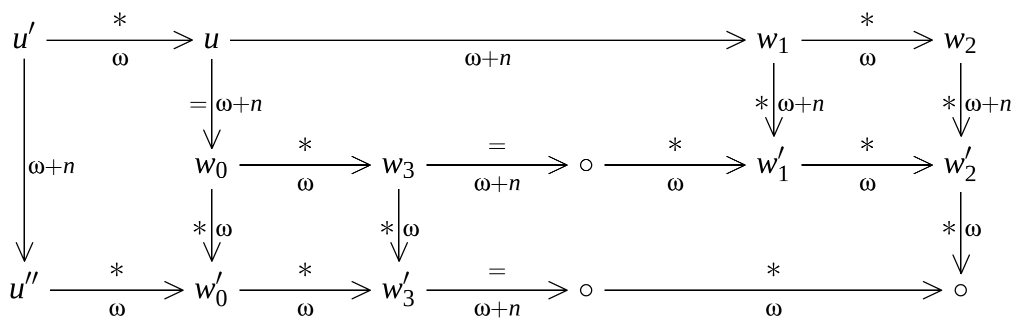
Claim 1: If $\stackrel{*}{\longrightarrow}_{\omega}\circ\longrightarrow_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}$ strongly commutes over $\stackrel{*}{\longrightarrow}_{\omega+n}$, then $\longrightarrow_{\omega+n}$ is confluent.
Proof of Claim 1: $\stackrel{*}{\longrightarrow}_{\omega}\circ\longrightarrow_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}$ and $\stackrel{*}{\longrightarrow}_{\omega+n}$ are commuting by Lemma 3.3. Since by Lemma 2.12 we have $\longrightarrow_{\omega+n}\subseteq\stackrel{*}{\longrightarrow}_{\omega}\circ\longrightarrow_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}\subseteq\stackrel{*}{\longrightarrow}_{\omega+n}$, now $\longrightarrow_{\omega+n}$ and $\longrightarrow_{\omega+n}$ are commuting, too. Q.e.d. (Claim 1)

For $n\prec\omega$ we are going to show by induction on n the following property:

$$w_0\longleftarrow_{\omega+n}u\longrightarrow_{\omega+n}w_1 \quad\Rightarrow\quad w_0\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{=}{\longrightarrow}_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{*}{\longleftarrow}_{\omega+n}w_1.$$



Claim 2: Let $\delta\prec\omega$. If $\forall n\preceq\delta.\ \forall w_0,w_1,u.\ \left(\left(\begin{array}{c}w_0\longleftarrow_{\omega+n}u\longrightarrow_{\omega+n}w_1\\ \Rightarrow\ w_0\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{=}{\longrightarrow}_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{*}{\longleftarrow}_{\omega+n}w_1\end{array}\right)\right)$, then $\forall n\preceq\delta.\left(\stackrel{*}{\longrightarrow}_{\omega}\circ\longrightarrow_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}\text{ strongly commutes over }\stackrel{*}{\longrightarrow}_{\omega+n}\right)$, and R,X is ω-level confluent up to δ.
Proof of Claim 2: First we show the strong commutation. Assume $n\preceq\delta$. By Lemma 3.3 it suffices to show that $\stackrel{*}{\longrightarrow}_{\omega}\circ\longrightarrow_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}$ strongly commutes over $\longrightarrow_{\omega+n}$. Assume $u''\longleftarrow_{\omega+n}u'\stackrel{*}{\longrightarrow}_{\omega}u\longrightarrow_{\omega+n}w_1\stackrel{*}{\longrightarrow}_{\omega}w_2$ (cf. diagram below). By the strong commutation assumed for our lemma, there are w_0 and w_0' with $u''\stackrel{*}{\longrightarrow}_{\omega}w_0'\stackrel{*}{\longleftarrow}_{\omega}w_0\stackrel{=}{\longleftarrow}_{\omega+n}u$. By the above property there are some w_3, w_1' with $w_0\stackrel{*}{\longrightarrow}_{\omega}w_3\stackrel{=}{\longrightarrow}_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}w_1'\stackrel{*}{\longleftarrow}_{\omega+n}w_1$. By Claim 0 we can close the peak $w_1'\stackrel{*}{\longleftarrow}_{\omega+n}w_1\stackrel{*}{\longrightarrow}_{\omega}w_2$ according to $w_1'\stackrel{*}{\longrightarrow}_{\omega}w_2'\stackrel{*}{\longleftarrow}_{\omega+n}w_2$ for some w_2'. By Claim 0 again, we can close the peak $w_0'\stackrel{*}{\longleftarrow}_{\omega}w_0\stackrel{*}{\longrightarrow}_{\omega}w_3$ according to $w_0'\stackrel{*}{\longrightarrow}_{\omega}w_3'\stackrel{*}{\longleftarrow}_{\omega}w_3$ for some w_3'. To close the whole diagram, we only have to show that we can close the peak $w_3'\stackrel{*}{\longleftarrow}_{\omega}w_3\stackrel{=}{\longrightarrow}_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}w_2'$ according to $w_3'\stackrel{=}{\longrightarrow}_{\omega+n}\circ\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{*}{\longleftarrow}_{\omega}w_2'$, which is possible due to the strong commutation assumed for our lemma or due to Claim 0.



Finally we show ω-level confluence up to δ. Assume $n_0, n_1 \prec \omega$ with $\max\{n_0,n_1\} \preceq \delta$ and $w_0 \overset{*}{\longleftarrow}_{\omega+n_0} u \overset{*}{\longrightarrow}_{\omega+n_1} w_1$. By Lemma 2.12 we get $w_0 \overset{*}{\longleftarrow}_{\omega+\max\{n_0,n_1\}} u \overset{*}{\longrightarrow}_{\omega+\max\{n_0,n_1\}} w_1$. Since $\max\{n_0,n_1\} \preceq \delta$, above we have shown that $\overset{*}{\longrightarrow}_{\omega} \circ \longrightarrow_{\omega+\max\{n_0,n_1\}} \circ \overset{*}{\longrightarrow}_{\omega}$ strongly commutes over $\overset{*}{\longrightarrow}_{\omega+\max\{n_0,n_1\}}$. By Claim 1 we finally get $w_0 \overset{*}{\longrightarrow}_{\omega+\max\{n_0,n_1\}} \circ \overset{*}{\longleftarrow}_{\omega+\max\{n_0,n_1\}} w_1$ as desired. <u>Q.e.d. (Claim 2)</u>

Note that for $n=0$ our property follows from Claim 0.

The benefit of Claim 2 is twofold: First, it says that our lemma is valid if the above property holds for all $n \prec \omega$. Second, it strengthens the property when used as induction hypothesis. Thus (writing $n+1$ instead of n since we may assume $0 \prec n$) it now suffices to show for $n \prec \omega$ that

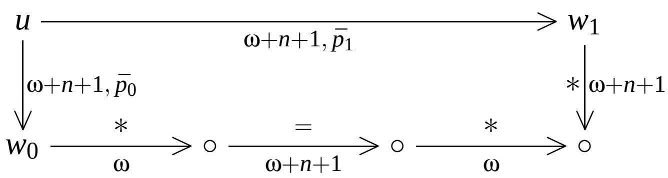
$$w_0 \longleftarrow_{\omega+n+1,\bar{p}_0} u \longrightarrow_{\omega+n+1,\bar{p}_1} w_1$$

together with our induction hypotheses that

$$\mathrm{R}, \mathrm{X} \text{ is } \omega\text{-level confluent up to } n$$

implies

$$w_0 \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} w_1 .$$



Now for each $i \prec 2$ there are $((l_i,r_i),C_i) \in \mathrm{R}$ and $\mu_i \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $u/\bar{p}_i = l_i\mu_i$, $w_i = u[\,\bar{p}_i \leftarrow r_i\mu_i\,]$, and $C_i\mu_i$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$.

<u>Claim 5:</u> We may assume $\forall i \prec 2.\ l_i \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$.
<u>Proof of Claim 5:</u> In case of $l_i \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ we get $u \longrightarrow_{\omega} w_i$ by Lemma 13.2 (matching both its μ and ν to our μ_i). In case of "$i=0$" our property follows from the strong commutation assumption of our lemma. In case of "$i=1$" our property follows from Claim 0. <u>Q.e.d. (Claim 5)</u>

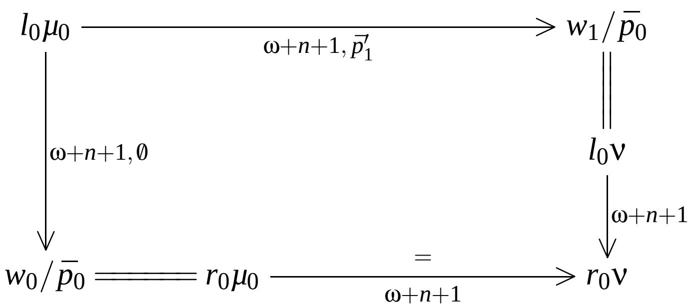
In case of $\bar{p}_0 \parallel \bar{p}_1$ we have $w_i/\bar{p}_{1-i} = u[\,\bar{p}_i \leftarrow r_i\mu_i\,]/\bar{p}_{1-i} = u/\bar{p}_{1-i} = l_{1-i}\mu_{1-i}$ and therefore $w_i \longrightarrow_{\omega+n+1} u[\,\bar{p}_k \leftarrow r_k\mu_k \mid k \prec 2\,]$, i.e. our proof is finished. Thus, according to whether $\bar{p}_0$ is a prefix of $\bar{p}_1$ or vice versa, we have the following two cases left:

<u>There is some $\overline{p'_1}$ with $\overline{p_0}\,\overline{p'_1} = \overline{p_1}$ and $\overline{p'_1} \neq \emptyset$:</u>

We have two cases:

<u>"The variable overlap case":</u>

<u>There are $x \in \mathrm{V}$ and p', p'' such that $l_0/p' = x \wedge p'p'' = \overline{p'_1}$:</u>



<u>Claim 6:</u> We have $x\mu_0/p'' = l_1\mu_1$ and may assume $x \in \mathrm{V_{SIG}}$.
<u>Proof of Claim 6:</u> We have $x\mu_0/p'' = l_0\mu_0/p'p'' = u/\overline{p_0}p'p'' = u/\overline{p_0}\,\overline{p'_1} = u/\overline{p_1} = l_1\mu_1$.
If $x \in \mathrm{V}_\mathcal{C}$, then $x\mu_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_\mathcal{C})$, then $x\mu_0/p'' \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_\mathcal{C})$, then $l_1\mu_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_\mathcal{C})$, and then $l_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ which we may assume not to be the case by Claim 5. <u>Q.e.d. (Claim 6)</u>

<u>Claim 7:</u> We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_0[p'' \leftarrow r_1\mu_1]$ and $\forall y \in \mathrm{V} \setminus \{x\}.\ y\nu = y\mu_0$. Then we have $x\mu_0 \longrightarrow_{\omega+n+1} x\nu$.
<u>Proof of Claim 7:</u> This follows directly from Claim 6. <u>Q.e.d. (Claim 7)</u>

<u>Claim 8:</u> $l_0\nu = w_1/\overline{p_0}$.
<u>Proof of Claim 8:</u> By the left-linearity assumption of our lemma and claims 5 and 6 we may assume $\{\, p''' \mid l_0/p''' = x \,\} = \{p'\}$. Thus, by Claim 7 we get $w_1/\overline{p_0} = u/\overline{p_0}[\overline{p'_1} \leftarrow r_1\mu_1] =$
$l_0[p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V}][\overline{p'_1} \leftarrow r_1\mu_1] =$
$l_0[p''' \leftarrow y\mu_0 \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_0][p'p'' \leftarrow r_1\mu_1] =$
$l_0[p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_0[p'' \leftarrow r_1\mu_1]] =$
$l_0[p''' \leftarrow y\nu \mid l_0/p''' = y \in \mathrm{V}] = l_0\nu$. <u>Q.e.d. (Claim 8)</u>

<u>Claim 9:</u> $w_0/\overline{p_0} \overset{=}{\longrightarrow}_{\omega+n+1} r_0\nu$.
<u>Proof of Claim 9:</u> By the right-linearity assumption of our lemma and claims 5 and 6 we may assume $|\{\, p''' \mid r_0/p''' = x \,\}| \preceq 1$. Thus by Claim 7 we get: $w_0/\overline{p_0} = r_0\mu_0 =$
$r_0[p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V} \setminus \{x\}][p''' \leftarrow x\mu_0 \mid r_0/p''' = x] \overset{=}{\longrightarrow}_{\omega+n+1}$
$r_0[p''' \leftarrow y\mu_0 \mid r_0/p''' = y \in \mathrm{V} \setminus \{x\}][p''' \leftarrow x\nu \mid r_0/p''' = x] =$
$r_0[p''' \leftarrow y\nu \mid r_0/p''' = y \in \mathrm{V} \setminus \{x\}][p''' \leftarrow x\nu \mid r_0/p''' = x] = r_0\nu$. <u>Q.e.d. (Claim 9)</u>

By claims 8 and 9 it now suffices to show $l_0\nu \longrightarrow_{\omega+n+1} r_0\nu$, which again follows from Lemma A.7 since R, X is ω-level confluent up to n by our induction hypothesis and since $\forall y \in \mathrm{V}.\ y\mu_0 \overset{*}{\longrightarrow}_{\omega+n+1} y\nu$ by Claim 7. <u>Q.e.d. ("The variable overlap case")</u>

"The critical peak case": $\overline{p'_1} \in \mathcal{POS}(l_0) \wedge l_0/\overline{p'_1} \notin \mathrm{V}$:

$$\begin{array}{ccccccc}
l_0\mu_0 & & & \xrightarrow[\omega+n+1,\,\overline{p'_1}]{} & & & w_1/\overline{p_0} \\
\Big\downarrow{\scriptstyle \omega+n+1,\emptyset} & & & & & & {\scriptstyle *}\Big\downarrow{\scriptstyle \omega+n+1} \\
w_0/\overline{p_0} & \xrightarrow[\omega]{*} & \circ & \xrightarrow[\omega+n+1]{=} & \circ & \xrightarrow[\omega]{*} & \circ
\end{array}$$

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_1,r_1),C_1))] \cap \mathcal{V}(((l_0,r_0),C_0)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_1,r_1),C_1))] \cup \mathcal{V}(((l_0,r_0),C_0))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_0 & \text{if } x \in \mathcal{V}(((l_0,r_0),C_0)) \\ x\xi^{-1}\mu_1 & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_1\xi\rho = l_1\xi\xi^{-1}\mu_1 = u/\overline{p_1} = u/\overline{p_0}\,\overline{p'_1} = l_0\mu_0/\overline{p'_1} = l_0\rho/\overline{p'_1} = (l_0/\overline{p'_1})\rho$
let $\sigma := \mathrm{mgu}(\{(l_1\xi, l_0/\overline{p'_1})\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
If $l_0[\overline{p'_1} \leftarrow r_1\xi]\sigma = r_0\sigma$, then the proof is finished due to

$$w_0/\overline{p_0} = r_0\mu_0 = r_0\sigma\varphi = l_0[\overline{p'_1} \leftarrow r_1\xi]\sigma\varphi = l_0\mu_0[\overline{p'_1} \leftarrow r_1\mu_1] = w_1/\overline{p_0}.$$

Otherwise we have $((l_0[\overline{p'_1} \leftarrow r_1\xi], C_1\xi, 1), (r_0, C_0, 1), l_0, \sigma, \overline{p'_1}) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $\overline{p'_1} \neq \emptyset$ (due the global case assumption); $C_1\xi\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_0\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R,X is ω-level confluent up to n (by our induction hypothesis) and ω-shallow confluent up to ω, due to our assumed ω-level anti-closedness (matching the definition's n to our $n+1$) we have $w_1/\overline{p_0} = l_0\mu_0[\overline{p'_1} \leftarrow r_1\mu_1] = l_0[\overline{p'_1} \leftarrow r_1\xi]\sigma\varphi \stackrel{*}{\longrightarrow}_{\omega+n+1} \circ \stackrel{*}{\longleftarrow}_{\omega} \circ \stackrel{=}{\longleftarrow}_{\omega+n+1} \circ \stackrel{*}{\longleftarrow}_{\omega} r_0\sigma\varphi = r_0\mu_0 = w_0/\overline{p_0}$.

Q.e.d. ("The critical peak case")

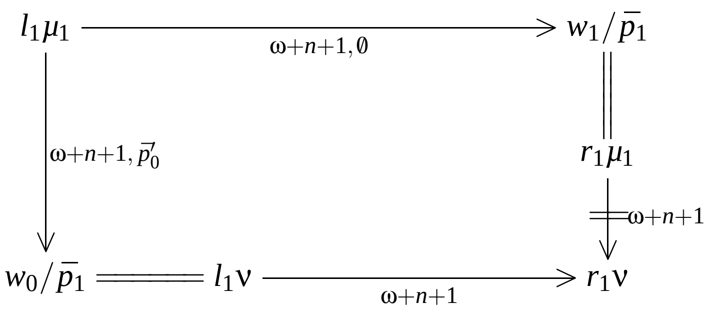
Q.e.d. ("There is some $\overline{p'_1}$ with $\overline{p_0}\,\overline{p'_1} = \overline{p_1}$ and $\overline{p'_1} \neq \emptyset$")

There is some $\bar{p}_0'$ with $\bar{p}_1\bar{p}_0' = \bar{p}_0$:

We have two cases:

"The second variable overlap case":

There are $x \in \mathrm{V}$ and p', p'' such that $l_1/p' = x \wedge p'p'' = \bar{p}_0'$:



Claim 11a: We have $x\mu_1/p'' = l_0\mu_0$ and may assume $x \in \mathrm{V}_{\mathrm{SIG}}$.
Proof of Claim 11a: We have $x\mu_1/p'' = l_1\mu_1/p'p'' = u/\bar{p}_1p'p'' = u/\bar{p}_1\bar{p}_0' = u/\bar{p}_0 = l_0\mu_0$.
If $x \in \mathrm{V}_{\mathcal{C}}$, then $x\mu_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $x\mu_1/p'' \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, then $l_0\mu_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, and then $l_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which we may assume not to be the case by Claim 5. Q.e.d. (Claim 11a)

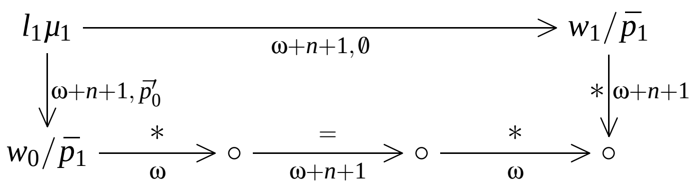
Claim 11b: We can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by $x\nu = x\mu_1[p'' \leftarrow r_0\mu_0]$ and $\forall y \in \mathrm{V} \setminus \{x\}.\ y\nu = y\mu_1$. Then we have $x\mu_1 \longrightarrow_{\omega+n+1} x\nu$.
Proof of Claim 11b: This follows directly from Claim 11a. Q.e.d. (Claim 11b)

Claim 12: $w_0/\bar{p}_1 = l_1\nu$.
Proof of Claim 12:
By the left-linearity assumption of our lemma and claims 5 and 11a we may assume $\{\, p''' \mid l_1/p''' = x \,\} = \{p'\}$. Thus, by Claim 11b we get $w_0/\bar{p}_1 = u/\bar{p}_1[\bar{p}_0' \leftarrow r_0\mu_0] =$
$l_1[p''' \leftarrow y\mu_1 \mid l_1/p''' = y \in \mathrm{V}][\bar{p}_0' \leftarrow r_0\mu_0] =$
$l_1[p''' \leftarrow y\mu_1 \mid l_1/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_1][p'p'' \leftarrow r_0\mu_0] =$
$l_1[p''' \leftarrow y\nu \mid l_1/p''' = y \in \mathrm{V} \wedge y \neq x][p' \leftarrow x\mu_1[p'' \leftarrow r_0\mu_0]] =$
$l_1[p''' \leftarrow y\nu \mid l_1/p''' = y \in \mathrm{V}] = l_1\nu$. Q.e.d. (Claim 12)

Claim 13: $r_1\nu \leftarrow\!\!\!+\!\!\!+\!\!-_{\omega+n+1} w_1/\bar{p}_1$.
Proof of Claim 13: Since $r_1\mu_1 = w_1/\bar{p}_1$, this follows directly from Claim 11b. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_1\nu \longrightarrow_{\omega+n+1} r_1\nu$, which again follows from Claim 11b, Lemma A.7 (matching its n_0 to our $n+1$ and its n_1 to our n), and our induction hypothesis that R, X is ω-level confluent up to n.
Q.e.d. ("The second variable overlap case")

“The second critical peak case”: $\overline{p'_0} \in \mathcal{POS}(l_1) \wedge l_1/\overline{p'_0} \notin \mathrm{V}$:



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_0,r_0),C_0))] \cap \mathcal{V}(((l_1,r_1),C_1)) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_0,r_0),C_0))] \cup \mathcal{V}(((l_1,r_1),C_1))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{matrix} x\mu_1 & \text{if } x \in \mathcal{V}(((l_1,r_1),C_1)) \\ x\xi^{-1}\mu_0 & \text{else} \end{matrix}\right\}$ $(x \in \mathrm{V})$.
By $l_0\xi\rho = l_0\xi\xi^{-1}\mu_0 = u/\overline{p_0} = u/\overline{p_1}\overline{p'_0} = l_1\mu_1/\overline{p'_0} = l_1\rho/\overline{p'_0} = (l_1/\overline{p'_0})\rho$
let $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/\overline{p'_0})\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}{\upharpoonleft}(\sigma\varphi) = {}_{\mathrm{Y}}{\upharpoonleft}\rho$.
If $l_1[\overline{p'_0} \leftarrow r_0\xi]\sigma = r_1\sigma$, then the proof is finished due to

$$w_0/\overline{p_1} = l_1\mu_1[\overline{p'_0} \leftarrow r_0\mu_0] = l_1[\overline{p'_0} \leftarrow r_0\xi]\sigma\varphi = r_1\sigma\varphi = r_1\mu_1 = w_1/\overline{p_1}.$$

Otherwise we have $((l_1[\overline{p'_0} \leftarrow r_0\xi], C_0\xi, 1), (r_1, C_1, 1), l_1, \sigma, \overline{p'_0}) \in \mathrm{CP(R)}$ (due to Claim 5); $C_0\xi\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $C_1\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Since R, X is ω-level confluent up to n (by our induction hypothesis) and ω-shallow confluent up to ω, due to our assumed ω-level strong joinability (matching the definition’s n to our $n{+}1$) we have $w_0/\overline{p_1} = l_1\mu_1[\overline{p'_0} \leftarrow r_0\mu_0] = l_1[\overline{p'_0} \leftarrow r_0\xi]\sigma\varphi \overset{*}{\longrightarrow}_{\omega} \circ \overset{=}{\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega+n+1} r_1\sigma\varphi = r_1\mu_1 = w_1/\overline{p_1}$.
Q.e.d. (“The second critical peak case”) **Q.e.d. (Lemma A.10)**

Proof of Lemma B.1

Due to $\mathcal{T}$-monotonicity of $>$ and $> \subseteq \rhd$, it is easy to show by induction over β in $\prec$ that $\forall\beta \preceq \omega{+}\alpha.\ \longrightarrow_{\mathrm{R,X},\beta} \subseteq \rhd$ using Lemma 2.12.

