1 Multi-Sets

Multi-sets differ from sets only in their possibility to contain an element more than once. While we do not want to tell what multi-sets are, we denote them very similar to sets: $\emptyset$ denotes the empty multi-set as well as the empty set. We use '$\langle$' instead of '$\{$', '$\rangle$' instead of '$\}$', ' $\sqsubseteq$ ' instead of '$\subseteq$', ' $\sqcup$ ' instead of '$\cup$'. and '$\setminus\!\!\setminus$' instead of '$\setminus$'. $\mathrm{FMul}(S)$ denotes the set of finite multi-sets whose elements are in the set S; $\mathrm{SMul}(S)$ the set of singleton[1] multi-sets whose elements are in the set S.

It would be nice to keep multi-sets as abstract as they are now. But as we do not want to introduce a duplicate of set-theoretic mathematics for multi-sets we have to be more concrete: We assume a multi-set to be a set that contains for each multi-set-element a occurring n-times in it n different representation set-elements[2] $\hat{a}_0, \ldots, \hat{a}_{n-1}$ with $\forall i < n$: $a = \mathrm{set}(\hat{a}_i)$. To avoid confusion we do not introduce an analogue of "$\ldots \in \ldots$" for multi-sets, but write "$\ldots \in \mathrm{set}[\ldots]$" instead.

For defining a multi-set via some property $P(x_0, \ldots, x_{n-1})$ and some term $f(x_0, \ldots, x_{n-1})$ we use

$$\langle\ f(x_0, \ldots, x_{n-1})\ |_{x_0,\ldots,x_{n-1}}\ P(x_0, \ldots, x_{n-1})\ \rangle$$

which denotes the multi-set containing an element 'a' exactly

$$|\{\ (x_0, \ldots, x_{n-1})\ |_{x_0,\ldots,x_{n-1}}\ f(x_0, \ldots, x_{n-1}) = a\ \wedge\ P(x_0, \ldots, x_{n-1})\ \}|\text{-times.}$$

In a context where it is obvious that exactly "$x_0, \ldots, x_{n-1}$" are the quantified variables we will drop[3] the quantifying subscript just as it is usual for sets. Furthermore, for a multi-set A and a term $g(x_0, \ldots, x_{n-1})$ we allow some occurrences of " $(g(x_0, \ldots, x_{n-1}) \text{ in } A)$ " in " $P(x_0, \ldots, x_{n-1})$ ". Schematically denoting this by " $P_{[(g(x_0,\ldots,x_{n-1}) \text{ in } A)]}(x_0, \ldots, x_{n-1})$ " we will use

$$\langle\ f(x_0, \ldots, x_{n-1})\ |_{x_0,\ldots,x_{n-1}}\ P_{[(g(x_0,\ldots,x_{n-1}) \text{ in } A)]}(x_0, \ldots, x_{n-1})\ \rangle$$

to denote

$$\langle\ f(x_0, \ldots, x_{n-1})\ |_{x_0,\ldots,x_{n-1},\hat{x}}\ \ g(x_0, \ldots, x_{n-1}) = \mathrm{set}(\hat{x})\ \wedge\ P_{[\hat{x} \in A]}(x_0, \ldots, x_{n-1})\ \rangle.$$

Note that $\langle\ x \mid (x \text{ in } A)\ \wedge\ Q(x)\ \rangle$ differs from $\langle\ x \mid x \in \mathrm{set}[A]\ \wedge\ Q(x)\ \rangle$ where the number of occurrences of each element is reduced to 1. Finally we use

$$\langle\ f(x_0, \ldots, x_{n-1}) \text{ in } A \mid P(x_0, \ldots, x_{n-1})\ \rangle$$

as an abbreviation for

$$\langle\ f(x_0, \ldots, x_{n-1}) \mid (f(x_0, \ldots, x_{n-1}) \text{ in } A)\ \wedge\ P(x_0, \ldots, x_{n-1})\ \rangle.$$

All in all, this allows us to use a rather intuitive notion for multi-sets.

Let S be a set. We now come to *multi-set extensions on* $\mathrm{FMul}(S)$ of equivalences, quasi-orderings, and orderings, resp., on S, which are equivalences, quasi-orderings, and orderings, resp., on $\mathrm{FMul}(S)$ and are denoted by the symbol 'R' for the relation on S written twice with very little blank space between. The name "multi-set *extension*" is

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[1] i.e. containing a single element (exactly once)

[2] generated in some fixed manner (e.g. $\hat{a}_i := (a, i)$) for intensional equality of multi-sets to imply extensional equality.

