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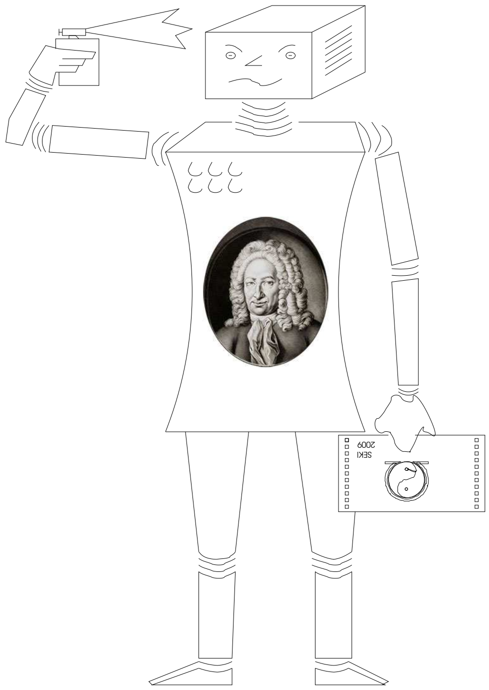

Notes to Wolfgang Spohn's "The Laws of Belief"

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Notes to WOLFGANG SPOHN's "The Laws of Belief"

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Abstract

This are some private notes to [SPOHN, 2012].

Keywords: Ranking Semantics

1 (Negative) Ranking Functions

The set W in the following definition is meant to be a set of mutually exclusive cases, typically formed by a conjunction of one concrete value for each observed variable, such as

$$\mathsf{temp} = 225.5\,\mathrm{K} \quad\wedge\quad \mathsf{uptime} = 503\,\mathrm{s}$$

or

$$\mathsf{emu(edna)} = \mathsf{true} \quad\wedge\quad \mathsf{bird(edna)} = \mathsf{true} \quad\wedge\quad \mathsf{flies(edna)} = \mathsf{false}.$$

A subset of W is called a *proposition* and is meant to be the case that consists in any of the cases of its elements, i.e. the disjunction of its elements. The name "proposition" for a subset of W is justified insofar as it is meant to be something similar to a full disjunctive normal form.

It is assumed that $W \neq \emptyset$, so that $\emptyset$ can represent FALSE, and W can represent TRUE.

Definition 1.1 (Algebra)
$\mathcal{A}$ is an [*complete*] *algebra over* W if we have $W \neq \emptyset$, $\mathcal{A}$ is a subset of the power set of W, and the following holds:

1. $W \in \mathcal{A}$.
2. $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. [Moreover, for each $\mathcal{B} \subseteq \mathcal{A}$, we have $\bigcup \mathcal{B} \in \mathcal{A}$.]
3. $A \in \mathcal{A}$ implies $\overline{A} \in \mathcal{A}$, where "$\overline{A}$" denotes $\mathcal{A} \setminus A$.

Using Definition 1.1(3) and $\overline{\overline{A} \cup \overline{B}} = A \cap B$, &c., we get:

Corollary 1.2 *Let $\mathcal{A}$ be an [complete] algebra over W. Then:*

1. $\emptyset \in \mathcal{A}$.
2. $A, B \in \mathcal{A}$ *implies* $A \cap B \in \mathcal{A}$. *[Moreover, for each* $\mathcal{B} \subseteq \mathcal{A}$, *we have* $\bigcap \mathcal{B} \in \mathcal{A}$.*]*

Let "ω" denote the first infinite ordinal number and let $\omega{+}1$ be its successor $\omega \cup \{\omega\}$. Let "inf" denote the infimum over subsets of $\omega{+}1$, i.e. $\inf A = \min A$ for $A \subseteq \omega{+}1$ with $A \neq \emptyset$, and $\inf \emptyset = \omega$. Moreover, let us define, for any $n \prec \omega$: $n + \omega := \omega$ and $\omega - n := \omega$, and $\omega - \omega := 0$.

Definition 1.3 (Set-Wise (Negative) Ranking Function)
Let $\mathcal{A}$ be an algebra over W.
A *set-wise (negative) ranking function for* $\mathcal{A}$ is a function $\kappa : \mathcal{A} \to \omega{+}1$ with

1. $\kappa(W) = 0$. "Global disjunction is believed."
2. $\kappa(\bigcup \mathcal{B}) = \inf\{\ \kappa(B) \mid B \in \mathcal{B}\ \}$ for each $\mathcal{B} \subseteq \mathcal{A}$ with $\bigcup \mathcal{B} \in \mathcal{A}$. "Law of Disjunction"