Proof of Lemma B.2

Claim 0: $\forall u \in \mathcal{TERMS}(C).\ \forall \hat{u} \in \mathcal{T}(\mathrm{sig}, \mathrm{X}).\ \left(\ u\mu \overset{*}{\longrightarrow} \hat{u} \ \Rightarrow\ u\nu{\downarrow}\hat{u}\ \right).$

Proof of Claim 1: We get the following cases:

$l\mu \rhd u\mu$: $u\nu \overset{*}{\longleftarrow} u\mu \overset{*}{\longrightarrow} \hat{u}$ implies $u\nu{\downarrow}\hat{u}$ by the assumed confluence below $u\mu$.

$u\mu \notin \mathrm{dom}(\longrightarrow)$: $u\nu \overset{*}{\longleftarrow} u\mu \overset{*}{\longrightarrow} \hat{u}$ implies $u\nu = u\mu = \hat{u}$.

[$\mathcal{V}(u) \subseteq \mathrm{V}_{\mathcal{C}}$: By Lemma 2.10 we get $\forall x \in \mathcal{V}(u).\ x\mu \overset{*}{\longrightarrow}_{\omega} x\nu$. Thus from $u\nu \overset{*}{\longleftarrow}_{\omega} u\mu \overset{*}{\longrightarrow} \hat{u}$ due to the assumed $\overset{*}{\longleftarrow}_{\mathrm{R,X},\omega} \circ \overset{*}{\longrightarrow}_{\mathrm{R,X}} \subseteq {\downarrow}$ we get $u\nu{\downarrow}\hat{u}$.] Q.e.d. (Claim 0)

By Lemma 2.7 it suffices to show that $C\nu$ is fulfilled. For each L in C we have to show that $L\nu$ is fulfilled. Note that we already know that $L\mu$ is fulfilled.

$L = (u{=}v)$: There is some $\hat{u}$ with $u\mu \overset{*}{\longrightarrow} \hat{u} \overset{*}{\longleftarrow} v\mu$. By Claim 0 there is some $\hat{v}$ with $u\nu \overset{*}{\longrightarrow} \hat{v} \overset{*}{\longleftarrow} \hat{u} \overset{*}{\longleftarrow} v\mu$. Thus, by Claim 0 we get $u\nu \overset{*}{\longrightarrow} \hat{v}{\downarrow}v\nu$.

$L = (\mathrm{Def}\,u)$: We know the existence of $\hat{u} \in \mathcal{GT}(\mathrm{cons})$ with $u\mu \overset{*}{\longrightarrow} \hat{u}$. By Claim 0 we get $u\nu \overset{*}{\longrightarrow} u' \overset{*}{\longleftarrow} \hat{u}$ for some u'. By Lemma 2.10 we get $u' \in \mathcal{GT}(\mathrm{cons})$.

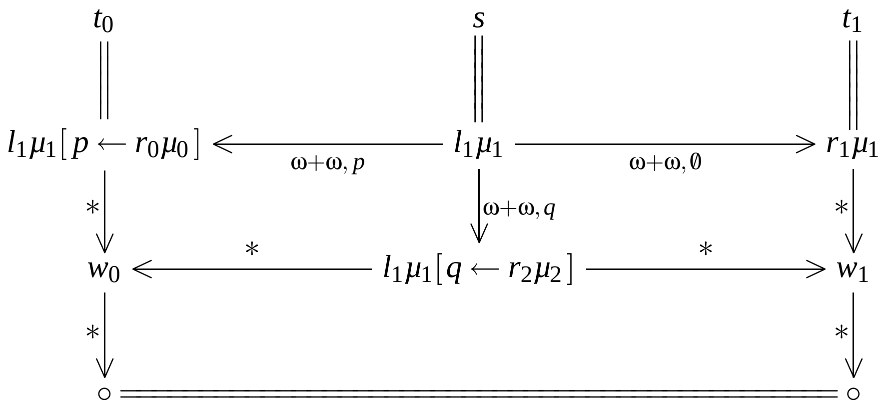
$L = (u{\neq}v)$: We know the existence of $\hat{u}, \hat{v} \in \mathcal{GT}(\mathrm{cons})$ with $u\mu \overset{*}{\longrightarrow} \hat{u} {\ddagger} \hat{v} \overset{*}{\longleftarrow} v\mu$. Just like above we get $u', v' \in \mathcal{GT}(\mathrm{cons})$ with $u\nu \overset{*}{\longrightarrow} u' \overset{*}{\longleftarrow} \hat{u}$ and $\hat{v} \overset{*}{\longrightarrow} v' \overset{*}{\longleftarrow} v\nu$. Due to $\hat{u} {\ddagger} \hat{v}$ we finally get $u' {\ddagger} v'$. **Q.e.d. (Lemma B.2)**

Proof of Lemma B.3

First notice that the usual modularization of the proof for the unconditional analogue of the theorem (by showing first that local confluence is guaranteed except for the cases that are matched by critical peaks (the so-called "critical pair lemma")) is not possible here because we need the confluence property to hold for the condition terms even for the cases that are not matched by critical peaks. Now to the proof: For all $s \in \mathcal{T}(\mathrm{sig}, \mathrm{X})$ we are going to prove confluence below s by induction over s in $\lhd$. Let s be minimal in $\lhd$ such that $\longrightarrow$ is not confluent below s. Because of $\longrightarrow \subseteq \rhd$ (by Lemma B.1) and minimality of s, $\longrightarrow$ is not even locally confluent below s. Let $p, q \in \mathcal{POS}(s)$; $t_0 \longleftarrow_{\omega+\omega,p} s \longrightarrow_{\omega+\omega,q} t_1$; $t_0 {\ddagger} t_1$. Now as one of p, q must be a prefix of the other, w.l.o.g. say that q is a prefix of p. As $s \trianglerighteq s/q$, by the minimality of s we have $q = \emptyset$. We start a second level of induction on p in $\lll_s$. Thus assume that p is minimal such that there are $p \in \mathcal{POS}(s)$ and $t_0, t_1 \in \mathcal{T}(\mathrm{sig}, \mathrm{X})$ with $t_0 \longleftarrow_{\omega+\omega,p} s \longrightarrow_{\omega+\omega,\emptyset} t_1$ and $t_0 {\ddagger} t_1$.

Now for $k < 2$ there must be $((l_k, r_k), C_k) \in \mathrm{R}$; $\mu_k \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; with $C_k\mu_k$ fulfilled; $s = l_1\mu_1$; $s/p = l_0\mu_0$; $t_0 = l_1\mu_1[p \leftarrow r_0\mu_0]$; $t_1 = r_1\mu_1$. Moreover, for $k < 2$ we define $\Lambda_k := \left\{ \begin{array}{ll} 0 & \text{if } l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}}) \\ 1 & \text{otherwise} \end{array} \right\}.$

<u>Claim 0:</u> We may assume that $\forall q\in\mathcal{POS}(s).\ \big(\ \emptyset\neq q \lll_s p\ \Rightarrow\ s\notin\mathrm{dom}(\longrightarrow_{\omega+\omega,q})\ \big).$
<u>Proof of Claim 0:</u> Otherwise there must be some $q\in\mathcal{POS}(s)$; $((l_2,r_2),C_2)\in\mathrm{R}$; $\mu_2\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $C_2\mu_2$ fulfilled; $s/q=l_2\mu_2$; and $\emptyset\neq q\lll_s p$. By our second induction level we get $l_1\mu_1[q\leftarrow r_2\mu_2]\overset{*}{\longrightarrow}w_1\overset{*}{\longleftarrow}r_1\mu_1$ for some w_1; cf. the diagram below. Next we are going to show that there is some w_0 with $l_1\mu_1[p\leftarrow r_0\mu_0]\overset{*}{\longrightarrow}w_0\overset{*}{\longleftarrow}l_1\mu_1[q\leftarrow r_2\mu_2]$. Note that (since $\longrightarrow\subseteq\rhd$ implies $s\rhd l_1\mu_1[q\leftarrow r_2\mu_2]$) this finishes the proof of Claim 0 since then $w_0\overset{*}{\longleftarrow}l_1\mu_1[q\leftarrow r_2\mu_2]\overset{*}{\longrightarrow}w_1$ by our first level of induction implies the contradictory $t_0\overset{*}{\longrightarrow}w_0\downarrow w_1\overset{*}{\longleftarrow}t_1$.



In case of $p\parallel q$ we simply can choose $w_0:=l_1\mu_1[p\leftarrow r_0\mu_0][q\leftarrow r_2\mu_2]$. Otherwise, there must be some $\bar{p}$, $\hat{p}$, $\hat{q}$, with $p=\bar{p}\hat{p}$, $q=\bar{p}\hat{q}$, and $(\hat{p}=\emptyset\vee\hat{q}=\emptyset)$. Now it suffices to show

$$s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\overset{*}{\longrightarrow}w_0'\overset{*}{\longleftarrow}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]$$

for some w_0', because by $\mathcal{T}(\mathrm{sig},\mathrm{X})$-monotonicity of $\overset{*}{\longrightarrow}$ we then have
$l_1\mu_1[p\leftarrow r_0\mu_0]=s[\bar{p}\hat{p}\leftarrow r_0\mu_0]=s[\bar{p}\leftarrow s/\bar{p}][\bar{p}\hat{p}\leftarrow r_0\mu_0]=s[\bar{p}\leftarrow s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]]\overset{*}{\longrightarrow}$
$s[\bar{p}\leftarrow w_0']$
$\overset{*}{\longleftarrow}s[\bar{p}\leftarrow s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]]=s[\bar{p}\leftarrow s/\bar{p}][\bar{p}\hat{q}\leftarrow r_2\mu_2]=s[\bar{p}\hat{q}\leftarrow r_2\mu_2]=l_1\mu_1[q\leftarrow r_2\mu_2]$.
Note that

$$s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\longleftarrow_{\omega+\omega,\hat{p}}s/\bar{p}\longrightarrow_{\omega+\omega,\hat{q}}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2].$$

In case of $\bar{p}\neq\emptyset$ (since then $\rhd_{\mathrm{ST}}\subseteq\rhd$ implies $s\rhd s/\bar{p}$) we get some w_0' with $s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\overset{*}{\longrightarrow}w_0'\overset{*}{\longleftarrow}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]$ by our first level of induction. Otherwise, in case of $\bar{p}=\emptyset$, our disjunction from above means $(p=\emptyset\vee q=\emptyset)$. Since we have $\emptyset\neq q$ by our initial assumption, we may assume $q=\hat{q}\neq\emptyset$ and $p=\hat{p}=\bar{p}=\emptyset$. Then the above divergence reads $s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\longleftarrow_{\omega+\omega,\emptyset}s\longrightarrow_{\omega+\omega,q}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]$ and we get the required joinability by our second induction level due to $q\lll_s p$. <u>Q.e.d. (Claim 0)</u>

Claim 1: In case of $\longleftarrow_{\omega} \circ \longrightarrow \subseteq \downarrow$ we may assume $s \notin \mathrm{dom}(\longrightarrow_{\omega})$.
Proof of Claim 1: Assume $\longleftarrow_{\omega} \circ \longrightarrow \subseteq \downarrow$. If there is a t_2 with $s \longrightarrow_{\omega} t_2$ then we get some t'_0, t'_1 with $t_0 \overset{*}{\longrightarrow} t'_0 \overset{*}{\longleftarrow} t_2 \overset{*}{\longrightarrow} t'_1 \overset{*}{\longleftarrow} t_1$. Due $\longrightarrow_{\omega} \subseteq \longrightarrow \subseteq \rhd$ by our first level of induction we get the contradictory $t_0 \overset{*}{\longrightarrow} t'_0 \downarrow t'_1 \overset{*}{\longleftarrow} t_1$. Q.e.d. (Claim 1)

Claim 2: In case of $\longleftarrow_{\omega} \circ \longrightarrow \subseteq \downarrow$ for each $k \prec 2$ we may assume:
$l_k\mu_k \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ and

$$(\, l_k \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}}) \vee \mathcal{TERMS}(C_k\mu_k) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}}) \,).$$

Proof of Claim 2: By Lemma 2.10 and $l_k\mu_k \longrightarrow r_k\mu_k$, $l_k\mu_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ implies $l_k\mu_k \longrightarrow_{\omega} r_k\mu_k$ which we may assume not to be the case by Claim 1. In case of $l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ and $\mathcal{TERMS}(C_k\mu_k) \subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ by Lemma 2.10 $C_k\mu_k$ is fulfilled w.r.t. $\longrightarrow_{\omega}$ and then Corollary 2.6 implies $l_k\mu_k \longrightarrow_{\omega} r_k\mu_k$ again, which we may assume not to be the case by Claim 1. Q.e.d. (Claim 2)

Now we have two cases:

The variable overlap case: $p = q_0 q_1$; $l_1/q_0 = x \in \mathrm{V}$:

We have $x\mu_1/q_1 = l_1\mu_1/q_0 q_1 = s/p = l_0\mu_0$. By Lemma 2.10 (in case of $x \in \mathrm{V}_{\mathcal{C}}$), we can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by ($y \in \mathrm{V}$):

$$y\nu := \left\{ \begin{array}{ll} x\mu_1[q_1 \leftarrow r_0\mu_0] & \text{if } y = x \\ y\mu_1 & \text{otherwise} \end{array} \right\} \text{ and get } y\mu_1 \overset{=}{\longrightarrow} y\nu \text{ for } y \in \mathrm{V}. \text{ By Corollary 2.8:}$$

$t_0 = l_1\mu_1[q_0 q_1 \leftarrow r_0\mu_0] = l_1[q_0 \leftarrow x\nu][q' \leftarrow y\mu_1 \mid l_1/q' = y \in \mathrm{V} \wedge q' \neq q_0] \overset{*}{\longrightarrow}$
$l_1[q' \leftarrow y\nu \mid l_1/q' = y \in \mathrm{V}] = l_1\nu$;
$t_1 = r_1\mu_1 \overset{*}{\longrightarrow} r_1\nu$. It suffices to show $l_1\nu \longrightarrow r_1\nu$, which follows from Lemma B.2 because of [$\overset{*}{\longleftarrow}_{\omega} \circ \overset{*}{\longrightarrow} \subseteq \downarrow$,] $l_1\mu_1 = s$ and our first level of induction. Q.e.d. (The variable overlap case)

The critical peak case: $p \in \mathcal{POS}(l_1);\ l_1/p \notin \mathrm{V}$: Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)] \cap \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) = \emptyset$. Define $\mathrm{Y} := \mathcal{V}((l_0{=}r_0{\longleftarrow}C_0)\xi, l_1{=}r_1{\longleftarrow}C_1)$. Let ρ be given by $x\rho = \begin{cases} x\mu_1 & \text{if } x \in \mathcal{V}(l_1{=}r_1{\longleftarrow}C_1) \\ x\xi^{-1}\mu_0 & \text{else} \end{cases}$ $(x \in \mathrm{V})$. By $l_0\xi\rho = l_0\xi\xi^{-1}\mu_0 = s/p = l_1\mu_1/p = l_1\rho/p = (l_1/p)\rho$ let $\sigma := \mathrm{mgu}(\{(l_0\xi, l_1/p)\}, \mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.

Claim A: We may assume $\left(p{=}\emptyset \ \vee\ \forall y{\in}\mathcal{V}(l_1).\ y\sigma\varphi \notin \mathrm{dom}(\longrightarrow) \right)$.
Proof of Claim A: Otherwise, when $p \neq \emptyset$ holds but $\forall y{\in}\mathcal{V}(l_1).\ y\sigma\varphi \notin \mathrm{dom}(\longrightarrow)$ is not the case, there are some $x \in \mathcal{V}(l_1)$, $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $x\sigma\varphi \longrightarrow x\nu$ and $\forall y{\in}\mathrm{V}\setminus\{x\}.\ y\mu_1 = y\nu$. Due to $l_1\mu_1/p \lhd l_1\mu_1 = s$ by our first level of induction from $r_0\xi\sigma\varphi \longleftarrow l_0\xi\sigma\varphi = l_1\sigma\varphi/p = l_1\mu_1/p \overset{*}{\longrightarrow} l_1\nu/p$ we know that there must be some u with $r_0\xi\sigma\varphi \overset{*}{\longrightarrow} u \overset{*}{\longleftarrow} l_1\nu/p$. Due to $l_1\mu_1 \overset{+}{\longrightarrow} l_1\nu$ and $\longrightarrow\ \subseteq\ \rhd$ we get $l_1\nu \lhd l_1\mu_1 = s$. Thus, by our first level of induction, from $l_1\nu[p \leftarrow u] \overset{*}{\longleftarrow} l_1\nu \longrightarrow r_1\nu$ (which is due to Lemma B.2, $[\overset{*}{\longleftarrow}_{\omega} \circ \overset{*}{\longrightarrow}\ \subseteq\ \downarrow,]$ $l_1\mu_1 = s$ and our first level of induction) we get $t_0 = l_1\mu_1[p \leftarrow r_0\xi\sigma\varphi] \overset{*}{\longrightarrow} l_1\nu[p \leftarrow r_0\xi\sigma\varphi] \overset{*}{\longrightarrow} l_1\nu[p \leftarrow u] \downarrow r_1\nu \overset{*}{\longleftarrow} r_1\mu_1 = t_1$. Q.e.d. (Claim A)

If $l_1[p \leftarrow r_0\xi]\sigma = r_1\sigma$, then we are finished due to $t_0 = l_1[p \leftarrow r_0\xi]\sigma\varphi = r_1\sigma\varphi = t_1$. Otherwise $((l_1[p \leftarrow r_0\xi], C_0\xi, \Lambda_0),\ (r_1, C_1, \Lambda_1),\ l_1,\ \sigma,\ p)$ is a critical peak in CP(R).

Now $(C_0\xi C_1)\sigma\varphi = C_0\mu_0 C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow$. Due to $l_1\sigma\varphi = l_1\rho = l_1\mu_1 = s$, by our first level of induction we get $\forall u \lhd l_1\sigma\varphi.$ ($\longrightarrow$ is confluent below u). [By Claim 1 we get $l_1\sigma\varphi \notin \mathrm{dom}(\longrightarrow_{\omega})$.] By Claim 0 we get $\forall q{\in}\mathcal{POS}(l_1\sigma\varphi).$ $\left(\emptyset \neq q \lll_{l_1\sigma\varphi} p \ \Rightarrow\ l_1\sigma\varphi \notin \mathrm{dom}(\longrightarrow_{\omega+\omega,q}) \right)$. This means $l_1\sigma\varphi \notin A(p)$. [Define $D_0 := C_0\xi$ and $D_1 := C_1$. If $\Lambda_k = 0$ for some $k \prec 2$, then $l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, which by Claim 2 implies $\mathcal{TERMS}(D_k\sigma\varphi) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, and then $\mathcal{TERMS}(D_k\sigma) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$.] Thus, in case of $\forall y{\in}\mathrm{V}.\ y\varphi \notin \mathrm{dom}(\longrightarrow)$, by Claim A and the assumed $\rhd$-weak joinability w.r.t. R, X besides A we get $t_0 = l_1[p \leftarrow r_0\xi]\sigma\varphi \downarrow r_1\sigma\varphi = t_1$.

Otherwise, when $\forall y{\in}\mathrm{V}.\ y\varphi \notin \mathrm{dom}(\longrightarrow)$ is not the case, by $\longrightarrow\ \subseteq\ \rhd$ and the Axiom of Choice there is some $\varphi' \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $\forall y{\in}\mathrm{V}.\ y\varphi \overset{*}{\longrightarrow} y\varphi' \notin \mathrm{dom}(\longrightarrow)$. Then, of course, $\forall y{\in}\mathrm{V}.\ y\xi\sigma\varphi \overset{*}{\longrightarrow} y\xi\sigma\varphi'$ and $\forall y{\in}\mathrm{V}.\ y\sigma\varphi \overset{*}{\longrightarrow} y\sigma\varphi'$. By Lemma B.2 (due to $[\overset{*}{\longleftarrow}_{\omega} \circ \overset{*}{\longrightarrow}\ \subseteq\ \downarrow;]$ $l_0\xi\sigma\varphi, l_1\sigma\varphi \trianglelefteq_{\mathrm{ST}} s$; $\trianglelefteq_{\mathrm{ST}}\ \subseteq\ \trianglelefteq$; and our first level of induction) we know that $C_0\xi\sigma\varphi'$ and $C_1\sigma\varphi'$ are fulfilled. Furthermore, we have $l_1[p \leftarrow r_0\xi]\sigma\varphi \overset{*}{\longrightarrow} l_1[p \leftarrow r_0\xi]\sigma\varphi'$ and $r_1\sigma\varphi' \overset{*}{\longleftarrow} r_1\sigma\varphi$. Therefore, in case of $l_1\sigma\varphi = l_1\sigma\varphi'$ the proof succeeds like above with φ' instead of φ. Otherwise we have $l_1\sigma\varphi \overset{+}{\longrightarrow} l_1\sigma\varphi'$. Then due to $\longrightarrow\ \subseteq\ \rhd$ we get $s = l_1\sigma\varphi \rhd l_1\sigma\varphi'$. Therefore, by our first level of induction, from $l_1[p \leftarrow r_0\xi]\sigma\varphi' \longleftarrow l_1[p \leftarrow l_0\xi]\sigma\varphi' = l_1\sigma\varphi' \longrightarrow r_1\sigma\varphi'$ (which is due to $[\overset{*}{\longleftarrow}_{\omega} \circ \overset{*}{\longrightarrow}\ \subseteq\ \downarrow;]$ $l_0\xi\sigma\varphi, l_1\sigma\varphi \trianglelefteq_{\mathrm{ST}} s$; $\trianglelefteq_{\mathrm{ST}}\ \subseteq\ \trianglelefteq$; and our first level of induction) we conclude $t_0 = l_1[p \leftarrow r_0\xi]\sigma\varphi \overset{*}{\longrightarrow} l_1[p \leftarrow r_0\xi]\sigma\varphi' \downarrow r_1\sigma\varphi' \overset{*}{\longleftarrow} r_1\sigma\varphi = t_1$. Q.e.d. (The critical peak case)

Q.e.d. (Lemma B.3)

Proof of Lemma B.4 and Lemma B.5

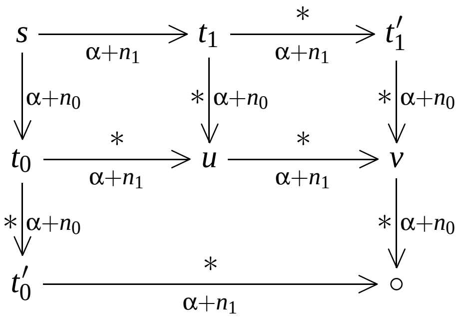
Since the proofs of the two lemmas are very similar, we treat them together, indicating the differences where necessary and using 'α' to denote ω in the proof of Lemma B.4.

