Review of the paper: The Explicit Definition of Quantifiers via Hilbert's ϵ is Confluent and Terminating, by Claus-Peter Wirth

January 12, 2016

1 Summary

The author of this paper proves the folklore observation that the rewriting relation induced by the definition of $\exists$ and $\forall$ by means of Hilbert's ϵ operator is strongly normalizing and confluent. This rewriting relation is defined over first-order formulas with ϵ-terms and its basic reductions are:

$$\exists x\, A \mapsto A[x := \epsilon x\, A]$$

$$\forall x\, A \mapsto A[x := \epsilon x \neg A]$$

A formula reduces to another formula A', and one writes $A \mapsto A'$, if A' is obtained from A by applying one of the basic reductions above to a subformula of A.

The main results of the paper are strong normalization, which says that there are no infinite reduction sequences

$$A_1 \mapsto A_2 \mapsto A_3 \mapsto \ldots \mapsto A_n \mapsto \ldots$$

and confluence, which says that if A reduces in some steps to A' and A'', then there is a formula B such that both A' and A'' reduce in some steps to B.

The author first discusses whether it is possible to deduce these results by applying modern rewriting theory. He hesitantly concludes that this is likely, but provides no formal proof. He then goes on and provides his own proof of the results by applying a beautiful abstract theorem of Klop. The author deduces strong normalization from weak normalization, which is a standard idea, but in this setting overly complicated. The proof of confluence is then a corollary, as usual, of strong normalization and local confluence, which is very easy to prove.

2 Overall Judgement

The results of the paper are without any doubt correct and the main proofs clear and sound. But I will be frank: by all modern standards, the results are quite trivial. Nowadays strong normalization and confluence are extremely well-studied and there is a huge literature on them. As I explain in the technical comments, the results of this paper are just corollaries of strong normalization and confluence of simply typed lambda calculus. It is even possible to provide quite simpler proofs than the ones provides here, just by specializing to this paper's setting some known easy proofs of strong normalization for typed lambda calculus.

All considered, in this paper there is no original research result. Theorem 2.2, which is dubbed "a new theorem", is not at all new: it is an equivalent, unnecessary reformulation of the cited theorem of Klop, which could have been applied straightaway instead of being reformulated. The author's proof literally follows Klop's one. Local confluence is proved with a complicated notation and it is a standard exercise. The deduction of strong normalization from weak normalization is also standard, but in this setting is quite cumbersome.

For all these reasons, I cannot recommend this paper for publication and I would not recommend this paper to any journal or collections of proceedings that only accept original research results. Nevertheless, this paper is sound and if the proceedings of the Epsilon 2015 workshop are not supposed to contain only papers with original results, this one could be accepted. But several revisions are needed, as I explain in the technical comments.

3 Technical Comments

- Section 1.4

  In this section the king of rewriting systems is totally missing: lambda calculus! Yet, as standard in Church simple theory of types, any formula A can be mapped to a term A^* of atomic type o in such a way that the reduction on lambda terms represents the reduction relation on formulas. Predicate symbols, constant symbols, propositional connectives and the epsilon symbol are just mapped to constants. For example,

  $$(A \vee B)^* = \mathsf{or}^{o \to o \to o}\, A^*\, B^*$$

  Quantifiers are just defined as

  $$(\exists x\, A)^* = (\lambda x^o A^*)(\epsilon^{o \to o \to o}\, x\, A^*)$$

  $$(\forall x\, A)^* = (\lambda x^o A^*)(\epsilon^{o \to o \to o}\, x\, (\mathsf{not}^{o \to o}\, A^*))$$

  Therefore the confluence and strong normalization of the rewriting relation considered in the paper are corollaries of the correspondent results for typed lambda calculus.

- Section 2.1

  It should be said somewhere that by $\circ$ one means composition.

- Theorem 2.2

  The use of two relations $\longrightarrow_0$ and $\longrightarrow_1$ is redundant. In general, the decomposition of the rewriting relation $\longrightarrow$ into two relations, as throughout the paper, is of no use and makes more complicated than necessary the proofs and the formulations of the theorems. It would be better to keep Klop's original theorem. In any case, the adjective "new" must be eliminated from Theorem 2.2: it is plainly equivalent to Klop's theorem.

- Section 1.3.

  I do not agree with the claims.

  First, a confluence proof for lambda calculus was published by Church and Rosser in 1936, and it happens that Bernays visited the IAS of Princeton in the academic year 1935/1936. It is unlikely that Church and Rosser did not inform him about the issue of rewriting in lambda calculus and about their new proof of confluence. In any case, in 1936 there was already all the mathematical technology needed to prove difficult confluence results, let alone the much easier confluence for epsilon calculus.

  Secondly, the strong normalization theorem could have very well been proved by Hilbert-Bernays. No sophisticated technique is needed. One can prove the following lemma very easily: If A is a strongly normalizing formula and $t_1, \ldots, t_n$ are strongly normalizing terms, then the formula

$$A[x_1 := t_1 \ldots x_n := t_n]$$

  is strongly normalizing. As usual the substitution does not allow capture of the free variables of the terms t_i. The lemma is directly proved by double induction on the size of the reduction tree of A and the sum of the sizes of all the reduction trees of all the occurrences of the terms t_i in A: just consider the first reduction step of the formula above and conclude by induction hypothesis that the reduct is strongly normalizing. With this lemma one can prove directly by induction on formulas that each formula is strongly normalizing. When we have $\exists x A$, by induction hypothesis A is strongly normalizing and thus by the lemma

$$A[x := \epsilon x\, A]$$

  is strongly normalizing. Any sufficiently long reduction path of $\exists x\, A$ is of the form

$$\exists x\, A \mapsto^* \exists x\, A' \mapsto A'[x := \epsilon x\, A']$$

  with $A \mapsto^* A'$. Moreover,

$$A[x := \epsilon x\, A] \mapsto A'[x := \epsilon x\, A']$$

  therefore the latter term is strongly normalizing.