Let $\mathcal{A}|\mathcal{A}$ denote the set of words of length 3 consisting of a proposition from $\mathcal{A}$, the sign "|" and a further proposition from $\mathcal{A}$. We will read $B|A$ as "B under condition of A".
We extend any set-wise ranking function κ for $\mathcal{A}$ from $\mathcal{A}$ to $\mathcal{A}|\mathcal{A}$ by ($A, B \in \mathcal{A}$):

$$\kappa(B|A) := \left\{ \begin{array}{ll} \omega & \text{if } A \cap B = \emptyset \text{ and } \kappa(A) = \omega \\ \kappa(A \cap B) - \kappa(A) & \text{otherwise} \end{array} \right\}.$$

Note that — to make clear that a higher rank means a higher level of disbelief instead of belief — the defined ranking functions are sometimes called "negative" ranking functions.[1] For instance, the Law of Disjunction says that our disbelief in a disjunction of propositions is equal to our smallest disbelief in any of the propositions.

Taking $\mathcal{B}$ in Definition 1.3(2) to be one of $\emptyset$, $\{B, \overline{B}\}$, $\{B, C \cap \overline{B}\}$, we get the following three items, respectively:

Corollary 1.4
Let $\mathcal{A}$ be an algebra over W. Let κ be a set-wise ranking function for $\mathcal{A}$. Then:

1. $\kappa(\emptyset) = \omega$. "Empty disjunction is not believed."
2. $\kappa(B) \neq 0$ *implies* $\kappa\left(\overline{B}\right) = 0$ *for each* $B \in \mathcal{A}$. "Law of Negation"
3. *For* $B, C \in \mathcal{A}$ *with* $B \subseteq C$, *we have* $\kappa(B) \succeq \kappa(C)$.

Note that the Law of Negation says that if we disbelieve a proposition then we have to believe the complementary proposition.

[1] This negativeness is in opposition to standard probability theory and also to positive and two-sided ranking functions, cf. [Spohn, 2012, § 5.2, pp. 69–77]. Positive and two-sided ranking functions do not provide any extra in formation, but their theory is typically more awkward w.r.t. the arithmetic expressions.

Lemma 1.5 *Let $\mathcal{A}$ be an algebra over W. Let κ be a set-wise ranking function for $\mathcal{A}$. For $A, B, C \in \mathcal{A}$:*

1. $\kappa(B|A) = \kappa(A \cap B) - \kappa(A)$
   if $A \cap B \neq \emptyset$ *or* $\kappa(A) \neq \omega$.

2. $\kappa(A \cap B) = \kappa(A) + \kappa(B|A)$
   if $A \cap B \neq \emptyset$ *or* $\kappa(A) \neq \omega$. "Law of Conjunction"

3. $\kappa(B|A) = \omega$ *iff* $A \cap B = \emptyset$ *or* $\kappa(A) \prec \kappa(A \cap B) = \omega$.

4. $\kappa_A(C) := \kappa(C|A)$ $(C \in \mathcal{A})$ *defines* κ_A *as a set-wise ranking function for* $\mathcal{A}$ *unless* $A = \emptyset$.

5. $\kappa(B|A) = \big(\kappa(B) + \kappa(A|B)\big) - \kappa(A)$
   if $A \cap B \neq \emptyset$ *or* $\kappa(A) \neq \omega \neq \kappa(B)$. "BAYES' Theorem[2]"

6. $\kappa(B) = \inf\{\ \kappa(A) + \kappa(B|A) \mid A \in \mathcal{P} \wedge A \cap B \neq \emptyset\ \}$
   if $\mathcal{P} \subseteq \mathcal{A}$ *with* $\bigcup \mathcal{P} = W$. "Formula of the Total Rank"

7. *Assume* $A \cap C \neq \emptyset \vee \kappa(A) \neq \omega$, $B \cap C \neq \emptyset \vee \kappa(B) \neq \omega$, *and* $\kappa(C|A) \preceq \kappa(C|B)$. *Then:*

   $$\kappa(C|A) \preceq \kappa(C|A \cup B) \preceq \kappa(C|B). \qquad \text{"Law of Disjunctive Conditions"}$$

   More precisely, we have:

   (a) *If* $\kappa(A) \preceq \kappa(B)$, *then* $\kappa(A \cap C) \preceq \kappa(B \cap C)$.
   (b) $\kappa(C|A) = \kappa(C|A \cup B)$ *in case of* $\kappa(A) \preceq \kappa(B)$.
   (c) $\kappa(C|A) \prec \kappa(C|A \cup B)$ *in case of* $\kappa(A) \succ \kappa(B)$ *and* $\kappa(A \cap C) \preceq \kappa(A \cap B)$.
   (d) *If* $\kappa(A \cap C) \succeq \kappa(B \cap C)$, *then* $\kappa(A) \succeq \kappa(B)$.
   (e) $\kappa(C|A \cup B) = \kappa(C|B)$ *in case of* $\kappa(A \cap C) \succeq \kappa(B \cap C)$.
   (f) $\kappa(C|A \cup B) \prec \kappa(C|B)$ *in case of* $\kappa(A \cap C) \prec \kappa(B \cap C)$ *and* $\kappa(A) \succeq \kappa(B)$.