For $(\delta,s)\,\underline{\prec\lhd}\,(\beta,\hat{s})$ we are going to show that R,X is α-shallow confluent up to δ and s in $\lhd$ by induction over (δ,s) in $\prec\lhd$. Suppose that for $n_0,n_1\prec\omega$ we have $(n_0{+_\alpha}n_1,s)\,\underline{\prec\lhd}\,(\beta,\hat{s})$ and $t_0'\overset{*}{\longleftarrow}_{\alpha+n_0}s\overset{*}{\longrightarrow}_{\alpha+n_1}t_1'$. We have to show $t_0'\overset{*}{\longrightarrow}_{\alpha+n_1}\circ\overset{*}{\longleftarrow}_{\alpha+n_0}t_1'$.
In case of $\exists i\prec 2.\ t_i'=s$ this is trivially true.

Thus, for
$$t_0'\overset{*}{\longleftarrow}_{\alpha+n_0}t_0\longleftarrow_{\alpha+n_0,p}s\longrightarrow_{\alpha+n_1,q}t_1\overset{*}{\longrightarrow}_{\alpha+n_1}t_1'$$
using the induction hypothesis that
$$\forall(\delta,w')\prec\lhd(n_0{+_\alpha}n_1,s).\ \mathrm{R},\mathrm{X}\text{ is }\alpha\text{-shallow confluent up to }\delta\text{ and }w'\text{ in }\lhd$$
we have to show
$$t_0'\overset{*}{\longrightarrow}_{\alpha+n_1}\circ\overset{*}{\longleftarrow}_{\alpha+n_0}t_1'.$$

Note that due to Lemma B.1 we have $\longrightarrow_{\omega+\alpha}\subseteq\rhd$.

<u>Claim 0:</u> Now it is sufficient to show $t_0\overset{*}{\longrightarrow}_{\alpha+n_1}u\overset{*}{\longleftarrow}_{\alpha+n_0}t_1$ for some u.
<u>Proof of Claim 0:</u> Due to $\longrightarrow_{\omega+\alpha}\subseteq\rhd$ we have $s\rhd t_0,t_1$. Thus by our induction hypotheses $u\overset{*}{\longleftarrow}_{\alpha+n_0}t_1\overset{*}{\longrightarrow}_{\alpha+n_1}t_1'$ (cf. diagram below) implies the existence of some v with $u\overset{*}{\longrightarrow}_{\alpha+n_1}v\overset{*}{\longleftarrow}_{\alpha+n_0}t_1'$ and then $t_0'\overset{*}{\longleftarrow}_{\alpha+n_0}t_0\overset{*}{\longrightarrow}_{\alpha+n_1}v$ implies $t_0'\overset{*}{\longrightarrow}_{\alpha+n_1}\circ\overset{*}{\longleftarrow}_{\alpha+n_0}v$.



<u>Q.e.d. (Claim 0)</u>

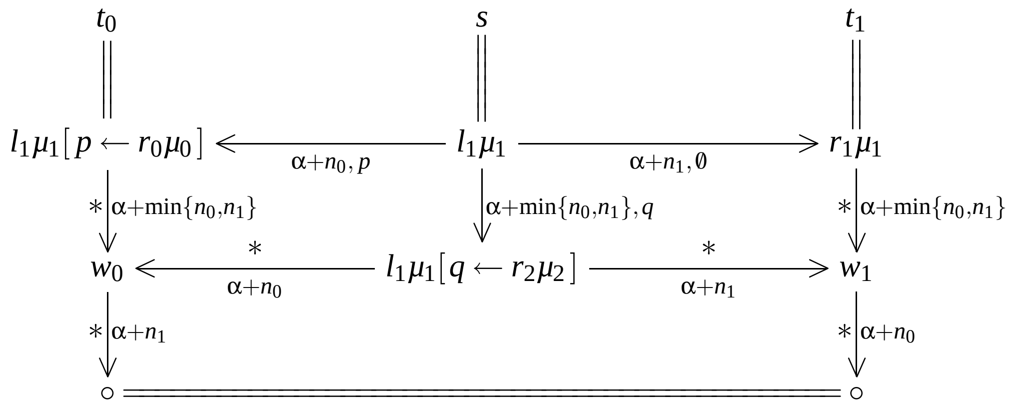
In case of $p\parallel q$ we have $t_0/q=s[p\leftarrow t_0/p]/q=s/q$ and $t_1/p=s[q\leftarrow t_1/q]/p=s/p$ and therefore $t_0\longrightarrow_{\alpha+n_1,q}s[p\leftarrow t_0/p][q\leftarrow t_1/q]\longleftarrow_{\alpha+n_0,p}t_1$, i.e. our proof is finished. Otherwise one of p,q must be a prefix of the other, w.l.o.g. say that q is a prefix of p. In case of $q\neq\emptyset$ due to $\rhd_{\mathrm{ST}}\subseteq\rhd$ we get $s/q\lhd s$ and the proof finished by our induction hypothesis and $\mathcal{T}(\mathrm{sig},\mathrm{X})$-monotonicity of $\overset{*}{\longrightarrow}_{\alpha+n_k}$. Thus we may assume $q=\emptyset$. We start a second level of induction on p in $\lll_s$. Thus we may assume the following induction hypothesis:
$\forall q\in\mathcal{POS}(s).\ \forall t_0',t_1'.\ \forall n_0',n_1'.$
$$\left(\left(\begin{array}{ll} & t_0'\longleftarrow_{\alpha+n_0',q}s\longrightarrow_{\alpha+n_1',\emptyset}t_1'\\ \wedge & n_0'{+_\alpha}n_1'\preceq n_0{+_\alpha}n_1\\ \wedge & q\lll_s p\end{array}\right)\Rightarrow t_0'\overset{*}{\longrightarrow}_{\alpha+n_1'}\circ\overset{*}{\longleftarrow}_{\alpha+n_0'}t_1'\right)$$

Now for $k\prec 2$ there must be $((l_k,r_k),C_k)\in\mathrm{R}$; $\mu_k\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $C_k\mu_k$ fulfilled w.r.t. $\longrightarrow_{\alpha+(n_k\dot{-}1)}$; $s=l_1\mu_1$; $s/p=l_0\mu_0$; $t_0=l_1\mu_1[p\leftarrow r_0\mu_0]$; $t_1=r_1\mu_1$; and $\Lambda_k\preceq n_k$ and $\alpha=0\Rightarrow\left(\begin{array}{ll} & 1\preceq n_k\\ \wedge & \Lambda_k=0\end{array}\right)$ for $\Lambda_k:=\left\{\begin{array}{ll}0 & \text{if } l_k\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_\mathcal{C})\\ 1 & \text{otherwise}\end{array}\right\}$.

Claim 1: We may assume that $\forall q \in \mathcal{POS}(s).\ \Big(\ \emptyset \neq q \lll_s p\ \Rightarrow\ s \notin \mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\},q})\ \Big).$

Proof of Claim 1: Otherwise there must be some $q \in \mathcal{POS}(s)$; $((l_2,r_2),C_2) \in \mathrm{R}$; $\mu_2 \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $C_2\mu_2$ fulfilled w.r.t. $\longrightarrow_{\alpha+(\min\{n_0,n_1\}\dot{-}1)}$; $s/q = l_2\mu_2$; $\emptyset \neq q \lll_s p$. By our second induction level we get $l_1\mu_1[q \leftarrow r_2\mu_2] \overset{*}{\longrightarrow}_{\alpha+n_1} w_1 \overset{*}{\longleftarrow}_{\alpha+\min\{n_0,n_1\}} r_1\mu_1$ for some w_1; cf. the diagram below. Next we are going to show that there is some w_0 with $l_1\mu_1[p \leftarrow r_0\mu_0] \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} w_0 \overset{*}{\longleftarrow}_{\alpha+n_0} l_1\mu_1[q \leftarrow r_2\mu_2]$. Note that (since $\longrightarrow_{\omega+\alpha} \subseteq \rhd$ implies $s \rhd l_1\mu_1[q \leftarrow r_2\mu_2]$) this finishes the proof since then $w_0 \overset{*}{\longleftarrow}_{\alpha+n_0} l_1\mu_1[q \leftarrow r_2\mu_2] \overset{*}{\longrightarrow}_{\alpha+n_1} w_1$ by our first level of induction implies

$$t_0 \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} w_0 \overset{*}{\longrightarrow}_{\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\alpha+n_0} w_1 \overset{*}{\longleftarrow}_{\alpha+\min\{n_0,n_1\}} t_1.$$



In case of $p \parallel q$ we simply can choose $w_0 := l_1\mu_1[p \leftarrow r_0\mu_0][q \leftarrow r_2\mu_2]$. Otherwise, there must be some $\bar{p}$, $\hat{p}$, $\hat{q}$, with $p = \bar{p}\hat{p}$, $q = \bar{p}\hat{q}$, and $(\hat{p} = \emptyset \vee \hat{q} = \emptyset)$. Now it suffices to show

$$s/\bar{p}[\hat{p} \leftarrow r_0\mu_0] \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} w_0' \overset{*}{\longleftarrow}_{\alpha+n_0} s/\bar{p}[\hat{q} \leftarrow r_2\mu_2]$$

for some w_0', because by $\mathcal{T}(\mathrm{sig},\mathrm{X})$-monotonicity of $\overset{*}{\longrightarrow}_{\alpha+n'}$ we then have
$l_1\mu_1[p \leftarrow r_0\mu_0] = s[\bar{p}\hat{p} \leftarrow r_0\mu_0] = s[\bar{p} \leftarrow s/\bar{p}][\bar{p}\hat{p} \leftarrow r_0\mu_0] =$
$s[\bar{p} \leftarrow s/\bar{p}[\hat{p} \leftarrow r_0\mu_0]] \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} s[\bar{p} \leftarrow w_0'] \overset{*}{\longleftarrow}_{\alpha+n_0} s[\bar{p} \leftarrow s/\bar{p}[\hat{q} \leftarrow r_2\mu_2]] =$
$s[\bar{p} \leftarrow s/\bar{p}][\bar{p}\hat{q} \leftarrow r_2\mu_2] = s[\bar{p}\hat{q} \leftarrow r_2\mu_2] = l_1\mu_1[q \leftarrow r_2\mu_2]$.
Note that

$$s/\bar{p}[\hat{p} \leftarrow r_0\mu_0] \longleftarrow_{\alpha+n_0,\hat{p}} s/\bar{p} \longrightarrow_{\alpha+\min\{n_0,n_1\},\hat{q}} s/\bar{p}[\hat{q} \leftarrow r_2\mu_2].$$

In case of $\bar{p} \neq \emptyset$ (since then $\rhd_{\mathrm{ST}} \subseteq \rhd$ implies $s \rhd s/\bar{p}$) we get some w_0' with $s/\bar{p}[\hat{p} \leftarrow r_0\mu_0] \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} w_0' \overset{*}{\longleftarrow}_{\alpha+n_0} s/\bar{p}[\hat{q} \leftarrow r_2\mu_2]$ by our first level of induction. Otherwise, in case of $\bar{p} = \emptyset$, our disjunction from above means $(p = \emptyset \vee q = \emptyset)$. Since we have $\emptyset \neq q$ by our initial assumption, we may assume $q = \hat{q} \neq \emptyset$ and $p = \hat{p} = \bar{p} = \emptyset$. Then the above divergence reads $s/\bar{p}[\hat{p} \leftarrow r_0\mu_0] \longleftarrow_{\alpha+n_0,\emptyset} s \longrightarrow_{\alpha+\min\{n_0,n_1\},q} s/\bar{p}[\hat{q} \leftarrow r_2\mu_2]$ and we get the required joinability by our second induction level due to $q \lll_s p$. Q.e.d. (Claim 1)

Claim 2 of the proof of Lemma B.4: We may assume that for some $i \prec 2$:
$n_i = 0 \prec n_{1-i}$; $l_i \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$; $l_{1-i}\mu_{1-i} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$;
and $(\ l_{1-i} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}}) \vee \mathcal{TERMS}(C_{1-i}\mu_{1-i}) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})\ )$.

Proof of Claim 2 of the proof of Lemma B.4: If $\forall i \prec 2.\ s \longrightarrow_{\omega} t_{1-i}$, then the whole proof is finished by confluence of $\longrightarrow_{\omega}$. Thus there is some $i \prec 2$ with $s \not\longrightarrow_{\omega} t_{1-i}$. Then we get $0 \prec n_{1-i}$. The case of $0 \prec n_i$ is empty, since then due to $\beta \preceq \omega \prec n_0 +_{\omega} n_1$ the globally supposed ordering property $(n_0 +_{\omega} n_1, s) \preceq\!\lhd (\beta, \hat{s})$ cannot hold. Thus we get $n_i = 0 \prec n_{1-i}$. Due

$\Lambda_i \preceq n_i = 0$ we get $l_i \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$. By Lemma 2.10 and $l_{1-i}\mu_{1-i} \longrightarrow_{\omega+n_{1-i}} r_{1-i}\mu_{1-i}$, $l_{1-i}\mu_{1-i} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ would imply the contradictory $l_{1-i}\mu_{1-i} \longrightarrow_\omega r_{1-i}\mu_{1-i}$. Finally, $l_{1-i} \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ and $\mathcal{TERMS}(C_{1-i}\mu_{1-i}) \subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ by Lemma 2.10 would imply that $C_{1-i}\mu_{1-i}$ is fulfilled w.r.t. $\longrightarrow_\omega$ and then Corollary 2.6 would imply the contradictory $l_{1-i}\mu_{1-i} \longrightarrow_\omega r_{1-i}\mu_{1-i}$ again. Q.e.d. (Claim 2 of the proof of Lemma B.4)

Claim 2 of the proof of Lemma B.5: For each $k \prec 2$ we may assume: $0 \prec n_k$;
$\alpha = 0 \ \Rightarrow\ l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$; and

$$\alpha = \omega \ \Rightarrow\ \left(\begin{array}{rl} & l_k\mu_k \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C}) \\ \wedge & \left(\begin{array}{rl} & l_k \notin \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C}) \\ \vee & \mathcal{TERMS}(C_k\mu_k) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C}) \end{array} \right) \end{array} \right).$$

Proof of Claim 2 of the proof of Lemma B.5: In case of $\alpha = 0$ we have $0 \prec n_k$ due to $1 \preceq n_k$ and have $l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ due to $\Lambda_k = 0$. Now we treat the case of $\alpha = \omega$: We may assume $\forall k \prec 2.\ s \not\longrightarrow_\omega t_k$, since otherwise the whole proof is finished by ω-shallow confluence up to ω. Thus we have $0 \prec n_0, n_1$. By Lemma 2.10 and $l_k\mu_k \longrightarrow_{\omega+n_k} r_k\mu_k$, $l_k\mu_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ would imply the contradictory $l_k\mu_k \longrightarrow_\omega r_k\mu_k$. Finally, $l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ and $\mathcal{TERMS}(C_k\mu_k) \subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$ by Lemma 2.10 would imply that $C_k\mu_k$ is fulfilled w.r.t. $\longrightarrow_\omega$ and then Corollary 2.6 would imply the contradictory $l_k\mu_k \longrightarrow_\omega r_k\mu_k$ again. Q.e.d. (Claim 2 of the proof of Lemma B.5)

Claim 3: For all $k \prec 2$ we may assume:
$(\ \alpha = 0 \ \Rightarrow\ l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})\ )$;
$(\ \min\{n_0, n_1\} \preceq (n_k \dot{-} 1) \ \vee\ ((l_k, r_k), C_k)$ is α-quasi-normal w.r.t. R, X $)$;
and R, X is α-shallow confluent up to $\min\{n_0, n_1\} +_\alpha (n_k \dot{-} 1)$.

Proof of Claim 3 of the proof of Lemma B.4: The first property is trivial due to $\alpha = \omega$. By Claim 2 we get $\min\{n_0, n_1\} = 0 \preceq (n_k \dot{-} 1)$ as well as $\min\{n_0, n_1\} +_\omega (n_k \dot{-} 1) = 0 +_\omega (n_k \dot{-} 1) = (n_k \dot{-} 1) \prec \max\{1, n_k\} \preceq \max\{n_0, n_1\} = n_0 +_\omega n_1$. Thus R, X is ω-shallow confluent up to $\min\{n_0, n_1\} +_\omega (n_k \dot{-} 1)$ by our first level of induction.
Q.e.d. (Claim 3 of the proof of Lemma B.4)

Proof of Claim 3 of the proof of Lemma B.5: The first property follows from Claim 2. Since R, X is α-quasi-normal, $((l_k, r_k), C_k)$ is α-quasi-normal w.r.t. R, X. By Claim 2 we have $\min\{n_0, n_1\} +_\alpha (n_k \dot{-} 1) \prec \min\{n_0, n_1\} +_\alpha n_k \preceq n_0 +_\alpha n_1$. Thus Claim 3 follows from our first level of induction. Q.e.d. (Claim 3 of the proof of Lemma B.5)

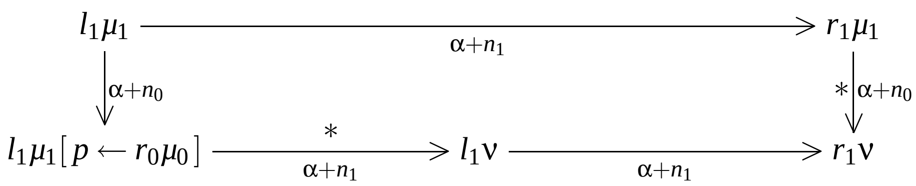
Claim 4: For any $k \prec 2$ and $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$, if $C_k\nu$ is fulfilled w.r.t. $\longrightarrow_{\alpha+(n_k \dot{-} 1)}$, then $l_k\nu \longrightarrow_{\alpha+n_k} r_k\nu$.

Proof of Claim 4 of the proof of Lemma B.4: By Claim 2 we have $0 \prec n_k$ or $n_k = 0 \ \wedge\ l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$. In the first case Claim 4 is trivial due to $(n_k \dot{-} 1) + 1 = n_k$. In the second case $C_k\nu$ is fulfilled w.r.t. $\longrightarrow_\omega$ and $l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$. Thus, by Corollary 2.6, we get $l_k\nu \longrightarrow_\omega r_k\nu$, which completes the proof of Claim 4 due to $n_k = 0$ in this case.
Q.e.d. (Claim 4 of the proof of Lemma B.4)

Proof of Claim 4 of the proof of Lemma B.5: By Claim 2 we have $0 \prec n_k$ and $\alpha = 0 \Rightarrow l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_\mathcal{C})$. Thus Claim 4 is trivial due to $(n_k \dot{-} 1) + 1 = n_k$.
Q.e.d. (Claim 4 of the proof of Lemma B.5)

Two cases:

The variable-overlap case: There are q_0', q_1' such that $p = q_0' q_1'$; $l_1/q_0' = x \in \mathrm{V}$:



We have $x\mu_1/q_1' = l_1\mu_1/q_0'q_1' = s/p = l_0\mu_0$.

Claim A of the proof of Lemma B.4:
In case of "$i{=}1$" for the 'i' of Claim 2 we may assume $x \in \mathrm{V_{SIG}}$.
Proof of Claim A of the proof of Lemma B.4: Otherwise we would have $x \in \mathrm{V}_{\mathcal{C}}$, which implies $x\mu_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_0\mu_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$. We may assume $l_{1-i}\mu_{1-i} \notin \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ for the i of Claim 2. Q.e.d. (Claim A of the proof of Lemma B.4)

Claim A of the proof of Lemma B.5:
We may assume $\left(\begin{array}{ll} & (\ \alpha = 0 \ \Rightarrow\ l_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})\ ) \\ \wedge & (\ \alpha = \omega \ \Rightarrow\ x \in \mathrm{V_{SIG}}\ ) \end{array} \right)$.
Proof of Claim A of the proof of Lemma B.5: The first statement follows from Claim 2. The second is show by contradiction: Suppose we would have $x \in \mathrm{V}_{\mathcal{C}}$, which implies $x\mu_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_0\mu_0 \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$. By Claim 2 we can assume that this is not the case for $\alpha = \omega$. Q.e.d. (Claim A of the proof of Lemma B.5)

By Lemma 2.10 (in case of $x \in \mathrm{V}_{\mathcal{C}}$), we can define $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ by ($y \in \mathrm{V}$):
$y\nu := \left\{ \begin{array}{ll} x\mu_1[q_1' \leftarrow r_0\mu_0] & \text{if } y = x \\ y\mu_1 & \text{otherwise} \end{array} \right\}$ and get $y\mu_1 \overset{=}{\longrightarrow}_{\alpha+n_0} y\nu$ for $y \in \mathrm{V}$.
By $\mathcal{T}(\mathrm{sig}, \mathrm{X})$-monotonicity of $\longrightarrow_{\alpha+n_0}$ we get $r_1\mu_1 \overset{*}{\longrightarrow}_{\alpha+n_0} r_1\nu$ and
$l_1\mu_1[q_0'q_1' \leftarrow r_0\mu_0] =$
$l_1[q_0' \leftarrow x\nu][q'' \leftarrow y\mu_1 \mid l_1/q'' = y \in \mathrm{V} \ \wedge\ q'' \neq q_0'] =$
$l_1[q_0' \leftarrow x\nu][q'' \leftarrow x\mu_1 \mid l_1/q'' = x \ \wedge\ q'' \neq q_0'][q'' \leftarrow y\nu \mid x \neq l_1/q'' = y \in \mathrm{V} \ \wedge\ q'' \neq q_0']$
$\overset{*}{\longrightarrow}_{\alpha+n_0}\ l_1[q'' \leftarrow y\nu \mid l_1/q'' = y \in \mathrm{V}] = l_1\nu$.