[3] Note however that we have to be a little bit more careful than with sets. E.g. for $<$ being the ordering on the natural numbers, the set $\{\ (x, z)\ |_{x,y,z}\ x < y < z\ \}$ does not differ from the set $\{\ (x, z)\ |_{x,z}\ \exists y\colon\ x < y < z\ \}$ which lazy people use to write as $\{\ (x, z)\ |_{x,z}\ x < y < z\ \}$. To the contrary, the multi-set $\langle\ (x, z)\ |_{x,y,z}\ x < y < z\ \rangle$ contains $(0, 3)$ twice while $\langle\ (x, z)\ |_{x,z}\ \exists y\colon\ x < y < z\ \rangle$ contains $(0, 3)$ only once. Thus a the notation $\langle\ (x, z)\ |_{x,z}\ x < y < z\ \rangle$ should only be used when y is global variable mentioned before.

justified by:

$$\langle a \rangle \ \mathrm{RR} \ \langle b \rangle \quad \text{iff} \quad a \ \mathrm{R} \ b \qquad\qquad \text{for} \ \ a, b \in S$$

Let $\lesssim$ be a quasi-ordering on S, $\approx$ its equivalence, $<$ its ordering[4].
We define (for $A, B \in \mathrm{FMul}(S)$):

$$A \approx\approx B \ \text{ if } \ \exists f\colon\ A \to B \text{ bijective: } \forall \hat{a} \in A\colon\ \mathrm{set}(\hat{a}) \approx \mathrm{set}(f(\hat{a})).$$

$$A \lesssim\!\lesssim B \ \text{ if } \ \exists X_A, Y_A, X_B, Y_B \in \mathrm{FMul}(S)\colon \begin{pmatrix} A = X_A \sqcup Y_A & \wedge \\ B = X_B \sqcup Y_B & \wedge \\ X_A \approx\approx X_B & \wedge \\ \forall a \in \mathrm{set}[Y_A]\colon\ \exists b \in \mathrm{set}[Y_B]\colon\ a < b & \end{pmatrix}.$$

$$A \ll B \ \text{ if } \ \exists X_A, Y_A, X_B, Y_B \in \mathrm{FMul}(S)\colon \begin{pmatrix} A = X_A \sqcup Y_A & \wedge \\ B = X_B \sqcup Y_B & \wedge \\ X_A \approx\approx X_B & \wedge^5 \\ Y_B \neq \emptyset & \wedge \\ \forall a \in \mathrm{set}[Y_A]\colon\ \exists b \in \mathrm{set}[Y_B]\colon\ a < b & \end{pmatrix}.$$

Now $\approx\approx$ is the equivalence and $\ll$ the ordering of $\lesssim\!\lesssim$.

If $<$ is well-founded, $\ll$ is well-founded, too. If $<$ is total on S, $\ll$ is total on $\mathrm{FMul}(S)$. For $\mathrm{field}(<) \subseteq \mathcal{T}$ the following holds: If $<$ is monotonic, $\ll$ is monotonic for replacing a subterm of a term in a multi-set. If $<$ is stable, $\ll$ is stable for application of the substitution to all terms in a multi-set.

[4]If there is no quasi-ordering but only an ordering on S given, just take the S-reflexive ordering of the ordering for the quasi-ordering.

[5]If there is no quasi-ordering but only an ordering on S given, just take '=' instead of $\approx\approx$ here.

2 Activated Terms

We are now going to describe a technique that enables the termination of non-terminating reduction relations. The idea is to activate each term with a certain amount of energy and to let each reduction step consume some energy such that the reduction must terminate when all energy has been consumed.