   This means that — in the irreflexive case $\kappa(C|A) \prec \kappa(C|B)$ *— we have:*

   (g) $\kappa(C|A) \prec \kappa(C|A \cup B)$ *iff* $\kappa(A) \succ \kappa(B)$.
   (h) $\kappa(C|A \cup B) \prec \kappa(C|B)$ *iff* $\kappa(A \cap C) \prec \kappa(B \cap C)$.

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[2]This is called "BAYES' Theorem" after the English philosopher THOMAS BAYES (1701–1761), who formulated a restricted version of what is called "BAYES' Theorem" in probability theory today:

$$P(B|A) = \frac{P(B) * P(A|B)}{P(A)}.$$

In general, theorems in probability theory strongly suggest theorems in ranking theory as describe as follows in [SPOHN, 2012, p. 82]:

> "adding probabilities translates into taking the minimum of negative ranks, multiplication of probabilities into addition of negative ranks, and division of probabilities into subtraction of negative ranks."

Example 1.6 Set $W := \{0,1,2\}$, $\mathcal{A} := \mathfrak{P}(W)$.

1. Set $\kappa(\{0\}) = 0$, $\kappa(\{1\}) = 1$, $\kappa(\{2\}) = 2$. Set $A := \{1\}$, $B := \{2\}$. Then, by Corollary 1.4, $\kappa(A \cap B) = \kappa(\emptyset) = \omega$. Moreover, by Lemma 1.5(1), we have $\kappa(B|A) = \kappa(A \cap B) - \kappa(A) = \omega - 1 = \omega$. As the second option of the condition of Lemma 1.5(2) is given, we indeed have $\kappa(A \cap B) = \kappa(A) + \kappa(B|A)$, namely $\omega = 1 + \omega$. But as $\omega \neq \omega + 1$, we do not have $\kappa(A \cap B) = \kappa(B|A) + \kappa(A)$.

   This means that the two arguments on the right-hand side of Lemma 1.5(2) cannot be commuted in case of $\kappa(A) \neq \omega$.

2. Set $\kappa(\{0\}) := 0$, $\kappa(\{1\}) := \omega$. Set $A := B := \{1\}$. Then $A \cap B = A = B \neq \emptyset$. Moreover, $\kappa(A|B) = \kappa(B|A) = 0$ by Lemma 1.5(1). Thus $\kappa(B|A) \neq \omega = \omega + 0 = \omega + \big(0 - \omega\big) = \kappa(B) + \big(\kappa(A|B) - \kappa(A)\big)$.

   This means that the parentheses on the right-hand side of Lemma 1.5(5) cannot be dropped in case of $A \cap B \neq \emptyset$.

Proof of Lemma 1.5

(1): By Definition 1.3.

(2): On the one hand, if $\kappa(A) \neq \omega$, then we have $\kappa(B|A) = \kappa(A \cap B) - \kappa(A)$ by (1), and then, in case of $\kappa(A \cap B) \neq \omega$, both $\kappa(A \cap B) = \kappa(A) + \kappa(B|A)$ and $\kappa(A \cap B) = \kappa(B|A) + \kappa(A)$; but the former even holds in case of $\kappa(A \cap B) = \omega$ because then $\kappa(B|A) = \omega$. If, on the other hand, $\kappa(A) = \omega$, then we have $\kappa(A \cap B) = \omega$ according to Corollary 1.4; moreover, we have $\kappa(B|A) = \kappa(A \cap B) - \kappa(A) = \omega - \omega = 0$ by Definition 1.3 and the restriction of (2). We then have $\kappa(A \cap B) = \omega = \omega + 0 = \kappa(A) + \kappa(B|A)$ again.

(3): Assume $A \cap B = \emptyset$. Then, in case of $\kappa(A) = \omega$, we have $\kappa(B|A) = \omega$ directly by Definition 1.3. Otherwise, in case of $\kappa(A) \neq \omega$, by (1) and Corollary 1.4(1) we get $\kappa(B|A) = \kappa(A \cap B) - \kappa(A) = \kappa(\emptyset) - \kappa(A) = \omega - \kappa(A) = \omega$ again.