Claim B: $l_1\mu_1[\,p \leftarrow r_0\mu_0\,] \overset{*}{\longrightarrow}_{\alpha+n_1} l_1\nu$.
Proof of Claim B of the proof of Lemma B.4: By case distinction over the 'i' of Claim 2:
"$i{=}0$": $n_0 = 0 \prec n_1$ implies $\longrightarrow_{\omega+n_0} \subseteq \longrightarrow_{\omega+n_1}$ by Lemma 2.12.
"$i{=}1$": In this case we have $l_1 \in \mathcal{T}(\mathrm{cons}, \mathrm{V_{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$. By Claim A we may assume $x \in \mathrm{V_{SIG}}$. Then l_1 is linear in x. Thus $\{\, q'' \mid l_1/q'' = x \ \wedge\ q'' \neq q_0' \,\} = \emptyset$, which means that the above reduction takes 0 steps, i.e. $l_1\mu_1[\,p \leftarrow r_0\mu_0\,] = l_1\nu$. Q.e.d. (Claim B of the proof of Lemma B.4)
Proof of Claim B of the proof of Lemma B.5: By Claim A and the assumption of our lemma we know that l_0 is linear in x. Thus $\{\, q'' \mid l_1/q'' = x \ \wedge\ q'' \neq q_0' \,\} = \emptyset$, which means that the above reduction takes 0 steps, i.e. $l_1\mu_1[\,p \leftarrow r_0\mu_0\,] = l_1\nu$. Q.e.d. (Claim B of the proof of Lemma B.5)

Claim C: $l_1\nu \longrightarrow_{\alpha+n_1} r_1\nu$.
Proof of Claim C of the proof of Lemma B.4: By case distinction over the 'i' of Claim 2:
"$i=0$": Due to $n_0=0\prec n_1$ this follows directly from Lemma 13.8 (matching its n_0 to our $n_0=0$ and its n_1 to our $n_1\dot{-}1$) (since $0\preceq n_1\dot{-}1$ and R,X is ω-shallow confluent up to $n_1\dot{-}1$ by our induction hypothesis).
"$i=1$": In this case we have $n_1=0$ and $l_1\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_C)$. Thus, since $C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_\omega$, by assumption of the lemma we know that $((l_1,r_1),C_1)$ is quasi-normal w.r.t. R,X and that for all $u\in\mathcal{TERMS}(C_1)$ we have $l_1\mu_1 \rhd u\mu_1$ or $u\mu_1\notin\mathrm{dom}(\longrightarrow)$ or $\mathcal{V}(u)\subseteq\mathrm{V}_C$. In the latter case, since we may assume $x\in\mathrm{V_{SIG}}$ by Claim A, we get $\forall y\in\mathcal{V}(u).\ y\mu_1=y\nu$ and, moreover, $\forall\delta\prec n_0+_\omega n_1.\ \mathrm{R},\mathrm{X}$ is ω-shallow confluent up to δ by our induction hypothesis. In the first case, due to $l_1\mu_1=s$ our induction hypothesis even implies that R,X is ω-shallow confluent up to $n_0+_\omega n_1$ and $u\mu_1$ in $\lhd$. Thus Lemma 13.8 (matching its n_0 to our n_0 and its n_1 to our n_1) implies that $C_1\nu$ is fulfilled w.r.t. $\longrightarrow_{\omega+n_1}$. Now since $n_1=0$, Corollary 2.6 implies $l_1\nu\longrightarrow_{\omega+n_1} r_1\nu$.
Q.e.d. (Claim C of the proof of Lemma B.4)

Proof of Claim C of the proof of Lemma B.5: Directly Lemma 13.8 (matching its n_0 to our n_0 and its n_1 to our $n_1\dot{-}1$) (by Claim 2 and since R,X is α-quasi-normal and α-shallow confluent up to $n_0+_\alpha(n_1\dot{-}1)$ by our first level of induction due to $n_1\dot{-}1\preceq n_1$ by Claim 2).
Q.e.d. (Claim C of the proof of Lemma B.5)

Q.e.d. (The variable-overlap case)

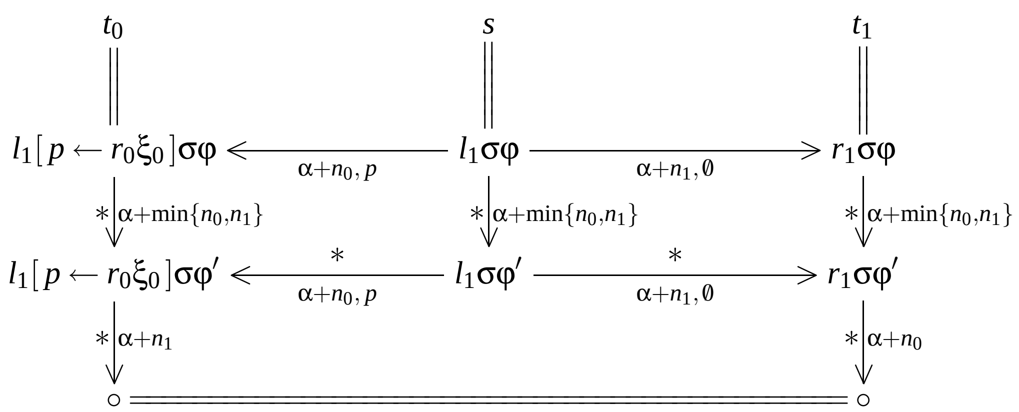
The critical peak case: $p\in\mathcal{POS}(l_1);\ l_1/p\notin\mathrm{V}$: Let $\xi_0\in\mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi_0[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)]\cap\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)=\emptyset$. Define $\mathrm{Y}:=\mathcal{V}((l_0{=}r_0{\longleftarrow}C_0)\xi_0,l_1{=}r_1{\longleftarrow}C_1)$. Define $\xi_1:={}_{\mathrm{V}}\!\upharpoonright\!\mathrm{id}$. Let ρ be given by $x\rho=\left\{\begin{array}{ll}x\mu_1 & \text{if } x\in\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)\\ x\xi_0^{-1}\mu_0 & \text{else}\end{array}\right\}\ (x\in\mathrm{V})$.
By $l_0\xi_0\rho=l_0\xi_0\xi_0^{-1}\mu_0=s/p=l_1\mu_1/p=l_1\rho/p=(l_1/p)\rho$ let $\sigma:=\mathrm{mgu}(\{(l_0\xi_0,l_1/p)\},\mathrm{Y})$ and $\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi)={}_{\mathrm{Y}}\!\upharpoonright\!\rho$.

Claim A: We may assume $\left(\ p=\emptyset\ \vee\ \forall y\in\mathcal{V}(l_1).\ y\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\}})\ \right)$.
Proof of Claim A: Otherwise, when $p\neq\emptyset$ holds but $\forall y\in\mathcal{V}(l_1).\ y\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\}})$ is not the case, there are some $x\in\mathcal{V}(l_1)$, $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $x\mu_1\longrightarrow_{\alpha+\min\{n_0,n_1\}}x\nu$ and $\forall y\in\mathrm{V}\setminus\{x\}.\ y\mu_1=y\nu$. Due to $l_1\mu_1/p\lhd l_1\mu_1=s$ by our first level of induction from $r_0\xi_0\sigma\varphi\longleftarrow_{\alpha+n_0}l_0\xi_0\sigma\varphi=l_1\sigma\varphi/p=l_1\mu_1/p\overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}}l_1\nu/p$ we know that there must be some u with $r_0\xi_0\sigma\varphi\overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}}u\overset{*}{\longleftarrow}_{\alpha+n_0}l_1\nu/p$. Due to Claim 3, by Lemma 13.8 (matching its n_0 to our $\min\{n_0,n_1\}$ and its n_1 to our $(n_1\dot{-}1)$) $C_1\nu$ is fulfilled w.r.t. $\longrightarrow_{\alpha+(n_1\dot{-}1)}$. Then Claim 4 implies $l_1\nu\longrightarrow_{\alpha+n_1}r_1\nu$. Due to $l_1\mu_1\overset{+}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}}l_1\nu$ and $\longrightarrow_{\omega+\alpha}\subseteq\rhd$ we get $l_1\nu\lhd l_1\mu_1=s$. Thus, by our first level of induction, from $l_1\nu[p\leftarrow u]\overset{*}{\longleftarrow}_{\alpha+n_0}l_1\nu\longrightarrow_{\alpha+n_1}r_1\nu$ we get $t_0=l_1\mu_1[p\leftarrow r_0\xi_0\sigma\varphi]\overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}}l_1\nu[p\leftarrow r_0\xi_0\sigma\varphi]\overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}}l_1\nu[p\leftarrow u]\overset{*}{\longrightarrow}_{\alpha+n_1}\circ\overset{*}{\longleftarrow}_{\alpha+n_0}r_1\nu\overset{*}{\longleftarrow}_{\alpha+\min\{n_0,n_1\}}r_1\mu_1=t_1$.
Q.e.d. (Claim A)

If $l_1[\,p \leftarrow r_0\xi_0\,]\sigma = r_1\sigma$, then we are finished due to $t_0 = l_1[\,p \leftarrow r_0\xi_0\,]\sigma\varphi = r_1\sigma\varphi = t_1$. Otherwise we have $(\,(l_1[\,p \leftarrow r_0\xi_0\,], C_0\xi_0, \Lambda_0),\,(r_1, C_1\xi_1, \Lambda_1),\,l_1,\,\sigma,\,p\,) \in \mathrm{CP}(\mathrm{R})$ with the following additional structure:
In the proof of Lemma B.4: By Claim 2 the critical peak cannot be of the form $(1,1)$. Moreover, if it is of the form $(0,0)$, then we have $\forall k \prec 2.\ l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, which by Claim 2 for some $i \prec 2$ implies $\mathcal{TERMS}(C_{1-i}\xi_{1-i}\sigma\varphi) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, and then $\mathcal{TERMS}(C_{1-i}\xi_{1-i}\sigma) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$, i.e. $\mathcal{TERMS}(C_0\xi_0\sigma C_1\xi_1\sigma) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$.
In the proof of Lemma B.5: For all $k \prec 2$ we have: $\alpha = 0 \Rightarrow \Lambda_k = 0$. If $\alpha = \omega$ and $\Lambda_k = 0$ for some $k \prec 2$, then $l_k \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, which by Claim 2 implies $\mathcal{TERMS}(C_k\xi_k\sigma\varphi) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$, and then $\mathcal{TERMS}(C_k\xi_k\sigma) \not\subseteq \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$.

Now $C_0\xi_0\sigma\varphi = C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow_{\alpha+(n_0\dot{-}1)}$; $C_1\xi_1\sigma\varphi = C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\alpha+(n_1\dot{-}1)}$. Since $l_1\sigma\varphi = l_1\mu_1 = s$, by our induction hypothesis we have $\forall(\delta, s') \prec\!\lhd (n_0 +_\alpha n_1, l_1\sigma\varphi)$. (R, X is α-shallow confluent up to δ and s' in $\lhd$). By Claim 1 we get $\forall q \in \mathcal{POS}(l_1\sigma\varphi)$. $\Big(\ \emptyset \neq q \lll_{l_1\sigma\varphi} p \ \Rightarrow\ l_1\sigma\varphi \notin \mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\},q})\ \Big)$. This means $l_1\sigma\varphi \notin A(p, \min\{n_0, n_1\})$. Furthermore, $(n_0 +_\alpha n_1, l_1\sigma\varphi) = (n_0 +_\alpha n_1, s) \underline{\preceq\!\lhd} (\beta, \hat{s})$. Therefore, in case of $\forall y \in \mathrm{V}.\ y\varphi \notin \mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\}})$, by Claim A and by the assumed form of α-shallow joinability up to β and $\hat{s}$ w.r.t. R, X and $\lhd$ [besides A], we get $t_0 = l_1[\,p \leftarrow r_0\xi_0\,]\sigma\varphi \overset{*}{\longrightarrow}_{\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\alpha+n_0} r_1\sigma\varphi = t_1$.

Otherwise, when $\forall y \in \mathrm{V}.\ y\varphi \notin \mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\}})$ is not the case, by $\longrightarrow_{\omega+\alpha} \subseteq \rhd$ and the Axiom of Choice there is some $\varphi' \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with $\forall y \in \mathrm{V}.\ y\varphi \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} y\varphi' \notin \mathrm{dom}(\longrightarrow_{\alpha+\min\{n_0,n_1\}})$. Then, of course, $\forall i \prec 2.\ \forall y \in \mathrm{V}.\ y\xi_i\sigma\varphi \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} y\xi_i\sigma\varphi'$. Due to Claim 3, by Lemma 13.8 (matching its n_0 to our $\min\{n_0, n_1\}$ and its n_1 to our $(n_i \dot{-} 1)$) we know that $\forall i \prec 2.\ C_i\xi_i\sigma\varphi'$ is fulfilled w.r.t. $\longrightarrow_{\alpha+(n_i\dot{-}1)}$. Then Claim 4 implies $\forall i \prec 2.$ $l_i\xi_i\sigma\varphi' \longrightarrow_{\alpha+n_i} r_i\xi_i\sigma\varphi'$. Furthermore, we have $l_1[\,p \leftarrow r_0\xi_0\,]\sigma\varphi \overset{*}{\longrightarrow}_{\alpha+\min\{n_0,n_1\}} l_1[\,p \leftarrow r_0\xi_0\,]\sigma\varphi'$ and $r_1\sigma\varphi' \overset{*}{\longleftarrow}_{\alpha+\min\{n_0,n_1\}} r_1\sigma\varphi$, cf. the diagram below. Therefore, in case of $l_1\sigma\varphi = l_1\sigma\varphi'$ the proof succeeds like above with φ' instead of φ. Otherwise we have $l_1\sigma\varphi \overset{+}{\longrightarrow}_{\omega+\alpha} l_1\sigma\varphi'$. Then due to $\longrightarrow_{\omega+\alpha} \subseteq \rhd$ we get $s = l_1\sigma\varphi \rhd l_1\sigma\varphi'$. Therefore, by our first level of induction, from $l_1[\,p \leftarrow r_0\xi_0\,]\sigma\varphi' \longleftarrow_{\alpha+n_0,p} l_1[\,p \leftarrow l_0\xi_0\,]\sigma\varphi' = l_1\sigma\varphi' \longrightarrow_{\alpha+n_1,\emptyset} r_1\sigma\varphi'$ we conclude $l_1[\,p \leftarrow r_0\xi_0\,]\sigma\varphi' \overset{*}{\longrightarrow}_{\alpha+n_1} \circ \overset{*}{\longleftarrow}_{\alpha+n_0} r_1\sigma\varphi'$.



Q.e.d. (The critical peak case) **Q.e.d. (Lemma B.4 and Lemma B.5)**

Proof of Lemma B.6

For $(\delta, s) \underline{\prec\lhd} (\beta, \hat{s})$ we are going to show that R, X is ω-level confluent up to δ and s in $\lhd$ by induction over (δ, s) in $\prec\lhd$. Suppose that for $\bar{n}_0, \bar{n}_1 \prec \omega$ we have $(\max\{\bar{n}_0, \bar{n}_1\}, s) \underline{\prec\lhd} (\beta, \hat{s})$ and $t'_0 \overset{*}{\longleftarrow}_{\omega+\bar{n}_0} s \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} t'_1$. We have to show $t'_0 \overset{*}{\longrightarrow}_{\omega+\max\{\bar{n}_0,\bar{n}_1\}} \circ \overset{*}{\longleftarrow}_{\omega+\max\{\bar{n}_0,\bar{n}_1\}} t'_1$.

In case of $\exists i \prec 2.\ t'_i = s$ this is trivially true by Lemma 2.12. In case of $\bar{n}_0 = \bar{n}_1 = 0$ this is true by confluence of $\longrightarrow_{\omega}$. Using symmetry in 0 and 1, w.l.o.g. we may assume $\bar{n}_0 \preceq \bar{n}_1$.

Thus, assuming $\bar{n}_0 \preceq \bar{n}_1 \succ 0$, for $t'_0 \overset{*}{\longleftarrow}_{\omega+\bar{n}_0} t_0 \longleftarrow_{\omega+\bar{n}_0} s \longrightarrow_{\omega+\bar{n}_1} t_1 \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} t'_1$ using the induction hypothesis that

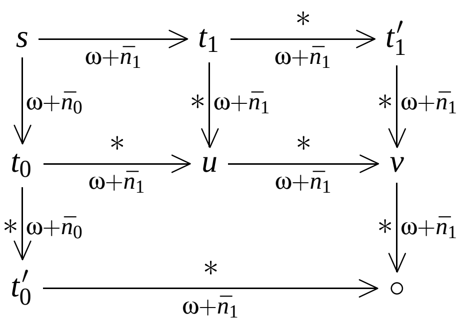
$$\forall (m, w') \prec\lhd (\max\{\bar{n}_0, \bar{n}_1\}, s).\ \text{R, X is ω-level confluent up to } m \text{ and } w' \text{ in } \lhd$$

we have to show

$$t'_0 \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} \circ \overset{*}{\longleftarrow}_{\omega+\bar{n}_1} t'_1.$$

<u>Claim 0:</u> Now it is sufficient to show $t_0 \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} u \overset{*}{\longleftarrow}_{\omega+\bar{n}_1} t_1$ for some u.

<u>Proof of Claim 0:</u> By Lemma B.1 we have $s \rhd t_0, t_1$. Thus, due to[35] $(\max\{\bar{n}_1, \bar{n}_1\}, t_1) \prec\lhd (\max\{\bar{n}_0, \bar{n}_1\}, s)$, by our induction hypotheses $u \overset{*}{\longleftarrow}_{\omega+\bar{n}_1} t_1 \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} t'_1$ (cf. diagram below) implies the existence of some v with $u \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} v \overset{*}{\longleftarrow}_{\omega+\bar{n}_1} t'_1$ and then $t'_0 \overset{*}{\longleftarrow}_{\omega+\bar{n}_0} t_0 \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} v$ implies $t'_0 \overset{*}{\longrightarrow}_{\omega+\bar{n}_1} \circ \overset{*}{\longleftarrow}_{\omega+\bar{n}_1} v$.



<u>Q.e.d. (Claim 0)</u>

Defining $n := \bar{n}_1 \dot{-} 1$ and using Lemma 2.12 we can now restate our proof task in the following symmetric way:

For $n \prec \omega$, $t_0 \longleftarrow_{\omega+n+1,p} s \longrightarrow_{\omega+n+1,q} t_1$ using the induction hypothesis that

$$\forall (m, w') \prec\lhd (n+1, s).\ \text{R, X is ω-level confluent up to } m \text{ and } w' \text{ in } \lhd$$

we have to show

$$t_0 \overset{*}{\longrightarrow}_{\omega+n+1} \circ \overset{*}{\longleftarrow}_{\omega+n+1} t_1.$$

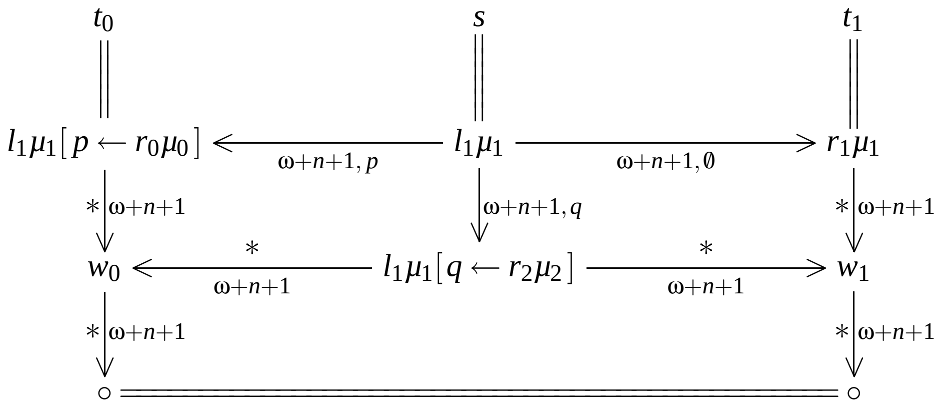
In case of $p \parallel q$ this is trivial. Otherwise one of p, q must be a prefix of the other, w.l.o.g. say that q is a prefix of p. In case of $q \neq \emptyset$ due to $\rhd_{\mathrm{ST}} \subseteq \rhd$ we get $s/q \lhd s$ and the proof finished by our induction hypothesis and $\mathcal{T}(\mathrm{sig}, \mathrm{X})$-monotonicity of $\overset{*}{\longrightarrow}_{\omega+n+1}$. Thus we may assume $q = \emptyset$. We start a second level of induction on p in $\lll_s$. Thus we may assume the following induction hypothesis:

$$\forall q \in \mathcal{POS}(s).\ \forall t'_0, t'_1.\ \left(\left(\begin{array}{l} q \lll_s p \\ \wedge\ t'_0 \longleftarrow_{\omega+n+1,q} s \longrightarrow_{\omega+n+1,\emptyset} t'_1 \end{array} \right) \Rightarrow t'_0 \downarrow_{\omega+n+1} t'_1 \right)$$

[35]Note that it is this change from $\bar{n}_0$ to $\bar{n}_1$ in $\max\{\bar{n}_0, \bar{n}_1\}$ that makes a two level treatment similar to that for ω-shallow confluence (i.e. considering $\bar{n}_0 +_{\omega} \bar{n}_1$ instead of $\bar{n}_0 + \bar{n}_1$) impossible because then for $\bar{n}_0 = 0 \prec \bar{n}_1$ we would get $\max_{\omega}\{\bar{n}_0, \bar{n}_1\} \prec \omega \preceq \max_{\omega}\{\bar{n}_1, \bar{n}_1\}$ and thus would not be allowed to apply our induction hypothesis here.

Now for $k<2$ there must be $((l_k,r_k),C_k)\in\mathrm{R}$; $\mu_k\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $C_k\mu_k$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $s=l_1\mu_1$; $s/p=l_0\mu_0$; $t_0=l_1\mu_1[p\leftarrow r_0\mu_0]$; $t_1=r_1\mu_1$. Moreover, for $k<2$ we define $\Lambda_k:=\left\{\begin{array}{ll}0 & \text{if } l_k\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\\ 1 & \text{otherwise}\end{array}\right\}$.

<u>Claim 1:</u> We may assume that $\forall q\in\mathcal{POS}(s).\ \big(\ \emptyset\neq q\lll_s p\ \Rightarrow\ s\notin\mathrm{dom}(\longrightarrow_{\omega+n+1,q})\ \big)$.
<u>Proof of Claim 1:</u> Otherwise there must be some $q\in\mathcal{POS}(s)$; $((l_2,r_2),C_2)\in\mathrm{R}$; $\mu_2\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$; with $C_2\mu_2$ fulfilled w.r.t. $\longrightarrow_{\omega+n}$; $s/q=l_2\mu_2$; and $\emptyset\neq q\lll_s p$. By our second induction level we get $l_1\mu_1[q\leftarrow r_2\mu_2]\overset{*}{\longrightarrow}_{\omega+n+1}w_1\overset{*}{\longleftarrow}_{\omega+n+1}r_1\mu_1$ for some w_1; cf. the diagram below. Next we are going to show that there is some w_0 with $l_1\mu_1[p\leftarrow r_0\mu_0]\overset{*}{\longrightarrow}_{\omega+n+1}w_0\overset{*}{\longleftarrow}_{\omega+n+1}l_1\mu_1[q\leftarrow r_2\mu_2]$. Note that (since $\longrightarrow\ \subseteq\ \rhd$ implies $s\rhd l_1\mu_1[q\leftarrow r_2\mu_2]$) this finishes the proof since then $w_0\overset{*}{\longleftarrow}_{\omega+n+1}l_1\mu_1[q\leftarrow r_2\mu_2]\overset{*}{\longrightarrow}_{\omega+n+1}w_1$ by our first level of induction implies $t_0\overset{*}{\longrightarrow}_{\omega+n+1}w_0\downarrow_{\omega+n+1}w_1\overset{*}{\longleftarrow}_{\omega+n+1}t_1$.