Let E be some set. An *activated term* from a set of terms T is a pair (t,A) of a term $t \in T$ and a finite multi-set $A \in \mathrm{FMul}(\mathcal{POS}(t) \times E)$ over pairs of positions of t and elements of E. A *path* of a term is a maximal position:

$$\mathcal{PATH}(t) := \{\ p \in \mathcal{POS}(t) \mid \forall i \in \mathbb{N}\colon\ pi \notin \mathcal{POS}(t)\ \}.$$

Now for a path $p \in \mathcal{PATH}(t)$ of an activated term (t, A) we define its energy to be the multi-set of the activations along p:

$$\text{path–energy}((t, A), p) := \langle\ (q, e) \text{ in } A \mid \exists q'\colon\ p = qq'\ \rangle.$$

Finally the *energy* of an activated term (t, A) is defined to be:

$$\text{energy}((t, A)) := \langle\ \text{path–energy}((t, A), p) \mid p \in \mathcal{PATH}(t)\ \rangle.$$

Now for some reduction relation $\longrightarrow_{\mathrm{R,X}}$ and some $e \in E$ we define its e-energetic version to be

$$\longrightarrow_{e,\mathrm{R,X}} := \longrightarrow_{e,\mathrm{propagate}} \cup \longrightarrow_{e,\mathrm{consume,R,X}}.$$

$\longrightarrow_{e,\mathrm{propagate}}$ propagates energy from a position to some of the positions directly below.
$\longrightarrow_{e,\mathrm{consume,R,X}}$ consumes energy at a certain position and then reduces does a reduction step via $\longrightarrow_{\mathrm{R,X}}$ at this position: More precisely, for some activated terms (t_i, A_i),

$(t_0, A_0) \longrightarrow_{e,\mathrm{propagate}} (t_1, A_1)$ if $t_0 = t_1$ and

$$\exists p \in \mathcal{POS}(t_0)\colon\ \exists B_0\colon\ \exists B_1 \sqsubseteq \langle\ (pi, e) \mid i \in \mathbb{N}\ \wedge\ pi \in \mathcal{POS}(t_0)\ \rangle\colon \Big(\ A_0 = \langle (p, e) \rangle \sqcup B_0\ \wedge\ A_1 = B_1 \sqcup B_0\ \Big).$$

and $(t_0, A_0) \longrightarrow_{e,\mathrm{consume,R,X}} (t_1, A_1)$ if

$$\exists p \in \mathcal{POS}(t_0)\colon\ \exists ((l, r), C) \in \mathrm{R}\colon\ \exists \sigma \in \mathcal{SUB}(\mathrm{V}, \mathcal{T}(\mathrm{X}))\colon$$
$$\left(\begin{array}{l} t_0/p = l\sigma\ \ \wedge\ \ t_1 = t_0[\,p \leftarrow r\sigma\,]\ \ \wedge\ \ (C\sigma \text{ fulfilled w.r.t. } \longrightarrow_{\mathrm{R,X}})\ \ \wedge \\ \exists B_0, B_1\colon\ \exists f\colon\ \{\ q \in \mathcal{PATH}(r) \mid r/q \in \mathcal{V}(l)\ \} \to \mathcal{PATH}(l)\colon \\ \left(\begin{array}{ll} A_0 = \langle (p, e) \rangle \sqcup B_0 \sqcup B_1 & \wedge \\ B_1 = \langle\ (pq, e') \text{ in } A_0 \mid q \neq \emptyset\ \rangle & \wedge \\ A_1 = B_0 \sqcup \langle\ (pqq', e') \mid (\,p\,f(q)\,q'\,, e') \text{ in } B_1\ \rangle & \\ \wedge \qquad\qquad \forall q \in \mathrm{dom}(f)\colon\ r/q = l/f(q) & \end{array}\right). \end{array}\right)$$

Note that the function f is uniquely determined (by p, (l, r)) when l is a linear term; in which case also A_1 is uniquely determined by (p, e), (l, r), and A_0.

Lemma 2.1
$\bigcup_{e \in E} \longrightarrow_{e,\mathrm{R,X}}$ is noetherian.

Proof of Lemma 2.1
Let $\lesssim\ :=\ _E|\mathrm{id}$. Since its ordering $<$ is wellfounded, so is its double multi-set extension $\lll$. $\bigcup_{e \in E} \longrightarrow_{e,\mathrm{propagate}}$ is obviously noetherian and for $(t_0, A_0) \longrightarrow_{e,\mathrm{propagate}} (t_1, A_1)$ we have $\text{energy}((t_0, A_0)) \gtrapprox\gtrapprox \text{energy}((t_1, A_1))$. Finally for $(t_0, A_0) \longrightarrow_{e,\mathrm{consume,R,X}} (t_1, A_1)$ we have $\text{energy}((t_0, A_0)) \ggg \text{energy}((t_1, A_1))$.