(4): Assume $A \neq \emptyset$. Because of $A \cap W = A \neq 0$, we have $\kappa_A(W) = \kappa(W|A) = \kappa(A \cap W) - \kappa(A) = \kappa(A) - \kappa(A) = 0$. So the global disjunction is disbelieved indeed.

Moreover, for showing the Law of Disjunction, suppose that $\mathcal{B} \subseteq \mathcal{A}$ with $\bigcup \mathcal{B} \in \mathcal{A}$.

(a) $A \cap \bigcup \mathcal{B} = \emptyset$ and $\kappa(A) = \omega$: Then: (α) For each $B \in \mathcal{B}$, we have $A \cap B = \emptyset$.

$\kappa_A(\bigcup \mathcal{B}) = \kappa(\bigcup \mathcal{B}|A) \overset{(a)}{=} \omega = \inf \{ \omega \,|\, B \in \mathcal{B} \} \overset{(\alpha)}{=} \inf \{ \kappa(B|A) \,|\, B \in \mathcal{B} \} = \inf \{ \kappa_A(B) \,|\, B \in \mathcal{B} \}$.

(b) $A \cap \bigcup \mathcal{B} \neq \emptyset$ or $\kappa(A) \neq \omega$: Set $\mathcal{B}' := \{ B \mid B \in \mathcal{B} \wedge A \cap B \neq \emptyset \}$.
Set $\mathcal{B}'' := \{ A \cap B \mid B \in \mathcal{B} \}$. Then:

(β) For each $B' \in \mathcal{B}'$, we have $A \cap B' \neq \emptyset$.

(γ) If $\kappa(A) = \omega$, then $\mathcal{B}' \neq \emptyset$.

(δ) According to (3), for $B \in \mathcal{B} \backslash \mathcal{B}'$, we have $\kappa(B|A) = \omega$.

(ε) $\mathcal{B}'' \subseteq \mathcal{A}$ and $\bigcup \mathcal{B}'' = A \cap \bigcup \mathcal{B}' = A \cap \bigcup \mathcal{B} \in \mathcal{A}$.

(ζ) From (ε), by Definition 1.3(2), we get: $\kappa(\bigcup \mathcal{B}'') = \inf \{ \kappa(B'') \mid B'' \in \mathcal{B}'' \}$.

$$\begin{aligned}
\kappa_A(\textstyle\bigcup \mathcal{B}) &= \kappa(\textstyle\bigcup \mathcal{B}|A) \overset{(b)}{=} \kappa(A \cap \textstyle\bigcup \mathcal{B}) - \kappa(A) \overset{(\varepsilon)}{=} \kappa(\textstyle\bigcup \mathcal{B}'') - \kappa(A) \\
&\overset{(\zeta)}{=} \inf \{ \kappa(B'') \mid B'' \in \mathcal{B}'' \} - \kappa(A) = \inf \{ \kappa(A \cap B') \mid B' \in \mathcal{B}' \} - \kappa(A) \\
&\overset{(\gamma)}{=} \inf \{ \kappa(A \cap B') - \kappa(A) \mid B' \in \mathcal{B}' \} \overset{(\beta)}{=} \inf \{ \kappa(B'|A) \mid B' \in \mathcal{B}' \} \\
&= \inf \big\{ \inf \{ \kappa(B'|A) \mid B' \in \mathcal{B}' \},\ \omega \big\} \\
&\overset{(\delta)}{=} \inf \big\{ \inf \{ \kappa(B'|A) \mid B' \in \mathcal{B}' \},\ \inf \{ \kappa(B|A) \mid B \in \mathcal{B} \backslash \mathcal{B}' \} \big\} \\
&= \inf \{ \kappa(B|A) \mid B \in \mathcal{B} \} = \inf \{ \kappa_A(B) \mid B \in \mathcal{B} \}.
\end{aligned}$$

(5): If $A \cap B \neq \emptyset$, then we have $\kappa(B|A) \overset{(1)}{=} \kappa(A \cap B) - \kappa(A) = \kappa(B \cap A) - \kappa(A) \overset{(2)}{=} \big(\kappa(B) + \kappa(A|B)\big) - \kappa(A)$. Otherwise, in case of $A \cap B = \emptyset$ and $\kappa(A) \neq \omega \neq \kappa(B)$, we have $\kappa(B|A) \overset{(3)}{=} \omega = \kappa(B) + \omega - \kappa(A) \overset{(3)}{=} \kappa(B) + \kappa(A|B) - \kappa(A)$.