In case of $p\parallel q$ we simply can choose $w_0:=l_1\mu_1[p\leftarrow r_0\mu_0][q\leftarrow r_2\mu_2]$. Otherwise, there must be some $\bar{p}$, $\hat{p}$, $\hat{q}$, with $p=\bar{p}\hat{p}$, $q=\bar{p}\hat{q}$, and $(\hat{p}=\emptyset\vee\hat{q}=\emptyset)$. Now it suffices to show

$$s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\overset{*}{\longrightarrow}_{\omega+n+1}w_0'\overset{*}{\longleftarrow}_{\omega+n+1}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]$$

for some w_0', because by $\mathcal{T}(\mathrm{sig},\mathrm{X})$-monotonicity of $\overset{*}{\longrightarrow}_{\omega+n+1}$ we then have
$l_1\mu_1[p\leftarrow r_0\mu_0]=s[\bar{p}\hat{p}\leftarrow r_0\mu_0]=s[\bar{p}\leftarrow s/\bar{p}][\bar{p}\hat{p}\leftarrow r_0\mu_0]=$
$s[\bar{p}\leftarrow s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]]\overset{*}{\longrightarrow}_{\omega+n+1}s[\bar{p}\leftarrow w_0']\overset{*}{\longleftarrow}_{\omega+n+1}s[\bar{p}\leftarrow s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]]=$
$s[\bar{p}\leftarrow s/\bar{p}][\bar{p}\hat{q}\leftarrow r_2\mu_2]=s[\bar{p}\hat{q}\leftarrow r_2\mu_2]=l_1\mu_1[q\leftarrow r_2\mu_2]$.
Note that

$$s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\longleftarrow_{\omega+n+1,\hat{p}}s/\bar{p}\longrightarrow_{\omega+n+1,\hat{q}}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2].$$

In case of $\bar{p}\neq\emptyset$ (since then $\rhd_{\mathrm{ST}}\subseteq\rhd$ implies $s\rhd s/\bar{p}$) we get some w_0' with $s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\overset{*}{\longrightarrow}_{\omega+n+1}w_0'\overset{*}{\longleftarrow}_{\omega+n+1}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]$ by our first level of induction. Otherwise, in case of $\bar{p}=\emptyset$, our disjunction from above means $(p=\emptyset\vee q=\emptyset)$. Since we have $\emptyset\neq q$ by our initial assumption, we may assume $q=\hat{q}\neq\emptyset$ and $p=\hat{p}=\bar{p}=\emptyset$. Then the above divergence reads $s/\bar{p}[\hat{p}\leftarrow r_0\mu_0]\longleftarrow_{\omega+n+1,\emptyset}s\longrightarrow_{\omega+n+1,q}s/\bar{p}[\hat{q}\leftarrow r_2\mu_2]$ and we get the required joinability by our second induction level due to $q\lll_s p$. <u>Q.e.d. (Claim 1)</u>

<u>Claim 2:</u> We may assume: $\exists i\prec 2.\ l_i\notin\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$.
<u>Proof of Claim 2:</u> Since $C_i\mu_i$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$, by Lemma 13.2 (matching both its μ and ν to our μ_1) $l_i\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$ implies $l_i\mu_i\longrightarrow_{\omega}r_i\mu_i$ and then $s\longrightarrow_{\omega}t_i$. Thus, if the claim does not hold, we have $t_0\longleftarrow_{\omega}s\longrightarrow_{\omega}t_1$ and the proof is finished by confluence of $\longrightarrow_{\omega}$.
<u>Q.e.d. (Claim 2)</u>

Now we have two cases:

The variable overlap case: $p=q_0q_1;\ l_1/q_0=x\in\mathrm{V}$:

We have $x\mu_1/q_1=l_1\mu_1/q_0q_1=s/p=l_0\mu_0$. By Lemma 2.10 (in case of $x\in\mathrm{V}_{\mathcal{C}}$), we can define $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ by ($y\in\mathrm{V}$):

$y\nu := \begin{cases} x\mu_1[q_1\leftarrow r_0\mu_0] & \text{if } y=x \\ y\mu_1 & \text{otherwise} \end{cases}$ and get $y\mu_1\overset{=}{\longrightarrow}_{\omega+n+1}y\nu$ for $y\in\mathrm{V}$. By Corollary 2.8:

$t_0=l_1\mu_1[q_0q_1\leftarrow r_0\mu_0]=l_1[q_0\leftarrow x\nu][q'\leftarrow y\mu_1\mid l_1/q'=y\in\mathrm{V}\wedge q'\neq q_0]\overset{*}{\longrightarrow}_{\omega+n+1}$
$l_1[q'\leftarrow y\nu\mid l_1/q'=y\in\mathrm{V}]=l_1\nu$;
$t_1=r_1\mu_1\overset{*}{\longrightarrow}_{\omega+n+1}r_1\nu$. It suffices to show $l_1\nu\longrightarrow_{\omega+n+1}r_1\nu$, which follows from our first level of induction saying that R,X is ω-level confluent up to n by Lemma A.7 (matching its n_0 to our $n+1$ and its n_1 to our n). Q.e.d. (The variable overlap case)

The critical peak case: $p\in\mathcal{POS}(l_1);\ l_1/p\notin\mathrm{V}$: Let $\xi\in\mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)]\cap\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)=\emptyset$. Define $\mathrm{Y}:=\mathcal{V}((l_0{=}r_0{\longleftarrow}C_0)\xi,l_1{=}r_1{\longleftarrow}C_1)$.
Let ρ be given by $x\rho=\begin{cases}x\mu_1 & \text{if } x\in\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)\\ x\xi^{-1}\mu_0 & \text{else}\end{cases}$ $(x\in\mathrm{V})$. By $l_0\xi\rho=l_0\xi\xi^{-1}\mu_0=s/p=l_1\mu_1/p=l_1\rho/p=(l_1/p)\rho$ let $\sigma:=\mathrm{mgu}(\{(l_0\xi,l_1/p)\},\mathrm{Y})$ and $\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}{\upharpoonleft}(\sigma\varphi)={}_{\mathrm{Y}}{\upharpoonleft}\rho$.

Claim A: We may assume $\big(\ p=\emptyset\ \vee\ \forall y\in\mathcal{V}(l_1).\,y\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\omega+n+1})\ \big)$.
Proof of Claim A: Otherwise, when $p\neq\emptyset$ holds but $\forall y\in\mathcal{V}(l_1).\,y\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\omega+n+1})$ is not the case, there are some $x\in\mathcal{V}(l_1)$, $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $x\sigma\varphi\longrightarrow_{\omega+n+1}x\nu$ and $\forall y\in\mathrm{V}\setminus\{x\}.\,y\mu_1=y\nu$. Due to $l_1\mu_1/p\lhd l_1\mu_1=s$ by our first level of induction from $r_0\xi\sigma\varphi\longleftarrow_{\omega+n+1}l_0\xi\sigma\varphi=l_1\sigma\varphi/p=l_1\mu_1/p\overset{*}{\longrightarrow}_{\omega+n+1}l_1\nu/p$ we know that there must be some u with $r_0\xi\sigma\varphi\overset{*}{\longrightarrow}_{\omega+n+1}u\overset{*}{\longleftarrow}_{\omega+n+1}l_1\nu/p$. Due to $l_1\mu_1\overset{+}{\longrightarrow}_{\omega+n+1}l_1\nu$ and $\longrightarrow\subseteq\rhd$ we get $l_1\nu\lhd l_1\mu_1=s$. Thus, by our first level of induction, from $l_1\nu[p\leftarrow u]\overset{*}{\longleftarrow}_{\omega+n+1}l_1\nu\longrightarrow_{\omega+n+1}r_1\nu$ (which is due to Lemma A.7 and our first level of induction saying that R,X is ω-level confluent up to n) we get $t_0=l_1\mu_1[p\leftarrow r_0\xi\sigma\varphi]\overset{*}{\longrightarrow}_{\omega+n+1}l_1\nu[p\leftarrow r_0\xi\sigma\varphi]\overset{*}{\longrightarrow}_{\omega+n+1}l_1\nu[p\leftarrow u]\downarrow_{\omega+n+1}r_1\nu\overset{*}{\longleftarrow}_{\omega+n+1}r_1\mu_1=t_1$. Q.e.d. (Claim A)

If $l_1[p\leftarrow r_0\xi]\sigma=r_1\sigma$, then we are finished due to $t_0=l_1[p\leftarrow r_0\xi]\sigma\varphi=r_1\sigma\varphi=t_1$. Otherwise $((l_1[p\leftarrow r_0\xi],C_0\xi,\Lambda_0),\ (r_1,C_1,\Lambda_1),\ l_1,\ \sigma,\ p\ )$ is a critical peak in CP(R). Furthermore, due to Claim 2, this critical peak is not of the form $(0,0)$.

Now $(C_0\xi C_1)\sigma\varphi=C_0\mu_0C_1\mu_1$ is fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Due to $l_1\sigma\varphi=l_1\rho=l_1\mu_1=s$, by our first level of induction we get $\forall(\delta,s')\prec\!\lhd(n{+}1,l_1\sigma\varphi)$. ($\mathrm{R},\mathrm{X}$ is ω-level confluent up to δ and s' in $\lhd$). By Claim 1 we get $\forall q\in\mathcal{POS}(l_1\sigma\varphi).\big(\ \emptyset\neq q\lll_{l_1\sigma\varphi}p\ \Rightarrow\ l_1\sigma\varphi\notin\mathrm{dom}(\longrightarrow_{\omega+n+1,q})\ \big)$. This means $l_1\sigma\varphi\notin A(p,n{+}1)$. Furthermore, $(n{+}1,l_1\sigma\varphi)=(\max\{n_0,n_1\},s)\ \underline{\prec\!\lhd}\ (\beta,\hat{s})$. Thus, in case of $\forall y\in\mathrm{V}.\,y\varphi\notin\mathrm{dom}(\longrightarrow_{\omega+n+1})$, by Claim A and the assumed by ω-level joinability up to β and $\hat{s}$ w.r.t. R,X and $\lhd$ [besides A] (matching the definition's n_0 and n_1 to our $n+1$) we get $t_0=l_1[p\leftarrow r_0\xi]\sigma\varphi\downarrow_{\omega+n+1}r_1\sigma\varphi=t_1$.

Otherwise, when $\forall y\in\mathrm{V}.\ y\varphi\not\in\mathrm{dom}(\longrightarrow_{\omega+n+1})$ is not the case, by $\longrightarrow\subseteq\rhd$ and the Axiom of Choice there is some $\varphi'\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $\forall y\in\mathrm{V}.\ y\varphi\overset{*}{\longrightarrow}_{\omega+n+1}y\varphi'\not\in\mathrm{dom}(\longrightarrow_{\omega+n+1})$. Then, of course, $\forall y\in\mathrm{V}.\ y\xi\sigma\varphi\overset{*}{\longrightarrow}_{\omega+n+1}y\xi\sigma\varphi'$ and $\forall y\in\mathrm{V}.\ y\sigma\varphi\overset{*}{\longrightarrow}_{\omega+n+1}y\sigma\varphi'$. By Lemma A.7 (due to our first level of induction saying that R,X is ω-level confluent up to n) we know that $C_0\xi\sigma\varphi'$ and $C_1\sigma\varphi'$ are fulfilled w.r.t. $\longrightarrow_{\omega+n}$. Furthermore, we have $l_1[p\leftarrow r_0\xi]\sigma\varphi\overset{*}{\longrightarrow}_{\omega+n+1}l_1[p\leftarrow r_0\xi]\sigma\varphi'$ and $r_1\sigma\varphi'\overset{*}{\longleftarrow}_{\omega+n+1}r_1\sigma\varphi$. Therefore, in case of $l_1\sigma\varphi=l_1\sigma\varphi'$ the proof succeeds like above with φ' instead of φ. Otherwise we have $l_1\sigma\varphi\overset{+}{\longrightarrow}l_1\sigma\varphi'$. Then due to $\longrightarrow\subseteq\rhd$ we get $s=l_1\sigma\varphi\rhd l_1\sigma\varphi'$. Therefore, by our first level of induction, from $l_1[p\leftarrow r_0\xi]\sigma\varphi'\longleftarrow_{\omega+n+1}l_1[p\leftarrow l_0\xi]\sigma\varphi'=l_1\sigma\varphi'\longrightarrow_{\omega+n+1}r_1\sigma\varphi'$ (which is due to Lemma A.7 and our first level of induction saying that R,X is ω-level confluent up to n) we conclude $t_0=l_1[p\leftarrow r_0\xi]\sigma\varphi\overset{*}{\longrightarrow}_{\omega+n+1}l_1[p\leftarrow r_0\xi]\sigma\varphi'\downarrow_{\omega+n+1}r_1\sigma\varphi'\overset{*}{\longleftarrow}_{\omega+n+1}r_1\sigma\varphi=t_1$.

Q.e.d. (The critical peak case) **Q.e.d. (Lemma B.6)**

Proof of Lemma B.7

1.: Since the direction "$\supseteq$" is trivial we only have to show "$\subseteq$" and begin with the first equation. For $t'\in\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]$ there are some $t\in\mathrm{T}$ and $p\in\mathcal{POS}(t)$ with $t/p=t'$. Now, in case of $t'\rightrightarrows t''$ by sort-invariance and T-monotonicity of $\rightrightarrows$ we get $t=t[p\leftarrow t']\rightrightarrows t[p\leftarrow t'']\in\mathrm{T}$, which implies $t''\in\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]$. Thus we have shown ${}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}\circ\rightrightarrows\ \subseteq\ {}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\,{}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}$. In case of $t'\in\mathrm{T}$ we can choose $p=\emptyset$ and get $t''\in\mathrm{T}$, which proves ${}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\ \subseteq\ {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\,{}_{\mathrm{T}}\!\rceil\mathrm{id}$.

2.: For $\mathrm{T}\ni t\rhd_{\mathrm{ST}}t'\rightrightarrows t''$ there is a $p\in\mathcal{POS}(t)$; $p\neq\emptyset$ with $t'=t/p$. By sort-invariance and T-monotonicity of $\rightrightarrows$ we get $t=t[p\leftarrow t']\rightrightarrows t[p\leftarrow t'']\rhd_{\mathrm{ST}}t''$ and $t[p\leftarrow t'']\in\mathrm{T}$.

3.: The subset relationship is simple:

$${}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}\circ(\rightrightarrows\cup\rhd_{\mathrm{ST}})^{+}\ \subseteq\ \trianglelefteq_{\mathrm{ST}}\circ\,{}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\trianglerighteq_{\mathrm{ST}}\circ(\rightrightarrows\cup\rhd_{\mathrm{ST}})^{+}\ \subseteq\ \trianglelefteq_{\mathrm{ST}}\circ\,{}_{\mathrm{T}}\!\rceil\mathrm{id}\circ(\rightrightarrows\cup\rhd_{\mathrm{ST}})^{+}.$$

The first equality follows from (1) and ${}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}}\ =\ {}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}}\circ\,{}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}\!\rceil\mathrm{id}$. For the second equality consider the following subset relationships as a word rewriting system over the alphabet $\{{}_{\mathrm{T}}\!\rceil\mathrm{id},\rightrightarrows,\rhd_{\mathrm{ST}}\}$ (containing three letters):

$$\begin{array}{lcll} {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}}\circ\rightrightarrows & \subseteq & {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\,{}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}} & ; \\ \rhd_{\mathrm{ST}}\circ\rhd_{\mathrm{ST}} & \subseteq & \rhd_{\mathrm{ST}} & ; \\ {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\rhd_{\mathrm{ST}} & \subseteq & {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\,{}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}} & ; \\ {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\rightrightarrows & \subseteq & {}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows\circ\,{}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows & . \end{array}$$

First note that the system is sound: The first rule was proved in (2). The second is transitivity of $\rhd_{\mathrm{ST}}$. The third and fourth are implied by (1). Since the number of substrings from $\{\rightrightarrows,\rhd_{\mathrm{ST}}\}^2$ is decreased by 1 by each of the rules, the word rewriting system is terminating. Thus, since all normal forms from ${}_{\mathrm{T}}\!\rceil\mathrm{id}\{\rightrightarrows,\rhd_{\mathrm{ST}}\}^{+}$ are in $\{{}_{\mathrm{T}}\!\rceil\mathrm{id}\rhd_{\mathrm{ST}}\}\cup\{{}_{\mathrm{T}}\!\rceil\mathrm{id}\rightrightarrows\}^{+}[\{{}_{\mathrm{T}}\!\rceil\mathrm{id}\rhd_{\mathrm{ST}}\}]$, we get ${}_{\mathrm{T}}\!\rceil\mathrm{id}\circ(\rightrightarrows\cup\rhd_{\mathrm{ST}})^{+}\ \subseteq\ \big({}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}}\big)\cup\big(({}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rightrightarrows)^{+}\circ({}_{\mathrm{T}}\!\rceil\mathrm{id}\circ\rhd_{\mathrm{ST}})^{=}\big)$. Using (1) again as well as $\rhd_{\mathrm{ST}}{}^{=}\subseteq\trianglerighteq_{\mathrm{ST}}$, this implies the one direction; the other direction as well as the special case are trivial.

4.: By the first equation of (3) we conclude $\rhd \subseteq \trianglerighteq_{\mathrm{ST}}[\mathrm{T}] \times \trianglerighteq_{\mathrm{ST}}[\mathrm{T}]$ as well as transitivity of $\rhd$. Suppose that $\rhd$ is not terminating. By the first equation of (3) there is some $r : \mathbf{N} \to \trianglerighteq_{\mathrm{ST}}[\mathrm{T}]$ with $\forall i \in \mathbf{N}.\ (\ r_i \rightrightarrows r_{i+1} \vee r_i \rhd_{\mathrm{ST}} r_{i+1}\ )$. There is some $t_0 \in \mathrm{T}$ and some $p_0 \in \mathcal{POS}(t_0)$ with $t_0/p_0 = r_0$. Moreover, there is also some $p : \mathbf{N}_+ \to \mathbf{N}^*$ such that

$$\forall i \in \mathbf{N}.\ \left(\begin{pmatrix} & r_i \rightrightarrows r_{i+1} \\ \wedge & p_{i+1} = \emptyset \end{pmatrix} \vee \begin{pmatrix} & r_i \rhd_{\mathrm{ST}} r_{i+1} \\ \wedge & r_i/p_{i+1} = r_{i+1} \end{pmatrix} \right).$$

Define $(t_n)_{n \in \mathbf{N}}$ inductively by $t_{n+1} := t_n[\,p_0 \ldots p_{n+1} \leftarrow r_{n+1}\,]$.

Claim 2: For each $n \in \mathbf{N}$ we get $\begin{pmatrix} & t_n, t_{n+1} \in \mathrm{T} \\ \wedge & t_n/p_0 \ldots p_n = r_n \\ \wedge & t_{n+1}/p_0 \ldots p_{n+1} = r_{n+1} \\ \wedge & \left(t_n \rightrightarrows t_{n+1} \vee \begin{pmatrix} & t_n = t_{n+1} \\ \wedge & r_n \rhd_{\mathrm{ST}} r_{n+1} \end{pmatrix} \right) \end{pmatrix}$.

Proof of Claim 2: We have $t_n \in \mathrm{T}$ and $t_n/p_0 \ldots p_n = r_n$ in case of $n = 0$ by our choice above and otherwise inductively by Claim 2. In case of $r_n \rightrightarrows r_{n+1} \wedge p_{n+1} = \emptyset$, since $\rightrightarrows$ is sort-invariant and T-monotonic, we thus get: $t_n = t_n[\,p_0 \ldots p_n \leftarrow r_n\,] \rightrightarrows t_n[\,p_0 \ldots p_n \leftarrow r_{n+1}\,] = t_n[\,p_0 \ldots p_n p_{n+1} \leftarrow r_{n+1}\,] = t_{n+1} \in \mathrm{T}$. Otherwise we have $r_n \rhd_{\mathrm{ST}} r_{n+1}$ and $r_n/p_{n+1} = r_{n+1}$ and get: $\mathrm{T} \ni t_n = t_n[\,p_0 \ldots p_n \leftarrow r_n\,] = t_n[\,p_0 \ldots p_n \leftarrow r_n[\,p_{n+1} \leftarrow r_{n+1}\,]\,] =$
$t_n[\,p_0 \ldots p_n \leftarrow r_n\,][\,p_0 \ldots p_n p_{n+1} \leftarrow r_{n+1}\,] = t_n[\,p_0 \ldots p_n p_{n+1} \leftarrow r_{n+1}\,] = t_{n+1}$. In both cases we have $t_{n+1}/p_0 \ldots p_{n+1} = t_n[\,p_0 \ldots p_{n+1} \leftarrow r_{n+1}\,]/p_0 \ldots p_{n+1} = r_{n+1}$. Q.e.d. (Claim 2)

Since $\rhd_{\mathrm{ST}}$ is terminating, Claim 2 contradicts $\rightrightarrows$ being terminating (below all $t \in \mathrm{T}$).

If $\rightrightarrows$ and T are X-stable, additionally, then $\rhd$ is X-stable too, because $\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]$, ${}_{\trianglerighteq_{\mathrm{ST}}[\mathrm{T}]}{\upharpoonleft}\mathrm{id}$, and $\rhd_{\mathrm{ST}}$ are.

Here is an example for $\rhd$ not sort-invariant and not T-monotonic: Let A, B be two different sorts. Let $\alpha(\mathsf{a}) = A$, $\alpha(\mathsf{f}) = A \to B$, $\alpha(\mathsf{g}) = A \to A$. Define $\rightrightarrows := \emptyset$ and $\mathrm{T} := \mathcal{T}$. Then we have $\rhd = \rhd_{\mathrm{ST}}$ and therefrom: $\mathsf{f}(\mathsf{a}) \rhd \mathsf{a}$ (hence not sort-invariant); and $\mathsf{g}(\mathsf{a}) \rhd \mathsf{a}$ but $\mathsf{f}(\mathsf{g}(\mathsf{a})) \not\rhd \mathsf{f}(\mathsf{a})$ (hence not T-monotonic).

5.: Take the signature from the example in the proof of (4). Define $\rightrightarrows := \{(\mathsf{a}, \mathsf{f}(\mathsf{a}))\}$ and $\mathrm{T} := \mathcal{T}$. Now $\rightrightarrows$ is a T-monotonic (indeed!), terminating relation on $\mathcal{T}$ that is not sort-invariant; whereas $\rhd$ is not irreflexive: $\mathsf{a} \rightrightarrows \mathsf{f}(\mathsf{a}) \rhd_{\mathrm{ST}} \mathsf{a}$. If one changes $\alpha(\mathsf{f})$ to be $\alpha(\mathsf{f}) = A \to A$, then $\rightrightarrows$ is a sort-invariant, terminating relation on $\mathcal{T}$ that is not T-monotonic but $\emptyset$-monotonic; whereas neither $\rhd$ nor $(\rightrightarrows \cup \rhd_{\mathrm{ST}})^+$ (in contrast to ${}_{\trianglerighteq_{\mathrm{ST}}[\emptyset]}{\upharpoonleft}\mathrm{id} \circ (\rightrightarrows \cup \rhd_{\mathrm{ST}})^+$) are irreflexive. **Q.e.d. (Lemma B.7)**

Proof of Lemma B.8

For the proof of Claim 3 below, we enrich the signatures by a new sort s_{new} and new constructor symbols $\mathsf{eq}_{\bar{s}}$ for each old sort $\bar{s}\in\mathbb{S}$ with arity $\bar{s}\bar{s}\rightarrow s_{\mathrm{new}}$ and $\bot$ with arity s_{new}. We take (in addition to R) the following set of new rules (with $X_{\bar{s}}\in \mathrm{V}_{\mathrm{SIG},\bar{s}}$ for $\bar{s}\in\mathbb{S}$):

$$\mathrm{R}' := \{\ \mathsf{eq}_{\bar{s}}(X_{\bar{s}},X_{\bar{s}}) = \bot \mid \bar{s}\in\mathbb{S}\ \}.$$

Since the sort restrictions do not allow $\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X},\beta}$ to make any use of terms of the sort s_{new} when rewriting terms of an "old" sort, we get

$$\forall\beta\preceq\omega{+}\omega.\ \longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X},\beta}\cap\,(\mathcal{T}(\mathrm{sig},\mathrm{X})\times\mathcal{T})\ =\ \longrightarrow_{\mathrm{R},\mathrm{X},\beta/\mathrm{sig}/\mathrm{cons}}$$

(the latter being defined over the non-enriched signatures). Thus, $\mathrm{T} := \overset{*}{\longrightarrow}[\{\hat{s}\}]$, ${}_{\mathrm{T}}{\upharpoonleft}\!\longrightarrow$, and ${}_{\unrhd[\mathrm{T}]}{\upharpoonleft}\!\longrightarrow$ do not change when we exchange the one $\longrightarrow$ with the other. We use '$\rhd_{\mathrm{ST}}$' to denote the subterm ordering over the enriched signature. For keeping the assumptions of our lemma valid for this subterm ordering (instead of the subterm ordering on the non-enriched signature) we have to extend $\rhd$ with $\mathsf{eq}_{\bar{s}}(t_0,t_1)\rhd t'$ if $\exists i\prec 2.\ t_i\unrhd_{\mathrm{ST}}t'$ for some $\bar{s}\in\mathbb{S}$ and $t_0,t_1\in\mathcal{T}(\mathrm{sig},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})_{\bar{s}}$. This extension neither changes $\unrhd[\mathrm{T}]$ nor ${}_{\unrhd[\mathrm{T}]}{\upharpoonleft}\!\longrightarrow$. Thus, since ${}_{\unrhd[\mathrm{T}]}{\upharpoonleft}\!\longrightarrow$ is not changed by any of the extensions, it now suffices to show its confluence after the extensions. Since the sort restrictions do not allow a term of the sort s_{new} to be a proper subterm of any other term, it is obvious that after the extension of $\rhd$ we still may assume either that ${}_{\mathrm{T}}{\upharpoonleft}\!\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X}}$ is terminating and $\rhd=\rhd_{\mathrm{ST}}$ or that ${}_{\unrhd[\mathrm{T}]}{\upharpoonleft}\!\longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X}}\subseteq\rhd$, $\rhd_{\mathrm{ST}}\subseteq\rhd$, and $\rhd$ is a wellfounded ordering on $\mathcal{T}$. Moreover, again due to the sort restrictions not allowing a term of the sort s_{new} to be a proper subterm of any other term, if $w\,(\longleftarrow\cup\lhd)^{+}\,(\hat{t}/p)\sigma\varphi$ holds for the extended $\longrightarrow$ and $\rhd$ and if $\hat{t}$ is an old term, then this also holds for the non-extended $\longrightarrow$ and $\rhd$. Therefore, (as no new critical peaks occur) the critical peaks keep being $\rhd$-quasi overlay joinable.

We define $\longrightarrow_{\beta} := \longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X},\beta}$ for any ordinal β with $\beta\prec\omega{+}\omega$; and $\longrightarrow := \longrightarrow_{\omega+\omega} := \longrightarrow_{\mathrm{R}\cup\mathrm{R}',\mathrm{X}}$. Since $\longrightarrow$ is sort-invariant, T-monotonic (cf. Corollary 2.8), and terminating below all $t\in\mathrm{T}$, by Lemma B.7(4), $\rhd' := {}_{\unrhd_{\mathrm{ST}}[\mathrm{T}]}{\upharpoonleft}\mathrm{id}\circ(\longrightarrow\cup\rhd_{\mathrm{ST}})^{+}$ is a wellfounded ordering on $\unrhd_{\mathrm{ST}}[\mathrm{T}]$. In case of $\rhd=\rhd_{\mathrm{ST}}$, we define $> := \rhd'$. Otherwise, in case that ${}_{\unrhd[\mathrm{T}]}{\upharpoonleft}\!\longrightarrow_{\mathrm{R},\mathrm{X}}\subseteq\rhd$, $\rhd_{\mathrm{ST}}\subseteq\rhd$, and $\rhd$ is a wellfounded ordering, we define $> := \rhd\cap(\unrhd[\mathrm{T}]\times\unrhd[\mathrm{T}])$. In any case, $>$ is a wellfounded ordering on $\unrhd[\mathrm{T}]$ containing ${}_{\unrhd[\mathrm{T}]}{\upharpoonleft}\mathrm{id}\circ(\longrightarrow\cup\rhd_{\mathrm{ST}}\cup\rhd)^{+}$. This means in particular that $\unrhd[\mathrm{T}]$ is closed under $\longrightarrow$, $\rhd_{\mathrm{ST}}$, and $\rhd$.

We say that $P(v,u,s,t,\Pi)$ holds if for $v,u,t\in\mathcal{T}(\mathrm{sig},\mathrm{X})$ and $s\in\unrhd[\mathrm{T}]$ with $v\overset{*}{\longleftarrow}u$; and $s\overset{*}{\longrightarrow}t$; $\Pi\subseteq\mathcal{POS}(u)$ with $\forall p,q{\in}\Pi.\ (\,p{\neq}q\Rightarrow p\parallel q\,)$ and $\forall o{\in}\Pi.\ u/o{=}s$; we have $v\downarrow u[o\leftarrow t\mid o{\in}\Pi]$. Now (by $\Pi := \{\emptyset\}$) it suffices to show that $P(v,u,s,t,\Pi)$ holds for all appropriate v,u,s,t,Π. We will show this by terminating induction over the lexicographic combination of the following orderings:

1. $>$
2. $\succ$
3. $\succ$

using the following measure on (v,u,s,t,Π):

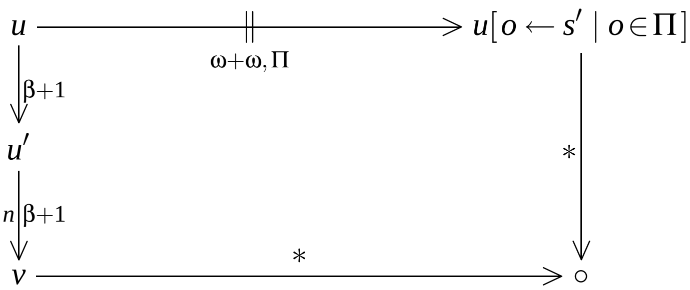
1. s
2. the smallest ordinal $\beta\preceq\omega{+}\omega$ for which $v\overset{*}{\longleftarrow}_{\beta}u$
3. the smallest $n\in\mathbf{N}$ for which $v\overset{n}{\longleftarrow}_{\beta}u$ for the β of (2)

For the limit ordinals 0, ω, ω+ω in the second position of the measure, the induction step is trivial ($\overset{*}{\longleftarrow}_0 \subseteq \mathrm{id}$; $\overset{*}{\longleftarrow}_\omega \subseteq \bigcup_{i\in\mathbf{N}} \overset{*}{\longleftarrow}_i$; $\overset{*}{\longleftarrow}_{\omega+\omega} \subseteq \bigcup_{i\in\mathbf{N}} \overset{*}{\longleftarrow}_{\omega+i}$). Thus, as we now suppose a smallest (v,u,s,t,Π) with $P(v,u,s,t,\Pi)$ not holding for, the second position of the measure must be a non-limit ordinal $\beta+1$.

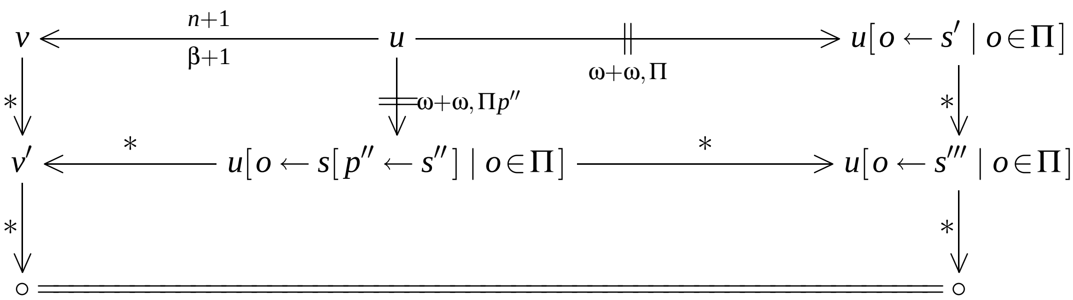
As $P(v,u,s,t,\Pi)$ holds trivially for $u=v$ or $s=t$ we have some u', s' with $v \overset{n}{\longleftarrow}_{\beta+1} u' \longleftarrow_{\beta+1} u$ $(n\in\mathbf{N})$ (with $\forall m\in\mathbf{N}.\ (v \overset{m}{\longleftarrow}_{\beta+1} u \ \Rightarrow\ m \succ n)$) and $s \longrightarrow s' \overset{*}{\longrightarrow} t$. Now for a contradiction it is sufficient to show

<u>**Claim**:</u> There is some z with $v \overset{*}{\longrightarrow} z \overset{*}{\longleftarrow} u[o \leftarrow s' \mid o \in \Pi]$.

because then we have $z \downarrow u[o \leftarrow t \mid o \in \Pi]$ by $P(z, u[o \leftarrow s' \mid o \in \Pi], s', t, \Pi)$, which is smaller than (v,u,s,t,Π) in the first position of the measure by $s \longrightarrow s'$.



<u>Claim 0:</u> We may assume $\forall p'' \in \mathcal{POS}(s)\setminus\{\emptyset\}.\ s/p'' \notin \mathrm{dom}(\longrightarrow)$.
<u>Proof of Claim 0:</u> Otherwise there are some $p'' \in \mathcal{POS}(s)\setminus\{\emptyset\}$ and some s'' with $s/p'' \longrightarrow s''$.



Then, by $P(s', s, s/p'', s'', \{p''\})$, which is smaller in the first position of the measure by $s \rhd_{\mathrm{ST}} s/p''$, we get $s' \overset{*}{\longrightarrow} s''' \overset{*}{\longleftarrow} s[p'' \leftarrow s'']$ for some s'''. Similarly, by $P(v, u, s/p'', s'', \Pi p'')$ we get $v \overset{*}{\longrightarrow} v' \overset{*}{\longleftarrow} u[p \leftarrow s'' \mid p \in \Pi p''] = u[o \leftarrow s[p'' \leftarrow s''] \mid o \in \Pi]$ for some v'. Finally, by $P(v',\ u[o \leftarrow s[p'' \leftarrow s''] \mid o \in \Pi],\ s[p'' \leftarrow s''],\ s''',\ \Pi\ )$, which is smaller in the first position of the measure by $s \longrightarrow s[p'' \leftarrow s'']$, we get $v' \downarrow u[o \leftarrow s''' \mid o \in \Pi] \overset{*}{\longleftarrow} u[o \leftarrow s' \mid o \in \Pi]$.

<u>Q.e.d. (Claim 0)</u>

By Claim 0 there are some $((l_0, r_0), C_0) \in \mathrm{R} \cup \mathrm{R}'$; $\mu_0 \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; with $s = l_0\mu_0$; $s' = r_0\mu_0$; and $C_0\mu_0$ is fulfilled w.r.t. $\longrightarrow$. Furthermore, we have some $q \in \mathcal{POS}(u)$; $((l_1, r_1), C_1) \in \mathrm{R} \cup \mathrm{R}'$; $\mu_1 \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$; with $u/q = l_1\mu_1$; $u' = u[q \leftarrow r_1\mu_1]$; $C_1\mu_1$ fulfilled w.r.t. $\longrightarrow_\beta$; and if C_1 contains some inequality $(u \neq v)$ then $\omega \preceq \beta$. By Claim 0 we may assume that q is not strictly below any $p \in \Pi$, i.e. that there are no p, p' with $pp' = q$, $p' \neq \emptyset$, and $p \in \Pi$.

Define $\Xi \;:=\; \Pi \setminus (q\mathbf{N}^*)$;
$\Pi' \;:=\; \{\, p' \mid\; qp' \in \Pi \;\wedge\; (p' \in \mathcal{POS}(l_1) \Rightarrow l_1/p' \in \mathrm{V})\,\}$;
$\Pi'' \;:=\; \{\, p' \mid qp' \in \Pi \setminus (q\Pi')\,\}$.

Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\, p'' \mid \exists p'.\, (l_1/p' = x \wedge p'p'' \in \Pi')\,\}$. Since for $p'' \in \Gamma(x)$ we always have some p' with $l_1/p' = x$; $x\mu_1/p'' = l_1\mu_1/p'p'' = u/qp'p'' = s$; we have

$$\forall x \in \mathrm{V}.\, \forall p'' \in \Gamma(x).\, x\mu_1/p'' = s. \quad (\#0)$$

Since the proper subterm ordering is irreflexive we cannot have $s \rhd_{\mathrm{ST}} s$, and therefore get

$$\forall x \in \mathrm{V}.\, \forall p', p'' \in \Gamma(x).\, (\, p' = p'' \vee p' \parallel p''\,). \quad (\#1)$$

Due to (#0) and (#1) we can define μ_1' by ($x \in \mathrm{V}$):

$$x\mu_1' := x\mu_1[\, p'' \leftarrow s' \mid p'' \in \Gamma(x)\,].$$

Define for $\bar{w} \in \mathcal{T}$:

$$\Theta_{\bar{w}} := \{\, p'p'' \mid \exists x.\, (\bar{w}/p' = x \wedge p'' \in \Gamma(x))\,\}.$$

By (#0) we get

$$\forall \bar{w} \in \mathcal{T}.\, \forall p' \in \Theta_{\bar{w}}.\, \bar{w}\mu_1/p' = s \quad (\#\Theta 1)$$

and by (#1)

$$\forall \bar{w} \in \mathcal{T}.\, \forall p', p'' \in \Theta_{\bar{w}}.\, (\, p' = p'' \;\vee p' \parallel p''\,) \quad (\#\Theta 2)$$

and

$$\forall \bar{w} \in \mathcal{T}.\, \bar{w}\mu_1' = \bar{w}\mu_1[\, p' \leftarrow s' \mid p' \in \Theta_{\bar{w}}]. \quad (\#\Theta 3)$$

Note that for $\Lambda := \Theta_{l_1} \setminus \Pi'$ we have

$$\Theta_{l_1} = \Pi' \uplus \Lambda. \quad (\#2)$$

By (#Θ1) and (#2) we get

$$\forall p' \in \Pi' \cup \Lambda \cup \Pi''.\, l_1\mu_1/p' = s \quad (\#3)$$

and by (#Θ2) and (#2)

$$\forall p', p'' \in \Pi' \cup \Lambda.\, (\, p' = p'' \;\vee\; p' \parallel p''\,). \quad (\#4)$$

Since

$$\forall p' \in \Pi' \cup \Lambda.\, (p' \in \mathcal{POS}(l_1) \;\Rightarrow\; l_1/p' \in \mathrm{V});$$
$$\forall p'' \in \Pi''.\, (\, p'' \in \mathcal{POS}(l_1) \;\wedge\; l_1/p'' \notin \mathrm{V}\,) \quad (\#5)$$

we get by (#3)

$$\forall p' \in \Pi' \cup \Lambda.\, \forall p'' \in \Pi''.\, p'' \parallel p' \quad (\#6)$$

and then together with (#2) and (#4)

$$\forall p', p'' \in \Pi' \uplus \Lambda \uplus \Pi''.\, (\, p' = p'' \;\vee\; p' \parallel p''\,). \quad (\#7)$$

Now due to (#2) and (#Θ3) we have

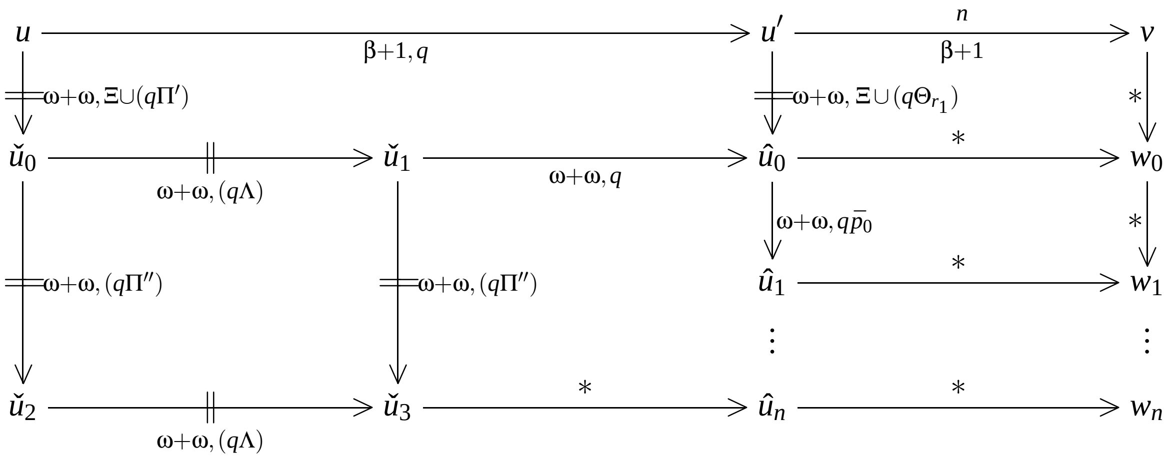
$$l_1\mu_1' = l_1\mu_1[\, p' \leftarrow s \mid p' \in \Pi' \cup \Lambda] \quad (\#8)$$

and then by (#6) and (#3)

$$\forall p'' \in \Pi''.\, l_1\mu_1'/p'' = s. \quad (\#9)$$

Summing up and defining we have:

$$
\begin{array}{llll}
 & \breve{u}_0 & := & u[q \leftarrow l_1\mu_1[p' \leftarrow s' \mid p' \in \Pi']] & [o \leftarrow s' \mid o \in \Xi] & ; \\
 & \breve{u}_1 & := & u[q \leftarrow l_1\mu_1[p' \leftarrow s' \mid p' \in \Pi' \cup \Lambda]] & [o \leftarrow s' \mid o \in \Xi] & \\
\text{(by (\#8))} & & = & u[q \leftarrow l_1\mu_1'] & [o \leftarrow s' \mid o \in \Xi] & ; \\
 & \breve{u}_2 & := & u[q \leftarrow l_1\mu_1[p' \leftarrow s' \mid p' \in \Pi' \cup \Pi'']] & [o \leftarrow s' \mid o \in \Xi] & ; \\
 & \breve{u}_3 & := & u[q \leftarrow l_1\mu_1[p' \leftarrow s' \mid p' \in \Pi' \cup \Lambda \cup \Pi'']] & [o \leftarrow s' \mid o \in \Xi] & \\
\text{(by (\#6), (\#8))} & & = & u[q \leftarrow l_1\mu_1'[p' \leftarrow s' \mid p' \in \Pi'']] & [o \leftarrow s' \mid o \in \Xi] & ; \\
 & u' & = & u[q \leftarrow r_1\mu_1] & & ; \\
 & \hat{u}_0 & := & u[q \leftarrow r_1\mu_1'] & [o \leftarrow s' \mid o \in \Xi] & \\
\text{(by Claim 2)} & & = & u[q \leftarrow \bar{u}_0] & [o \leftarrow s' \mid o \in \Xi] & ; \\
 & \hat{u}_{i+1} & := & u[q \leftarrow \bar{u}_{i+1}] & [o \leftarrow s' \mid o \in \Xi] & .
\end{array}
$$



Due to (#3) we have $\breve{u}_2 \leftarrow\!\!\!\!+\!\!\!\!+\!\!-_{\omega+\omega,(q\Pi'')} \breve{u}_0 -\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow_{\omega+\omega,(q\Lambda)} \breve{u}_1$.
Thus by (#6): $\breve{u}_2 -\!\!\!\!+\!\!\!\!+\!\!\!\rightarrow_{\omega+\omega,(q\Lambda)} \breve{u}_3 \leftarrow\!\!\!\!+\!\!\!\!+\!\!-_{\omega+\omega,(q\Pi'')} \breve{u}_1$.

We get $\breve{u}_1 \longrightarrow_{\omega+\omega,q} \hat{u}_0$ by Lemma 2.7 and

Claim 3: $C_1\mu_1'$ is fulfilled.

Moreover, we get $\hat{u}_0 \overset{*}{\longrightarrow} w_0 \overset{*}{\longleftarrow} v$ for some w_0 by (#Θ1), (#Θ2), (#Θ3), and $P(v, u', s, s', \Xi \cup (q\Theta_{r_1}))$, which is smaller in the second or third position of the measure.

Claim 1: We may assume that there is some $p \in \Pi''$ with $l_1\mu_1'[p \leftarrow s'] \neq r_1\mu_1'$.

Claim 2: There are some $\bar{n} \in \mathbf{N}$; $\bar{p} : \{0, \ldots, \bar{n}-1\} \rightarrow \mathbf{N}^*$; $\bar{u} : \{0, \ldots, \bar{n}\} \rightarrow \mathcal{T}(\mathrm{sig}, \mathrm{X})$; such that $l_1\mu_1'[p'' \leftarrow s' \mid p'' \in \Pi''] \overset{*}{\longrightarrow} \bar{u}_n$;
$\forall i \prec n. \left(\begin{array}{ll} & \bar{u}_{i+1} = \bar{u}_i[\bar{p}_i \leftarrow \bar{u}_{i+1}/\bar{p}_i] \\ \wedge & \bar{u}_{i+1}/\bar{p}_i \overset{*}{\longleftarrow} \bar{u}_i/\bar{p}_i < s \end{array} \right)$; and $\bar{u}_0 = r_1\mu_1'$.

Inductively for $i \prec n$ we now get some w_{i+1} with $\hat{u}_{i+1} \overset{*}{\longrightarrow} w_{i+1} \overset{*}{\longleftarrow} w_i$ due to Claim 2 and $P(w_i, \hat{u}_i, \bar{u}_i/\bar{p}_i, \bar{u}_{i+1}/\bar{p}_i, \{q\bar{p}_i\})$ which is smaller in the first position of the measure by Claim 2. Finally by Claim 2 we get $\breve{u}_3 \overset{*}{\longrightarrow} u[q \leftarrow \bar{u}_n][o \leftarrow s' \mid o \in \Xi] = \hat{u}_n$. This completes the proof of **Claim** due to $u[o \leftarrow s' \mid o \in \Pi] = \breve{u}_2 \overset{*}{\longrightarrow} w_n \overset{*}{\longleftarrow} v$.