(6): Set $\mathcal{P}' := \{ A \in \mathcal{P} \mid A \cap B \neq \emptyset \}$. Set $\mathcal{P}'' := \{ A' \cap B \mid A' \in \mathcal{P}' \}$. Then:

(α) For each $A' \in \mathcal{P}'$, we have $A' \cap B \neq \emptyset$, and then, by (2): $\kappa(A' \cap B) = \kappa(A') + \kappa(B|A')$.

(β) $\mathcal{P}'' \subseteq \mathcal{A}$ and $\bigcup \mathcal{P}'' = B \cap \bigcup \mathcal{P}' = B \cap \bigcup \mathcal{P} = B \cap W = B \in \mathcal{A}$.

(γ) From (β), by Definition 1.3(2), we get: $\kappa(\bigcup \mathcal{P}'') = \inf \{ \kappa(A'') \mid A'' \in \mathcal{P}'' \}$.

$$\begin{aligned}
\kappa_A(B) &\overset{(\beta)}{=} \kappa(\textstyle\bigcup \mathcal{P}'') \overset{(\gamma)}{=} \inf \{ \kappa(A'') \mid A'' \in \mathcal{P}'' \} = \inf \{ \kappa(A' \cap B) \mid A' \in \mathcal{P}' \} \\
&\overset{(\alpha)}{=} \inf \{ \kappa(A') + \kappa(B|A') \mid A' \in \mathcal{P}' \} \\
&= \inf \{ \kappa(A) + \kappa(B|A) \mid A \in \mathcal{P} \wedge A \cap B \neq \emptyset \}
\end{aligned}$$

(7): Let us assume Assume $A{\cap}C \neq \emptyset \vee \kappa(A) \neq \omega$ and $B{\cap}C \neq \emptyset \vee \kappa(B) \neq \omega$. Then, by Definition 1.3(2), we also have $(A{\cup}B){\cap}C \neq \emptyset$ or $\kappa(A{\cup}B) \neq \omega$.
Then, by (1) and Definition 1.3(2) we have:

(α) $\kappa(C|A) = \kappa(A{\cap}C) - \kappa(A)$.

(β) $\kappa(C|B) = \kappa(B{\cap}C) - \kappa(B)$.

(γ) $\kappa(C|A{\cup}B) = \kappa((A{\cup}B){\cap}C) - \kappa(A{\cup}B) = \kappa((A{\cap}C){\cup}(B{\cap}C)) - \kappa(A{\cup}B)$
$= \inf\{\kappa(A{\cap}C), \kappa(B{\cap}C)\} - \inf\{\kappa(A), \kappa(B)\}$.

Let us assume according to the assumption in the lemma:

(δ) $\kappa(C|A) \preceq \kappa(C|B)$.

By (α, β), we get from (δ):

(ε) $\kappa(A{\cap}C) - \kappa(A) \preceq \kappa(B{\cap}C) - \kappa(B)$.

(a), (d): For both items it suffices to refute $(\kappa(A) \preceq \kappa(B)) \wedge (\kappa(B{\cap}C) \preceq \kappa(A{\cap}C))$ and the additional assumption that one of the two steps is actually irreflexive (i.e. a step via $\prec$). In this case, however, we get $\kappa(A{\cap}C) - \kappa(A) \succeq \kappa(B{\cap}C) - \kappa(A) \succeq \kappa(B{\cap}C) - \kappa(B)$. Under the additional assumption one of these two steps is irreflexive and so we get $\kappa(A{\cap}C) - \kappa(A) \succ \kappa(B{\cap}C) - \kappa(B)$, which contradicts ($\varepsilon$).

(b): Let us assume $\kappa(A) \preceq \kappa(B)$. By ($a$) we get $\kappa(A{\cap}C) \preceq \kappa(B{\cap}C)$. Then:
$\kappa(C|A) \overset{(\alpha)}{=} \kappa(A{\cap}C) - \kappa(A) = \inf\{\kappa(A{\cap}C), \kappa(B{\cap}C)\} - \inf\{\kappa(A), \kappa(B)\} \overset{(\gamma)}{=} \kappa(C|A{\cup}B)$.

(e): Let us assume $\kappa(A{\cap}C) \succeq \kappa(B{\cap}C)$. By ($d$) we get $\kappa(A) \succeq \kappa(B)$. Then:
$\kappa(C|B) \overset{(\alpha)}{=} \kappa(B{\cap}C) - \kappa(B) = \inf\{\kappa(A{\cap}C), \kappa(B{\cap}C)\} - \inf\{\kappa(A), \kappa(B)\} \overset{(\gamma)}{=} \kappa(C|A{\cup}B)$.