<u>Proof of Claim 1:</u> In case of $p,p'\in\Pi''$ with $l_1\mu'_1[p\leftarrow s']=r_1\mu'_1$ and $l_1\mu'_1[p'\leftarrow s']=r_1\mu'_1$ we cannot have $p\parallel p'$ because then by (#9) we would get the contradiction $s=l_1\mu'_1/p=l_1\mu'_1[p'\leftarrow s']/p=r_1\mu'_1/p=l_1\mu'_1[p\leftarrow s']/p=s'<s$. Therefore, if Claim 1 does not hold, i.e. if $\forall p''\in\Pi''.\ l_1\mu'_1[p''\leftarrow s']=r_1\mu'_1$, by (#7) must have we have $|\Pi''|\preceq 1$. In case $\Pi''=\emptyset$, we have $\check{u}_3=\check{u}_1\overset{*}{\longrightarrow}w_0$. Otherwise, in case of $\Pi''=\{p\}$ and $l_1\mu'_1[p\leftarrow s']=r_1\mu'_1$, we have $l_1\mu'_1[p'\leftarrow s'\mid p'\in\Pi'']=l_1\mu'_1[p\leftarrow s']=r_1\mu'_1$, and then $\check{u}_3=\hat{u}_0\overset{*}{\longrightarrow}w_0$. In both cases we have shown **Claim** due to $u[o\leftarrow s'\mid o\in\Pi]=\check{u}_2\overset{*}{\longrightarrow}\check{u}_3\overset{*}{\longrightarrow}w_0\overset{*}{\longleftarrow}v$. <u>Q.e.d. (Claim 1)</u>

<u>Proof of Claim 2</u>: Let $\xi\in\mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(l_0{=}r_0{\longleftarrow}C_0)]\cap\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)=\emptyset$. Let ρ be given by ($x\in\mathrm{V}$): $x\rho:=\left\{\begin{array}{ll}x\mu'_1 & \text{if } x\in\mathcal{V}(l_1{=}r_1{\longleftarrow}C_1)\\ x\xi^{-1}\mu_0 & \text{otherwise}\end{array}\right\}$.

By (#9) and (#5) for the p of Claim 1 we have $l_0\xi\rho=l_0\xi\xi^{-1}\mu_0=s=l_1\mu'_1/p=(l_1/p)\rho$ and $l_1/p\not\in\mathrm{V}$. Thus, let $\mathrm{Y}:=\mathcal{V}((l_0{=}r_0{\longleftarrow}C_0)\xi,l_1{=}r_1{\longleftarrow}C_1)$; $\sigma:=\mathrm{mgu}(\{(l_0\xi,l_1/p)\},\mathrm{Y})$; and $\varphi\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonleft\!(\sigma\varphi)={}_{\mathrm{Y}}\!\upharpoonleft\!\rho$. Let $t_0:=l_1[p\leftarrow r_0\xi]$ and $t_1:=r_1$. By Claim 1 we may assume $t_0\sigma\neq t_1\sigma$ (since otherwise $l_1\mu'_1[p\leftarrow s']=l_1\mu'_1[p\leftarrow r_0\mu_0]=t_0\sigma\varphi=t_1\sigma\varphi=r_1\mu'_1$). Thus $((t_0,C_0\xi,\ldots),\ (t_1,C_1,\ldots),\ l_1,\ \sigma,\ p)$ is a critical peak. By Lemma 2.12, $(C_0\xi C_1)\sigma\varphi$ is fulfilled w.r.t. $\longrightarrow_{\omega+\omega}$. Since $(l_1/p)\sigma\varphi=s$ it makes sense to define $\Delta:=\{\,p'\in\mathcal{POS}(l_1)\setminus\{p\}\mid l_1/p'\not\in\mathrm{V}\wedge(l_1/p')\sigma\varphi=s\,\}$. Then by (#5) and (#9) we get $\Pi''\subseteq\{p\}\cup\Delta$. Thus by $p\in\Pi''$ we get $\Pi''\cup\Delta=\{p\}\cup\Delta$ and therefore

$$\begin{array}{rcll}
l_1\mu'_1[p''\leftarrow s'\mid p''\in\Pi''] & = & l_1\sigma\varphi[p''\leftarrow s'\mid p''\in\Pi''] & [p''\leftarrow s\mid p''\in\Delta\setminus\Pi'']\\
& \overset{*}{\longrightarrow} & l_1\sigma\varphi[p''\leftarrow s'\mid p''\in\Pi''] & [p''\leftarrow s'\mid p''\in\Delta\setminus\Pi'']\\
& = & l_1\sigma\varphi[p''\leftarrow s'\mid p''\in\{p\}\cup\Delta] & \\
& = & l_1[p\leftarrow r_0\xi] & [p''\leftarrow r_0\xi\mid p''\in\Delta]\quad\sigma\varphi\\
& = & t_0 & [p''\leftarrow t_0/p\mid p''\in\Delta]\quad\sigma\varphi.
\end{array}$$

Moreover, for w with $w\,(\longleftarrow\cup\lhd)^+\,(l_1/p)\sigma\varphi$ due to $(l_1/p)\sigma\varphi=s$ we have $w<s$ and therefore $\longrightarrow$ is confluent below w due to $P(?,w,w,?,\{\emptyset\})$ which is smaller in the first position of the measure. Finally, by Claim 0 we get $\forall p''\in\mathcal{POS}((l_1/p)\sigma\varphi)\setminus\{\emptyset\}.\ (l_1/p)\sigma\varphi/p''\not\in\mathrm{dom}(\longrightarrow_{\mathrm{R,X}})$. Thus, by $\rhd$-quasi overlay joinability, there are some $\bar{n}\in\mathbf{N}$; $\bar{p}:\{0,\ldots,\bar{n}{-}1\}\to\mathbf{N}^*$; $\bar{u}:\{0,\ldots,\bar{n}\}\to\mathcal{T}$; with $t_0[p''\leftarrow t_0/p\mid p''\in\Delta]\sigma\varphi\overset{*}{\longrightarrow}\bar{u}_{\bar{n}}$;
$\forall i\prec\bar{n}.\left(\begin{array}{l}\quad\bar{u}_{i+1}=\bar{u}_i[\bar{p}_i\leftarrow\bar{u}_{i+1}/\bar{p}_i]\\ \wedge\quad\bar{u}_{i+1}/\bar{p}_i\overset{*}{\longleftarrow}\bar{u}_i/\bar{p}_i\ (\longleftarrow\cup\lhd)^+\ (l_1/p)\sigma\varphi=s\end{array}\right)$ and $\bar{u}_0=t_1\sigma\varphi=r_1\mu'_1$. Finally, for all $\bar{v}$ due to $s\in\trianglerighteq[\mathrm{T}]$ we know that $s(\longrightarrow\cup\rhd)^+\bar{v}$ implies $s>\bar{v}$. <u>Q.e.d. (Claim 2)</u>

<u>Proof of Claim 3:</u> For $(\bar{u}{=}\bar{v})$ in C_1 we have $\bar{u}\mu_1\downarrow_\beta\bar{v}\mu_1$. In case of $\beta=0$, due to (#Θ1), (#Θ2), and (#Θ3), we have $\bar{u}\mu_1[p'\leftarrow s'\mid p'\in\Theta_{\bar{u}}]\leftarrow\!\!\!\!+\!\!\!\!\!+\!\!\!-_{\Theta_{\bar{u}}}\bar{u}\mu_1=\bar{v}\mu_1-\!\!\!\!\!+\!\!\!\!+\!\!\!\!\rightarrow_{\Theta_{\bar{v}}}\bar{v}\mu_1[p'\leftarrow s'\mid p'\in\Theta_{\bar{v}}]$ and then $\bar{u}\mu'_1=\bar{u}\mu_1[p'\leftarrow s'\mid p'\in\Theta_{\bar{u}}]-\!\!\!\!\!+\!\!\!\!+\!\!\!\!\rightarrow_{\Theta_{\bar{v}}\setminus\Theta_{\bar{u}}}$
$\bar{u}\mu_1[p'\leftarrow s'\mid p'\in\Theta_{\bar{u}}\cup\Theta_{\bar{v}}]=\bar{v}\mu_1[p'\leftarrow s'\mid p'\in\Theta_{\bar{v}}\cup\Theta_{\bar{u}}]\leftarrow\!\!\!\!+\!\!\!\!\!+\!\!\!-_{\Theta_{\bar{u}}\setminus\Theta_{\bar{v}}}\bar{v}\mu_1[p'\leftarrow s'\mid p'\in\Theta_{\bar{v}}]=\bar{v}\mu_1$.
Otherwise, in case of $0\prec\beta$, we have for the sort $\bar{s}\in\mathbb{S}$ of $\bar{u}$: $\bot\overset{+}{\longleftarrow}_\beta(\mathrm{eq}_{\bar{s}}(\bar{u},\bar{v}))\mu_1$. We get $\bot\downarrow(\mathrm{eq}_{\bar{s}}(\bar{u},\bar{v}))\mu'_1$ by $P(\bot,\ (\mathrm{eq}_{\bar{s}}(\bar{u},\bar{v}))\mu_1,\ s,\ s'\ ,\Theta_{\mathrm{eq}_{\bar{s}}(\bar{u},\bar{v})})$ which is smaller in the second position. Since there are no rules for $\bot$ and only one for $\mathrm{eq}_{\bar{s}}$, this means $\bar{u}\mu'_1\downarrow\bar{v}\mu'_1$. For (Def $\bar{u}$) in C_1 we know the existence of some $\vec{u}\in\mathcal{GT}(\mathrm{cons})$ with $\vec{u}\overset{*}{\longleftarrow}_\beta\bar{u}\mu_1$. We get some $\hat{u}$ with $\vec{u}\overset{*}{\longrightarrow}\hat{u}\overset{*}{\longleftarrow}\bar{u}\mu'_1$ by $P(\vec{u},\bar{u}\mu_1,s,s',\Theta_{\bar{u}})$ which is smaller in the second position. By Lemma 2.10 we get $\hat{u}\in\mathcal{GT}(\mathrm{cons})$. Finally, for $(\bar{u}{\neq}\bar{v})$ in C_1 we have some $\vec{u},\vec{v}\in\mathcal{GT}(\mathrm{cons})$ with $\bar{u}\mu_1\overset{*}{\longrightarrow}_\beta\vec{u}\,\natural\,\vec{v}\overset{*}{\longleftarrow}_\beta\bar{v}\mu_1$ (by Lemma 2.11 and $\omega\preceq\beta$). By applying the same procedure as before twice we get $\hat{u},\hat{v}\in\mathcal{GT}(\mathrm{cons})$ with $\bar{u}\mu'_1\overset{*}{\longrightarrow}\hat{u}\overset{*}{\longleftarrow}\vec{u}\,\natural\,\vec{v}\overset{*}{\longrightarrow}\hat{v}\overset{*}{\longleftarrow}\bar{v}\mu'_1$, i.e. $\bar{u}\mu'_1\overset{*}{\longrightarrow}\hat{u}\,\natural\,\hat{v}\overset{*}{\longleftarrow}\bar{v}\mu'_1$. <u>Q.e.d. (Claim 3)</u>

<u>Q.e.d. (Lemma B.8)</u>

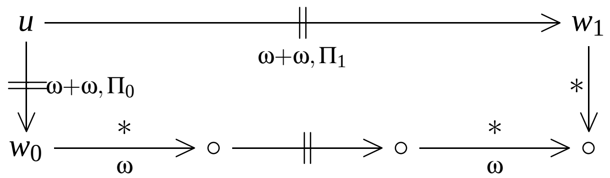
Proof of Lemma C.3

Claim 0: $\longrightarrow$ and $\longrightarrow_{\omega}$ are commuting.
Proof of Claim 0: By the assumed strong commutation assumption and Lemma 3.3 $\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}$ and $\stackrel{*}{\longrightarrow}_{\omega}$ are commuting. Since by Corollary 2.14 we have $\longrightarrow\ \subseteq\ \mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}\ \subseteq\ \stackrel{*}{\longrightarrow}$, now $\longrightarrow$ and $\longrightarrow_{\omega}$ are commuting, too. Q.e.d. (Claim 0)

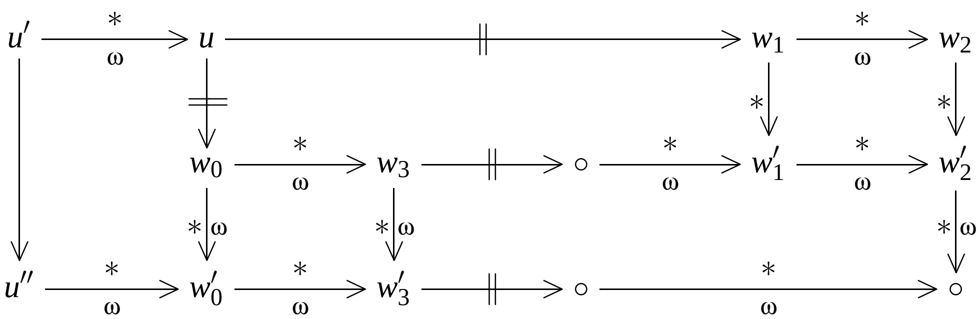
Claim 1: If $\stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}$ strongly commutes over $\stackrel{*}{\longrightarrow}$, then $\longrightarrow$ is confluent.
Proof of Claim 1: $\stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}$ and $\stackrel{*}{\longrightarrow}$ are commuting by Lemma 3.3. Since by Corollary 2.14 we have $\longrightarrow\ \subseteq\ \stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}\ \subseteq\ \stackrel{*}{\longrightarrow}$, now $\longrightarrow$ and $\longrightarrow$ are commuting, too.
Q.e.d. (Claim 1)

We are going to show the following property:

$$w_0 \mathrel{\leftarrow\!\!\!\Vert\!\!-}_{\omega+\omega,\Pi_0} u \mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+\omega,\Pi_1} w_1 \quad\Rightarrow\quad w_0 \stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{*}{\longleftarrow} w_1.$$



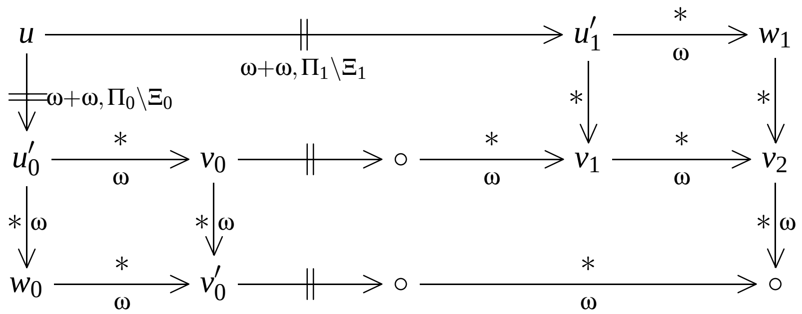
Claim 2: The above property implies that $\stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}$ strongly commutes over $\stackrel{*}{\longrightarrow}$ and that $\longrightarrow$ is confluent.
Proof of Claim 2: First we show the strong commutation. By Lemma 3.3 it suffices to show that $\stackrel{*}{\longrightarrow}_{\omega}\circ\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}$ strongly commutes over $\longrightarrow$. Assume $u''\longleftarrow u'\stackrel{*}{\longrightarrow}_{\omega} u\mathrel{-\!\!\Vert\!\!\!\rightarrow} w_1\stackrel{*}{\longrightarrow}_{\omega} w_2$ (cf. diagram below). By the strong commutation assumed for our lemma and Corollary 2.14, there are w_0 and w_0' with $u''\stackrel{*}{\longrightarrow}_{\omega} w_0'\stackrel{*}{\longleftarrow}_{\omega} w_0\mathrel{\leftarrow\!\!\!\Vert\!\!-} u$. By the above property there are some w_3, w_1' with $w_0\stackrel{*}{\longrightarrow}_{\omega} w_3\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega} w_1'\stackrel{*}{\longleftarrow} w_1$. By Claim 0 we can close the peak $w_1'\stackrel{*}{\longleftarrow} w_1\stackrel{*}{\longrightarrow}_{\omega} w_2$ according to $w_1'\stackrel{*}{\longrightarrow}_{\omega} w_2'\stackrel{*}{\longleftarrow} w_2$ for some w_2'. By the assumed confluence of $\longrightarrow_{\omega}$, we can close the peak $w_0'\stackrel{*}{\longleftarrow}_{\omega} w_0\stackrel{*}{\longrightarrow}_{\omega} w_3$ according to $w_0'\stackrel{*}{\longrightarrow}_{\omega} w_3'\stackrel{*}{\longleftarrow}_{\omega} w_3$ for some w_3'. To close the whole diagram, we only have to show that we can close the peak $w_3'\stackrel{*}{\longleftarrow}_{\omega} w_3\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega} w_2'$ according to $w_3'\mathrel{-\!\!\Vert\!\!\!\rightarrow}\circ\stackrel{*}{\longrightarrow}_{\omega}\circ\stackrel{*}{\longleftarrow}_{\omega} w_2'$, which is possible due to the strong commutation assumed for our lemma.



Finally, confluence of $\longrightarrow$ follows from Claim 1. Q.e.d. (Claim 2)

W.l.o.g. let the positions of Π_i be maximal in the sense that for any $p\in\Pi_i$ and $\Xi\subseteq\mathcal{POS}(u)\cap(p\mathbf{N}^+)$ we do not have $u\mathrel{-\!\!\Vert\!\!\!\rightarrow}_{\omega+\omega,(\Pi_i\setminus\{p\})\cup\Xi} w_i$ anymore. Then for each $i\prec 2$ and $p\in\Pi_i$ there are $((l_{i,p},r_{i,p}),C_{i,p})\in\mathrm{R}$ and $\mu_{i,p}\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with $u/p=l_{i,p}\mu_{i,p}$, $r_{i,p}\mu_{i,p}=w_i/p$, $C_{i,p}\mu_{i,p}$ fulfilled w.r.t. $\longrightarrow$. Finally, for each $i\prec 2$: $w_i=u[p\leftarrow r_{i,p}\mu_{i,p}\mid p\in\Pi_i]$.

<u>Claim 5:</u> We may assume $\forall i\prec 2.\ \forall p\in\Pi_i.\ l_{i,p}\notin\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$.
<u>Proof of Claim 5:</u> Define $\Xi_i := \{\, p\in\Pi_i \mid l_{i,p}\in\mathcal{T}(\mathrm{cons},\mathrm{V_{SIG}}\uplus\mathrm{V}_{\mathcal{C}})\,\}$ and $u'_i := u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i\setminus\Xi_i\,]$. If we have succeeded with our proof under the assumption of Claim 5, then we have shown $u'_0 \xrightarrow{*}_{\omega} v_0 \mathrel{-\!\!\|\!\!\rightarrow} \circ \xrightarrow{*}_{\omega} v_1 \xleftarrow{*} u'_1$ for some v_0, v_1 (cf. diagram below). By Lemma 13.2 (matching both its μ and ν to our $\mu_{i,p}$) we get $\forall i\prec 2.\ \forall p\in\Xi_i.\ l_{i,p}\mu_{i,p}\longrightarrow_{\omega} r_{i,p}\mu_{i,p}$ and therefore $\forall i\prec 2.\ u'_i \xrightarrow{*}_{\omega} w_i$. Thus from $v_1 \xleftarrow{*} u'_1 \xrightarrow{*}_{\omega} w_1$ we get $v_1 \xrightarrow{*}_{\omega} v_2 \xleftarrow{*} w_1$ for some v_2 by Claim 0. Due to the assumed confluence of $\longrightarrow_{\omega}$, we can close the peak $w_0 \xleftarrow{*}_{\omega} u'_0 \xrightarrow{*}_{\omega} v_0$ according to $w_0 \xrightarrow{*}_{\omega} v'_0 \xleftarrow{*}_{\omega} v_0$ for some v'_0. By the strong commutation assumption of our lemma, from $v'_0 \xleftarrow{*}_{\omega} v_0 \mathrel{-\!\!\|\!\!\rightarrow} \circ \xrightarrow{*}_{\omega} v_1 \xrightarrow{*}_{\omega} v_2$ we can finally conclude $v'_0 \mathrel{-\!\!\|\!\!\rightarrow} \circ \xrightarrow{*}_{\omega} \circ \xleftarrow{*}_{\omega} v_2$.



<u>Q.e.d. (Claim 5)</u>

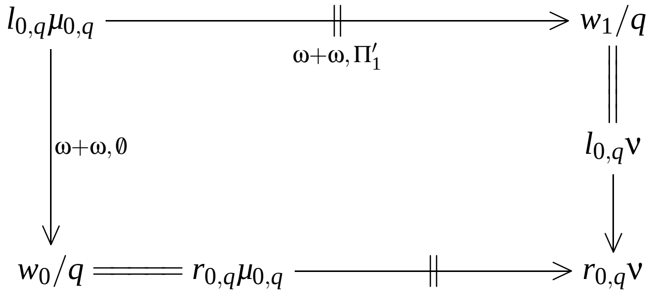
Define the set of inner overlapping positions by
$$\Omega(\Pi_0,\Pi_1) := \bigcup_{i\prec 2}\{\, p\in\Pi_{1-i} \mid \exists q\in\Pi_i.\ \exists q'\in\mathbf{N}^{*}.\ p=qq'\,\},$$
and the length of a term by $\lambda(f(t_0,\ldots,t_{m-1})) := 1+\sum_{j\prec m}\lambda(t_j)$.

Now we start an induction on $\displaystyle\sum_{p'\in\Omega(\Pi_0,\Pi_1)}\lambda(u/p')$ in $\prec$.

Define the set of top positions by
$$\Theta := \{\, p\in\Pi_0\cup\Pi_1 \mid \neg\exists q\in\Pi_0\cup\Pi_1.\ \exists q'\in\mathbf{N}^{+}.\ p=qq'\,\}.$$
Since the prefix ordering is wellfounded we have $\forall i\prec 2.\ \forall p\in\Pi_i.\ \exists q\in\Theta.\ \exists q'\in\mathbf{N}^{*}.\ p=qq'$. Then $\forall i\prec 2.\ w_i=w_i[q\leftarrow w_i/q \mid q\in\Theta]=u[\,p\leftarrow r_{i,p}\mu_{i,p} \mid p\in\Pi_i][q\leftarrow w_i/q \mid q\in\Theta]=u[q\leftarrow w_i/q \mid q\in\Theta]$. Thus, it now suffices to show for all $q\in\Theta$
$$w_0/q \xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\|\!\!\rightarrow} \circ \xrightarrow{*}_{\omega} \circ \xleftarrow{*} w_1/q$$
because then we have
$$w_0=u[q\leftarrow w_0/q \mid q\in\Theta] \xrightarrow{*}_{\omega} \circ \mathrel{-\!\!\|\!\!\rightarrow} \circ \xrightarrow{*}_{\omega} \circ \xleftarrow{*} u[q\leftarrow w_1/q \mid q\in\Theta]=w_1.$$
Therefore we are left with the following two cases for $q\in\Theta$:

$\underline{\underline{q \notin \Pi_1}}$: Then $q \in \Pi_0$. Define $\Pi'_1 := \{\ p \mid qp \in \Pi_1\ \}$. We have two cases:

$\underline{\underline{\text{“The variable overlap (if any) case”: } \forall p \in \Pi'_1 \cap \mathcal{POS}(l_{0,q}).\ l_{0,q}/p \in \mathrm{V}:}}$



Define a function Γ on V by ($x \in \mathrm{V}$): $\Gamma(x) := \{\ (p', p'') \mid l_{0,q}/p' = x \wedge p'p'' \in \Pi'_1\ \}$.

<u>Claim 7:</u> There is some $\nu \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))$ with

$$\forall x \in \mathrm{V}. \left(\begin{array}{ll} & x\mu_{0,q} \dashv\!\!\vdash\!\!\rightarrow x\nu \\ \wedge & \forall p' \in \mathrm{dom}(\Gamma(x)).\ x\nu = x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \end{array} \right).$$

<u>Proof of Claim 7:</u>
In case of $\mathrm{dom}(\Gamma(x)) = \emptyset$ we define $x\nu := x\mu_{0,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x)) = \{p'\}$ we define $x\nu := x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,]$. This is appropriate since due to $\forall (p', p'') \in \Gamma(x).\ x\mu_{0,q}/p'' = l_{0,q}\mu_{0,q}/p'p'' = u/qp'p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ we have

$$x\mu_{0,q} = x\mu_{0,q}[\,p'' \leftarrow l_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \dashv\!\!\vdash\!\!\rightarrow$$
$$x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] = x\nu.$$

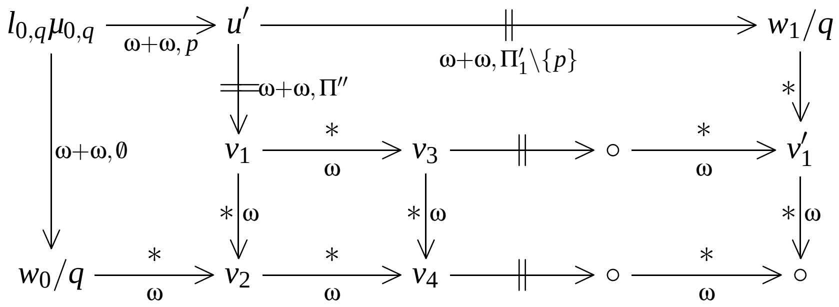
Finally, in case of $|\mathrm{dom}(\Gamma(x))| \succ 1$, $l_{0,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x \in \mathrm{V}_{\mathcal{C}}$. Since there is some $(p', p'') \in \Gamma(x)$ with $x\mu_{0,q}/p'' = l_{1,qp'p''}\mu_{1,qp'p''}$ this implies $l_{1,qp'p''}\mu_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathcal{C}})$ and then $l_{1,qp'p''} \in \mathcal{T}(\mathrm{cons}, \mathrm{V}_{\mathrm{SIG}} \uplus \mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. <u>Q.e.d. (Claim 7)</u>

<u>Claim 8:</u> $l_{0,q}\nu = w_1/q$.
<u>Proof of Claim 8:</u>
By Claim 7 we get $w_1/q = u/q[\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q} \mid l_{0,q}/p' = x \in \mathrm{V}\,][\,p'p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid \exists x \in \mathrm{V}.\ (p', p'') \in \Gamma(x)\,] =$
$l_{0,q}[\,p' \leftarrow x\mu_{0,q}[\,p'' \leftarrow r_{1,qp'p''}\mu_{1,qp'p''} \mid (p', p'') \in \Gamma(x)\,] \mid l_{0,q}/p' = x \in \mathrm{V}\,] =$
$l_{0,q}[\,p' \leftarrow x\nu \mid l_{0,q}/p' = x \in \mathrm{V}\,] = l_{0,q}\nu$. <u>Q.e.d. (Claim 8)</u>

<u>Claim 9:</u> $w_0/q \dashv\!\!\vdash\!\!\rightarrow r_{0,q}\nu$.
<u>Proof of Claim 9:</u> Since $w_0/q = r_{0,q}\mu_{0,q}$, this follows directly from Claim 7. <u>Q.e.d. (Claim 9)</u>

By claims 8 and 9 it now suffices to show $l_{0,q}\nu \longrightarrow r_{0,q}\nu$, which again follows from Lemma C.4 since $\longrightarrow$ and $\longrightarrow_\omega$ are commuting by Claim 0 and since $\forall x \in \mathrm{V}.\ x\mu_{0,q} \overset{*}{\longrightarrow} x\nu$ by Claim 7 and Corollary 2.14.
$\underline{\underline{\text{Q.e.d. (“The variable overlap (if any) case”)}}}$

"The critical peak case": There is some $p \in \Pi_1' \cap \mathcal{POS}(l_{0,q})$ with $l_{0,q}/p \notin \mathrm{V}$:



Claim 10: $p \neq \emptyset$.
Proof of Claim 10: If $p=\emptyset$, then $\emptyset \in \Pi_1'$, then $q \in \Pi_1$, which contradicts our global case assumption. Q.e.d. (Claim 10)

Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{1,qp},r_{1,qp}),C_{1,qp}))] \cap \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{1,qp},r_{1,qp}),C_{1,qp}))] \cup \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \left\{\begin{array}{ll} x\mu_{0,q} & \text{if } x \in \mathcal{V}(((l_{0,q},r_{0,q}),C_{0,q})) \\ x\xi^{-1}\mu_{1,qp} & \text{else} \end{array}\right\}$ $(x \in \mathrm{V})$.
By $l_{1,qp}\xi\rho = l_{1,qp}\xi\xi^{-1}\mu_{1,qp} = u/qp = l_{0,q}\mu_{0,q}/p = l_{0,q}\rho/p = (l_{0,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{1,qp}\xi, l_{0,q}/p)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\!\upharpoonright\!(\sigma\varphi) = {}_{\mathrm{Y}}\!\upharpoonright\!\rho$.
Define $u' := l_{0,q}\mu_{0,q}[p \leftarrow r_{1,qp}\mu_{1,qp}]$. We get

$$u' = u/q[p' \leftarrow l_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi_1'\setminus\{p\}][p \leftarrow r_{1,qp}\mu_{1,qp}] \;{+\!\!\!\Vdash\!\!\!\rightarrow}_{\omega+\omega,\Pi_1'\setminus\{p\}}$$
$$u/q[p' \leftarrow r_{1,qp'}\mu_{1,qp'} \mid p' \in \Pi_1'] = w_1/q.$$

If $l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma = r_{0,q}\sigma$, then the proof is finished due to

$$w_0/q = r_{0,q}\mu_{0,q} = r_{0,q}\sigma\varphi = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi = u' \;{+\!\!\!\Vdash\!\!\!\rightarrow}_{\omega+\omega,\Pi_1'\setminus\{p\}} w_1/q.$$

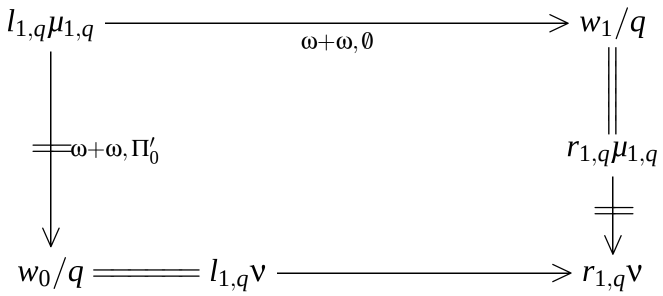
Otherwise we have $((l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma, C_{1,qp}\xi\sigma, 1), (r_{0,q}\sigma, C_{0,q}\sigma, 1), l_{0,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $p \neq \emptyset$ (due to Claim 10); $C_{1,qp}\xi\sigma\varphi = C_{1,qp}\mu_{1,qp}$ is fulfilled w.r.t. $\longrightarrow$; $C_{0,q}\sigma\varphi = C_{0,q}\mu_{0,q}$ is fulfilled w.r.t. $\longrightarrow$. Due to Claim 0 and our assumed ω-coarse level parallel closedness we have $u' = l_{0,q}[p \leftarrow r_{1,qp}\xi]\sigma\varphi \;{+\!\!\!\Vdash\!\!\!\rightarrow}\; v_1 \overset{*}{\longrightarrow}_{\omega} v_2 \overset{*}{\longleftarrow}_{\omega} r_{0,q}\sigma\varphi = r_{0,q}\mu_{0,q} = w_0/q$ for some v_1, v_2. We then have $v_1 \;{\leftarrow\!\!\!\Vdash\!\!\!+}_{\omega+\omega,\Pi''}\; u' \;{+\!\!\!\Vdash\!\!\!\rightarrow}_{\omega+\omega,\Pi_1'\setminus\{p\}}\; w_1/q$ for some Π''. By $\sum_{p'' \in \Omega(\Pi'',\Pi_1'\setminus\{p\})} \lambda(u'/p'') \preceq$

$\sum_{p'' \in \Pi_1'\setminus\{p\}} \lambda(u'/p'') = \sum_{p'' \in \Pi_1'\setminus\{p\}} \lambda(u/qp'') \prec$
$\sum_{p'' \in \Pi_1'} \lambda(u/qp'') = \sum_{p' \in q\Pi_1'} \lambda(u/p') = \sum_{p' \in \Omega(\{q\},\Pi_1)} \lambda(u/p') \preceq \sum_{p' \in \Omega(\Pi_0,\Pi_1)} \lambda(u/p')$, due to our induction hypothesis we get some v_1', v_3 with $v_1 \overset{*}{\longrightarrow}_{\omega} v_3 \;{+\!\!\!\Vdash\!\!\!\rightarrow}\; \circ \overset{*}{\longrightarrow}_{\omega} v_1' \overset{*}{\longleftarrow} w_1/q$. By confluence of $\longrightarrow_{\omega}$ we can close the peak at v_1 according to $v_2 \overset{*}{\longrightarrow}_{\omega} v_4 \overset{*}{\longleftarrow}_{\omega} v_3$ for some v_4. Finally by the strong commutation assumption of our lemma, the peak at v_3 can be closed according to $v_4 \;{+\!\!\!\Vdash\!\!\!\rightarrow}\; \circ \overset{*}{\longrightarrow}_{\omega} \circ \overset{*}{\longleftarrow}_{\omega} v_1'$.

Q.e.d. ("The critical peak case") Q.e.d. ("$q \notin \Pi_1$")

$\underline{\underline{q\in\Pi_1}}$: Define $\Pi_0' := \{\ p \mid qp\in\Pi_0\ \}$. We have two cases:

$\underline{\underline{\text{“The second variable overlap (if any) case”: } \forall p\in\Pi_0'\cap\mathcal{POS}(l_{1,q}).\ l_{1,q}/p\in\mathrm{V}:}}$



Define a function Γ on V by ($x\in\mathrm{V}$): $\Gamma(x) := \{\ (p',p'') \mid l_{1,q}/p'=x \wedge p'p''\in\Pi_0'\ \}$.

Claim 11: There is some $\nu\in\mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with

$$\forall x\in\mathrm{V}.\left(\begin{array}{ll} & x\nu \leftarrow\!\!\!+\!\!\!+\!\!\!- x\mu_{1,q} \\ \wedge & \forall p'\in\mathrm{dom}(\Gamma(x)).\ x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]=x\nu \end{array}\right).$$

Proof of Claim 11:
In case of $\mathrm{dom}(\Gamma(x))=\emptyset$ we define $x\nu := x\mu_{1,q}$. If there is some p' such that $\mathrm{dom}(\Gamma(x))=\{p'\}$ we define $x\nu := x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]$. This is appropriate since due to $\forall(p',p'')\in\Gamma(x).\ x\mu_{1,q}/p''=l_{1,q}\mu_{1,q}/p'p''=u/qp'p''=l_{0,qp'p''}\mu_{0,qp'p''}$ we have

$$x\mu_{1,q}=x\mu_{1,q}[\,p''\leftarrow l_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,] -\!\!\!+\!\!\!+\!\!\!\rightarrow$$
$$x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,]=x\nu.$$

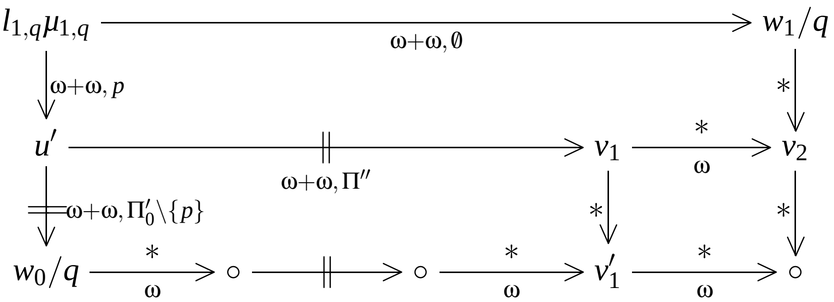
Finally, in case of $|\mathrm{dom}(\Gamma(x))|\succ 1$, $l_{1,q}$ is not linear in x. By the conditions of our lemma and Claim 5 this implies $x\in\mathrm{V}_{\mathcal{C}}$. Since there is some $(p',p'')\in\Gamma(x)$ with $x\mu_{1,q}/p''=l_{0,qp'p''}\mu_{0,qp'p''}$ this implies $l_{0,qp'p''}\mu_{0,qp'p''}\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathcal{C}})$ and then $l_{0,qp'p''}\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{\mathrm{SIG}}\uplus\mathrm{V}_{\mathcal{C}})$ which contradicts Claim 5. Q.e.d. (Claim 11)

Claim 12: $w_0/q=l_{1,q}\nu$.
Proof of Claim 12:
By Claim 11 we get $w_0/q=u/q[\,p'p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x\in\mathrm{V}.\ (p',p'')\in\Gamma(x)\,]=$
$l_{1,q}[\,p'\leftarrow x\mu_{1,q} \mid l_{1,q}/p'=x\in\mathrm{V}\,][\,p'p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid \exists x\in\mathrm{V}.\ (p',p'')\in\Gamma(x)\,]=$
$l_{1,q}[\,p'\leftarrow x\mu_{1,q}[\,p''\leftarrow r_{0,qp'p''}\mu_{0,qp'p''} \mid (p',p'')\in\Gamma(x)\,] \mid l_{1,q}/p'=x\in\mathrm{V}\,]=$
$l_{1,q}[\,p'\leftarrow x\nu \mid l_{1,q}/p'=x\in\mathrm{V}\,]=l_{1,q}\nu$. Q.e.d. (Claim 12)

Claim 13: $r_{1,q}\nu \leftarrow\!\!\!+\!\!\!+\!\!\!- w_1/q$.
Proof of Claim 13: Since $r_{1,q}\mu_{1,q}=w_1/q$, this follows directly from Claim 11. Q.e.d. (Claim 13)

By claims 12 and 13 using Corollary 2.14 it now suffices to show $l_{1,q}\nu\longrightarrow r_{1,q}\nu$, which again follows from Lemma C.4 since $\longrightarrow$ and $\longrightarrow_{\omega}$ are commuting by Claim 0 and since $\forall x\in\mathrm{V}.\ x\mu_{1,q}\overset{*}{\longrightarrow}x\nu$ by Claim 11 and Corollary 2.14.
Q.e.d. (“The second variable overlap (if any) case”)

<u>"The second critical peak case": There is some $p \in \Pi'_0 \cap \mathcal{POS}(l_{1,q})$ with $l_{1,q}/p \notin \mathrm{V}$:</u>



Let $\xi \in \mathcal{SUB}(\mathrm{V},\mathrm{V})$ be a bijection with $\xi[\mathcal{V}(((l_{0,qp},r_{0,qp}),C_{0,qp}))] \cap \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q})) = \emptyset$.
Define $\mathrm{Y} := \xi[\mathcal{V}(((l_{0,qp},r_{0,qp}),C_{0,qp}))] \cup \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q}))$.
Let $\rho \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ be given by $x\rho = \begin{Bmatrix} x\mu_{1,q} & \text{if } x \in \mathcal{V}(((l_{1,q},r_{1,q}),C_{1,q})) \\ x\xi^{-1}\mu_{0,qp} & \text{else} \end{Bmatrix}$ $(x \in \mathrm{V})$.
By $l_{0,qp}\xi\rho = l_{0,qp}\xi\xi^{-1}\mu_{0,qp} = u/qp = l_{1,q}\mu_{1,q}/p = l_{1,q}\rho/p = (l_{1,q}/p)\rho$
let $\sigma := \mathrm{mgu}(\{(l_{0,qp}\xi, l_{1,q}/p)\},\mathrm{Y})$ and $\varphi \in \mathcal{SUB}(\mathrm{V},\mathcal{T}(\mathrm{X}))$ with ${}_{\mathrm{Y}}\upharpoonright(\sigma\varphi) = {}_{\mathrm{Y}}\upharpoonright\rho$.
Define $u' := l_{1,q}\mu_{1,q}[p \leftarrow r_{0,qp}\mu_{0,qp}]$. We get

$$w_0/q = u/q[p' \leftarrow r_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0] \leftarrow\!\!\!\Vert\!\!\!\!-_{\omega+\omega,\Pi'_0\setminus\{p\}}$$
$$u/q[p' \leftarrow l_{0,qp'}\mu_{0,qp'} \mid p' \in \Pi'_0\setminus\{p\}][p \leftarrow r_{0,qp}\mu_{0,qp}] = u'.$$

If $l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma = r_{1,q}\sigma$, then the proof is finished due to

$$w_0/q \leftarrow\!\!\!\Vert\!\!\!\!-_{\omega+\omega,\Pi'_0\setminus\{p\}} u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi = r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q.$$

Otherwise we have $((l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma, C_{0,qp}\xi\sigma, 1), (r_{1,q}\sigma, C_{1,q}\sigma, 1), l_{1,q}\sigma, p) \in \mathrm{CP}(\mathrm{R})$ (due to Claim 5); $C_{0,qp}\xi\sigma\varphi = C_{0,qp}\mu_{0,qp}$ is fulfilled w.r.t. $\longrightarrow$; $C_{1,q}\sigma\varphi = C_{1,q}\mu_{1,q}$ is fulfilled w.r.t. $\longrightarrow$. Due to Claim 0 and our assumed ω-coarse level parallel joinability we have $u' = l_{1,q}[p \leftarrow r_{0,qp}\xi]\sigma\varphi \dashrightarrow\!\!\!\Vert\, v_1 \stackrel{*}{\longrightarrow}_{\omega} v_2 \stackrel{*}{\longleftarrow} r_{1,q}\sigma\varphi = r_{1,q}\mu_{1,q} = w_1/q$ for some v_1, v_2. We then have $w_0/q \leftarrow\!\!\!\Vert\!\!\!\!-_{\omega+\omega,\Pi'_0\setminus\{p\}} u' -\!\!\!\Vert\!\!\!\!\rightarrow_{\omega+\omega,\Pi''} v_1$ for some Π''. Since $\sum_{p'' \in \Omega(\Pi'_0\setminus\{p\},\Pi'')} \lambda(u'/p'') \preceq \sum_{p'' \in \Pi'_0\setminus\{p\}} \lambda(u'/p'') = \sum_{p'' \in \Pi'_0\setminus\{p\}} \lambda(u/qp'') \prec \sum_{p'' \in \Pi'_0} \lambda(u/qp'') = \sum_{p' \in q\Pi'_0} \lambda(u/p') = \sum_{p' \in \Omega(\Pi_0,\{q\})} \lambda(u/p') \preceq \sum_{p' \in \Omega(\Pi_0,\Pi_1)} \lambda(u/p')$ due to our induction hypothesis we get some v'_1 with $w_0/q \stackrel{*}{\longrightarrow}_{\omega} \circ -\!\!\!\Vert\!\!\!\!\rightarrow \circ \stackrel{*}{\longrightarrow}_{\omega} v'_1 \stackrel{*}{\longleftarrow} v_1$. Finally the peak at v_1 can be closed according to $v'_1 \stackrel{*}{\longrightarrow}_{\omega} \circ \stackrel{*}{\longleftarrow} v_2$ by Claim 0.

<u>Q.e.d. ("The second critical peak case")</u> **Q.e.d. (Lemma C.3)**

Proof of Lemma C.4

By Lemma 2.7 it suffices to show for each literal L in C that $L\nu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$. Note that we already know that $L\mu$ is fulfilled w.r.t. $\longrightarrow_{\mathrm{R,X}}$. Since $\mathcal{V}(C)\subseteq \mathrm{V}_{C}$, for all x in $\mathcal{V}(C)$ we have $x\mu\in\mathcal{T}(\mathrm{cons},\mathrm{V}_{C})$ and then by Lemma 2.10 $x\mu\overset{*}{\longrightarrow}_{\mathrm{R,X,\omega}}y\mu$.

$\underline{L=(s_0{=}s_1)\text{:}}$ We have $s_0\nu\overset{*}{\longleftarrow}_{\mathrm{R,X,\omega}}s_0\mu\overset{*}{\longrightarrow}_{\mathrm{R,X}}t_0\overset{*}{\longleftarrow}_{\mathrm{R,X}}s_1\mu\overset{*}{\longrightarrow}_{\mathrm{R,X,\omega}}s_1\nu$ for some t_0. By the inclusion assumption of the lemma we get some v with $s_0\nu\overset{*}{\longrightarrow}_{\mathrm{R,X}}v\overset{*}{\longleftarrow}_{\mathrm{R,X}}t_0$ and then (due to $v\overset{*}{\longleftarrow}_{\mathrm{R,X}}s_1\mu\overset{*}{\longrightarrow}_{\mathrm{R,X,\omega}}s_1\nu$) $v\overset{*}{\longrightarrow}_{\mathrm{R,X}}\circ\overset{*}{\longleftarrow}_{\mathrm{R,X}}s_1\nu$.

$\underline{L=(\mathrm{Def}\,s)\text{:}}$ We know the existence of $t\in\mathcal{GT}(\mathrm{cons})$ with $s\nu\overset{*}{\longleftarrow}_{\mathrm{R,X,\omega}}s\mu\overset{*}{\longrightarrow}_{\mathrm{R,X}}t$. By the above inclusion property again, there is some t' with $s\nu\overset{*}{\longrightarrow}_{\mathrm{R,X}}t'\overset{*}{\longleftarrow}_{\mathrm{R,X}}t$. By Lemma 2.10 we get $t'\in\mathcal{GT}(\mathrm{cons})$.

$\underline{L=(s_0{\neq}s_1)\text{:}}$ There exist some $t_0,t_1\in\mathcal{GT}(\mathrm{cons})$ with $\forall i\prec 2.\ s_i\nu\overset{*}{\longleftarrow}_{\mathrm{R,X,\omega}}s_i\mu\overset{*}{\longrightarrow}_{\mathrm{R,X}}t_i$ and $t_0\not\downarrow_{\mathrm{R,X}}t_1$. Just like above we get t_0', $t_1'\in\mathcal{GT}(\mathrm{cons})$ with $\forall i\prec 2.\ s_i\nu\overset{*}{\longrightarrow}_{\mathrm{R,X}}t_i'\overset{*}{\longleftarrow}_{\mathrm{R,X}}t_i$. Finally $t_0'\overset{*}{\longleftarrow}_{\mathrm{R,X}}t_0\not\downarrow_{\mathrm{R,X}}t_1\overset{*}{\longrightarrow}_{\mathrm{R,X}}t_1'$ implies $t_0'\not\downarrow_{\mathrm{R,X}}t_1'$.