$\kappa(A) \succeq \kappa(B)$ and $\kappa(A{\cap}C) \preceq \kappa(B{\cap}C)$: Then $\kappa(C|A{\cup}B) \overset{(\gamma)}{=} \inf\{\kappa(A{\cap}C), \kappa(B{\cap}C)\} - \inf\{\kappa(A), \kappa(B)\} = \kappa(A{\cap}C) - \kappa(B)$, i.e.:

(ζ) $\kappa(C|A{\cup}B) = \kappa(A{\cap}C) - \kappa(B)$.

(η) Moreover, $\kappa(B) \neq \omega$.

(c): Let us assume that the first element of our case assumption is an irreflexive step.
Then: $\kappa(C|A) \overset{(\alpha)}{=} \kappa(A{\cap}C) - \kappa(A) \prec \kappa(A{\cap}C) - \kappa(B) \overset{(\zeta)}{=} \kappa(C|A{\cup}B)$.

(f): Let us assume that the second element of our case assumption is an irreflexive step.
Then: $\kappa(C|A{\cup}B) \overset{(\zeta)}{=} \kappa(A{\cap}C) - \kappa(B) \overset{(\eta)}{\prec} \kappa(B{\cap}C) - \kappa(B) \overset{(\beta)}{=} \kappa(C|B)$.

"Law of Disjunctive Conditions": This holds for $\kappa(A) \preceq \kappa(B)$ by (b), and for $\kappa(A{\cap}C) \succeq \kappa(B{\cap}C)$ by (e). In the remaining case of $(\kappa(A) \succ \kappa(B)) \wedge (\kappa(A{\cap}C) \prec \kappa(B{\cap}C))$, it holds by (c, f).

(g): "$\Rightarrow$": By (b), the negation of $\kappa(A) \succ \kappa(B)$ implies the negation of $\kappa(C|A) \prec \kappa(C|A{\cup}B)$.

"$\Leftarrow$": In case of $\kappa(A{\cap}C) \preceq \kappa(B{\cap}C)$, this holds by ($c$). Otherwise, it holds by (e):
$\kappa(C|A) \prec \kappa(C|B) = \kappa(C|A{\cap}B)$.

(h): "$\Rightarrow$": By (e), the negation of $\kappa(A{\cap}C) \prec \kappa(B{\cap}C)$ implies the negation of $\kappa(C|A) \prec \kappa(C|A{\cup}B)$.

"$\Leftarrow$": In case of $\kappa(A) \succeq \kappa(B)$, this holds by ($f$). Otherwise, it holds by (b):
$\kappa(C|A{\cap}B) = \kappa(C|A) \prec \kappa(C|B)$. **Q.e.d. (Lemma 1.5)**

Definition 1.7 (Point-Wise (Negative) Ranking Function)
Let $\mathcal{A}$ be an algebra over W.

1. A *point-wise (negative) ranking function for* $\mathcal{A}$ is a function $\kappa : W \to \omega{+}1$ with $\kappa\langle\!\{0\}\!\rangle \neq \emptyset$, i.e. there is a $w \in W$ with $\kappa(w) = 0$. [3] "Something is believed."

2. The *set-wise ranking function* $\hat{\kappa}$ *of a point-wise ranking function* κ *for* $\mathcal{A}$ is defined by $\hat{\kappa}(B) = \inf\{\ \kappa(w) \mid w \in B\ \}$ $(B \in \mathcal{A})$.

3. Let $W|\mathcal{A}$ denote the set of words of length 3 consisting of an element from W, the sign "$|$" and a proposition from $\mathcal{A}$. We will read $w|A$ as "w under condition of A". We extend any point-wise ranking function κ for $\mathcal{A}$ from W to $W|\mathcal{A}$ by $(w \in W, A \in \mathcal{A})$:
$$\kappa(w|A) := \left\{ \begin{array}{ll} \omega & \text{if } w \notin A \\ \kappa(w) - \hat{\kappa}(A) & \text{otherwise} \end{array} \right\}.$$

4. Let $W|\mathcal{A}{\uparrow}\omega{+}1$ denote the set of words of length 3 consisting of an element from W, the sign "$|$", a proposition from $\mathcal{A}$, the sign "$\uparrow$", and an element from $\omega{+}1$. We will read $w|A{\uparrow}\alpha$ as "w under condition of A with evidence α". We extend any point-wise ranking function κ for $\mathcal{A}$ from W to $W|\mathcal{A}{\uparrow}\omega{+}1$ by $(w \in W, A \in \mathcal{A}, \alpha \in \omega{+}1)$:
$$\kappa(w|A{\uparrow}\alpha) := \left\{ \begin{array}{ll} \omega & \text{if } w \notin A \text{ and } \alpha = \omega \\ \alpha - \inf\{\hat{\kappa}(A), \alpha\} + \kappa(w) & \text{if } w \notin A \text{ and } \alpha \neq \omega \\ \kappa(w) - \inf\{\hat{\kappa}(A), \alpha\} & \text{if } w \in A \end{array} \right\}.$$

Corollary 1.8
Let $\mathcal{A}$ be an algebra over W. Let κ be a point-wise ranking function for $\mathcal{A}$.

1. *Let $\hat{\kappa}$ be the set-wise ranking function of κ. Then: For $w \in A \in \mathcal{A}$, we have $\kappa(w) \succeq \hat{\kappa}(A)$.*

2. *For $w \in W$, $A \in \mathcal{A}$: $\kappa(w|A{\uparrow}\omega) = \kappa(w|A)$ and $\kappa(w|A{\uparrow}0) = \kappa(w)$.*

Definition 1.9 (Point-Wise Ranking Function of Set-Wise Ranking Function)
Let $\mathcal{A}$ be a complete algebra over W.
The *point-wise ranking function* $\dot{\kappa}$ *of a set-wise ranking function* κ *for* $\mathcal{A}$ is defined by $\dot{\kappa}(w) = \kappa\big(\bigcap\{\ B \in \mathcal{A} \mid w \in B\ \}\big)$ $(w \in W)$.

Lemma 1.10
Let $\mathcal{A}$ be a complete algebra over W. Let κ be a set-wise ranking function for $\mathcal{A}$. Then:

1. *$\dot{\kappa}$ (the point-wise ranking function of κ) is indeed a point-wise ranking function for $\mathcal{A}$.*

2. *$\hat{\dot{\kappa}}$ (the set-wise ranking function of $\dot{\kappa}$) satisfies $\hat{\dot{\kappa}} = \kappa$.*

[3] In his related definition, [Spohn, 2012, Definition 5.5, p. 70] additionally requires $\kappa\langle\!\{\alpha\}\!\rangle \in \mathcal{A}$ for each $\alpha \in \omega{+}1$.

Proof of Lemma 1.10

(1): By Corollary 1.2(2), $\dot{\kappa} : W \to \omega{+}1$ is well-defined because $\mathcal{A}$ is a complete algebra over W. Moreover, by Definition 1.3(2,1), we have $\inf\{\ \dot{\kappa}(w) \mid w \in W\ \} = \inf\{\ \kappa\big(\bigcap\{\ B \in \mathcal{A} \mid w \in B\ \}\big) \mid w \in W\ \} = \kappa\big(\bigcup\{\ \bigcap\{\ B \in \mathcal{A} \mid w \in B\ \} \mid w \in W\ \}\big) = \kappa(W) = 0$. Thus, there must be some $w \in W$ with $\dot{\kappa}(w) = 0$, i.e. $\dot{\kappa}\langle\!\{0\}\!\rangle \neq \emptyset$.

(2): By Definitions 1.7 and 1.9 and Definition 1.3(2), we have for every $B' \in \mathcal{A}$:
$\hat{\dot{\kappa}}(B') = \inf\{\ \dot{\kappa}(w) \mid w \in B'\ \} = \inf\{\ \kappa\big(\bigcap\{\ B \in \mathcal{A} \mid w \in B\ \}\big) \mid w \in B'\ \} = \kappa\big(\bigcup\{\ \bigcap\{\ B \in \mathcal{A} \mid w \in B\ \} \mid w \in B'\ \}\big) = \kappa(B')$.
Indeed, we have $B' = \bigcup\{\ \bigcap\{\ B \in \mathcal{A} \mid w \in B\ \} \mid w \in B'\ \}$, because, for each $w \in B'$, we have $w \in \bigcap\{\ B \in \mathcal{A} \mid w \in B\ \} \subseteq B'$. **Q.e.d. (Lemma 1.10)**

Lemma 1.11
Let $\mathcal{A}$ be an algebra over W. Let κ be a point-wise ranking function for $\mathcal{A}$. Then:

1. *$\hat{\kappa}$ (the set-wise ranking function of κ) is indeed a set-wise ranking function for $\mathcal{A}$.*
2. *$\dot{\hat{\kappa}}$ (the set-wise ranking function of $\hat{\kappa}$) satisfies $\dot{\hat{\kappa}}(w') = \kappa(w')$ for each $w' \in W$ with $\inf\{\ \kappa(w) \mid w \in \bigcap\{\ B \in \mathcal{A} \mid w' \in B\ \}\ \} = \kappa(w')$, which is the case e.g. if $\{w'\} \in \mathcal{A}$ or if $\kappa\langle\!\{\kappa(w')\}\!\rangle \in \mathcal{A}$.*

The following example shows that the restriction in Lemma 1.11(2) is indeed necessary:

Example 1.12 Set $W := \{0,1\}$, $\mathcal{A} := \{\emptyset, W\}$, $\kappa(0) := 0$, $\kappa(1) := 1$.
Then $\hat{\kappa}(\emptyset) = \omega$, $\hat{\kappa}(W) = 0$. Then $\dot{\hat{\kappa}}(0) = \dot{\hat{\kappa}}(1) = 0$, i.e. $\dot{\hat{\kappa}} \neq \kappa$.

Proof of Lemma 1.11

(1): Because of $\kappa\langle\!\{0\}\!\rangle \neq \emptyset$, by Definition 1.7, we have $\hat{\kappa}(W) = \inf\{\ \kappa(w) \mid w \in W\ \} = \inf\big(\{\ \kappa(w) \mid w \in W\ \} \cup \{0\}\big) = 0$. Moreover, for $\mathcal{B} \subseteq \mathcal{A}$ with $\bigcup\mathcal{B} \in \mathcal{A}$, we have $\hat{\kappa}(\bigcup\mathcal{B}) = \inf\{\ \kappa(w) \mid w \in \bigcup\mathcal{B}\ \} = \inf\{\ \inf\{\ \kappa(w) \mid w \in B\ \} \mid B \in \mathcal{B}\ \} = \inf\{\ \hat{\kappa}(B) \mid B \in \mathcal{B}\ \}$.

(2): By Definitions 1.7 and 1.9, we have for every $w' \in W$ with $\inf\{\ \kappa(w) \mid w \in \bigcap\{\ B \in \mathcal{A} \mid w' \in B\ \}\ \} = \kappa(w')$:
$\dot{\hat{\kappa}}(w') = \hat{\kappa}\big(\bigcap\{\ B \in \mathcal{A} \mid w' \in B\ \}\big) = \inf\{\ \kappa(w) \mid w \in \bigcap\{\ B \in \mathcal{A} \mid w' \in B\ \}\ \} = \kappa(w')$.
Q.e.d. (Lemma 1.11)

Lemma 1.13 *Let $\mathcal{A}$ be an algebra over W. Let κ be a point-wise ranking function for $\mathcal{A}$. For $w \in W$, $A \in \mathcal{A}$:*

1. *$\kappa(w|A) = \kappa(w) - \hat{\kappa}(A)$ unless $w \notin A$.*
2. *$\kappa(w) = \hat{\kappa}(A) + \kappa(w|A)$ unless $w \notin A$.* "Law of Conjunction"
3. *$\kappa(w|A) = \omega$ iff $w \notin A$ or $\hat{\kappa}(A) \prec \kappa(w) = \omega$.*
4. *$\kappa_A(v) := \kappa(v|A)$ $(v \in W)$ defines κ_A as a point-wise ranking function for $\mathcal{A}$ unless $A = \emptyset$.*

Proof of Lemma 1.13

(1): By Definition 1.7(3).

(2): Let us assume $w \in A$. Then we have $\kappa(w|A) = \kappa(w) - \hat{\kappa}(A)$ by (1), and then, in case of $\kappa(w) \prec \omega$, we have $\hat{\kappa}(A) \preceq \kappa(w) \prec \omega$ by Corollary 1.8, and so $\kappa(w|A) \prec \omega$, which means that we have both $\kappa(w) = \hat{\kappa}(A) + \kappa(w|A)$ and $\kappa(w) = \kappa(w|A) + \hat{\kappa}(A)$. Both also hold in case of $\kappa(w) = \omega$ and $\hat{\kappa}(A) = \omega$ because then $\kappa(w|A) = 0$. In the remaining case of $\kappa(w) = \omega$ and $\hat{\kappa}(A) \prec \omega$, we get $\kappa(w|A) = \omega$, and so the former of the two equations holds even in this case.

(3): By Definition 1.7(3) and the same case analysis as in (2).

(4): Let us assume $A \neq \emptyset$. Set $\alpha := \hat{\kappa}(A)$. By $A \neq \emptyset$ and by Definition 1.7(2), there must be some $w \in A$ with $\kappa(w) = \alpha$. Then, by Definition 1.7(3), we have $\kappa_A(w) = \kappa(w|A) = \kappa(w) - \hat{\kappa}(A) = \alpha - \alpha = 0$. **Q.e.d. (Lemma 1.13)**

References

[Gillman, 1987] Leonard Gillman. *Writing Mathematics Well*. The Mathematical Association of America, 1987.

[Spohn, 2012] Wolfgang Spohn. *The Laws of Belief*. Oxford Univ. Press, 2012.