THE AUTOMATION OF MATHEMATICAL INDUCTION*

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1 A SNAPSHOT OF A DECISIVE MOMENT IN HISTORY

The automation of mathematical theorem proving for deductive *first-order logic* started in the 1950s, and it took about half a century to develop systems that are sufficiently strong and general to be successfully applied outside the community of automated theorem proving.[1] Surprisingly, the development of such strong systems for restricted logic languages was not achieved much earlier, neither for the *purely equational fragment* nor for *propositional logic*.[2] Moreover, automation of theorem proving for *higher-order logic* is making progress towards general usefulness just during the last ten years.[3]

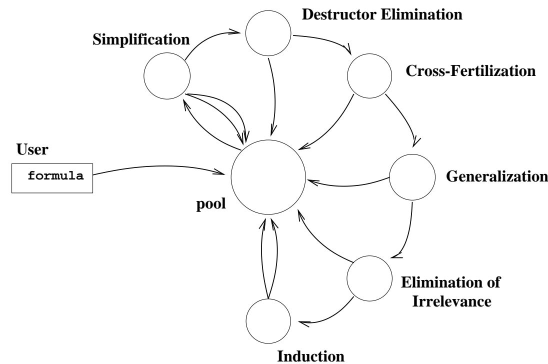

Figure 1. The Boyer–Moore Waterfall

Note that a formula falls back to the center pool after each successful application of one of the stages in the circle.

*Second Readers: Somebody from the history of math & logic. I wanna ask Wilfried Sieg. OK?

[1]The currently (i.e. in 2012) most successful first-order automated theorem prover is VAMPIRE, cf. e.g. [Riazanov and Voronkov, 2001].

[2]The most successful automated theorem prover for purely equational logic is WALDMEISTER, cf. e.g. [Buch and Hillenbrand, 1996], [Hillenbrand and Löchner, 2002]. For deciding propositional validity (i.e. sentential validity) (or its dual: propositional satisfiability) (which is decidable, but NP-complete), a breakthrough toward industrial strength was the SAT solver CHAFF, cf. e.g. [Moskewicz *et al.*, 2001].

[3]One of the driving forces in the automaton of higher-order theorem proving is the system LEO-II, cf. e.g. [Benzmüller *et al.*, 2008].

In this context, it is most astonishing that for the field of quantifier-free first-order *inductive* theorem proving based on recursive functions, the whole development — from the scratch to general usefulness — took place within the 1970s.

In this article we describe how this giant step took place, and sketch the further development of automated inductive theorem proving.

The work on this breakthrough in the automation of inductive theorem proving was started in September 1972 by Robert S. Boyer and J Strother Moore, and most of the crucial steps and their synergetic combination in the now famous "waterfall" (cf. Figure 1) are already implemented in their "PURE LISP THEOREM PROVER", presented at IJCAI in Stanford (CA) in August 1973,[4] and documented in Moore's PhD thesis [1973], defended in November 1973.

At that time, Boyer and Moore were both members of the Metamathematics Unit of the University of Edinburgh.[5]

Boyer and Moore had met for the first time in August 1971 in Edinburgh, and had worked together on structure sharing in resolution theorem proving, for which the inventor of resolution, J. Alan Robinson, invented and granted to them the "1971 Programming Prize" on December 17, 1971, half jokingly, half seriously.[6] In spite of this experience, the original plan to implement structure sharing in the PURE LISP THEOREM PROVER was dropped at some point.

[4] Cf. [Boyer and Moore, 1973].

[5] The Metamathematics Unit of the University of Edinburgh was renamed into "Dept. of Computational Logic" in late 1971, and was absorbed into the new "Dept. of Artificial Intelligence" in Oct. 1974. It was founded and headed by Bernard Meltzer. In the early 1970s, the University of Edinburgh hosted most remarkable scientists, of which the following are relevant in our context:

	Univ. Edinburgh (time, Dept.)	PhD (year, advisor)	life time (birth–death)
Donald Michie	(1965–1984, MI)	(1953, ?)	(1923–2007)
Bernard Meltzer	(1965–1978, CL)	(1953, Fürth)	(1916?–2008)
Robin J. Popplestone	(1965–1984, MI)	(no PhD?)	(1938–2004)
Rod M. Burstall	(1965–2000, MI & CL)	(1966, Dudley)	(*1934)
Robert A. Kowalski	(1967–1974, CL)	(1970, Meltzer)	(*1941)
Pat Hayes	(1967–1973, CL)	(1973, Meltzer)	(*1944)
Gordon Plotkin	(1968–today, CL & LFCS)	(1972, Burstall)	(*1946)
J Strother Moore	(1970–1973, CL)	(1973, Burstall)	(*1947)
Mike J. C. Gordon	(1970–1978, MI)	(1973, Burstall)	(*1948)
Robert S. Boyer	(1971–1974, CL)	(1971, Bledsoe)	(*1946)
Alan Bundy	(1971–today, CL)	(1971, Goodstein)	(*1947)
Robin Milner	(1973–1979, LFCS)	(no PhD)	(1934–2010)

CL = Metamathematics Unit (founded and headed by Bernard Meltzer)
(new name from late 1971 to Oct. 1974: Dept. of Computational logic)
(new name from Oct. 1974: Dept. of Artificial Intelligence)
MI = Experimental Programming Unit (founded and headed by Donald Michie)
(new name from 1966 to Oct. 1974: Dept. for Machine Intelligence and Perception)
(new name from Oct. 1974: Machine Intelligence Unit)
LFCS = Laboratory for Foundations of Computer Science

(Sources: [Meltzer, 1975], [Kowalski, 1988], etc.)

Boyer and Moore's achievements become even more surprising by the fact that they had to work with inferior computing machinery and that they did all the programming themselves.

Working with computers was pretty troublesome and difficult at that time: The storage medium was paper tapes, and the editor programs of the time still simulated paper-tape editing.

Moreover, the Metamathematics Unit did not even have a PDP–10, which was the top model of the time, but only an ICL–4130, which had only 64 kByte core memory (RAM). A funny feature was the loudspeaker connect to this core memory, which, however, did not produce a regular sound when the PURE LISP THEOREM PROVER was running (unless during garbage collection). An irregular sound was meant to indicate that it was not in a constant loop, but making "progress".

What made the situation worse was that Boyer and Moore were granted sufficient computation time only at nights, for which Boyer had to have a huge teletype terminal in his tiny house in Edinburgh.

It is easy to imagine what this meant for the wives of Boyer and Moore who lived with them in Edinburgh.

> "The PDP–10, the PDP–10,
> I've sung it before and I'll sing it again.
> I hav'nt seen my man since I don't know when —
> Oh, I lost my husband to a PDP–10.
>
> If he'd a woman, the fight would be fair.
> If he were out drinking, his liquor I'd share.
> But the odds are against me, you know what I mean —
> Even John Henry couldn't beat a machine.
>
> Guess I'll give up, I know when I am beat.
> Sure hate to say it, but this is defeat.
> There's just one more problem that I've got to face —
> Can you name a machine in an adultery case?"

The Boyers and Moores used to sing this text to the melody of the popular American folksong "John Henry", a legendary African-American railroad worker who died in a competition with a steam-powered hammer. The new text — which paradigmatically describes a situation well-known to probably all spouses of scientists until today — was written by Anne Boyer at a time when actually no PDP–10 was available to Boyer and Moore; but their situation had improved in this aspect when she published it [1980]. Needless to say that Anne and Bob Boyer are still living together today.

[6] The document, handwritten by J. Alan Robinson (*1930?) actually says:

> "In 1971, the prize is awarded, by unanimous agreement of the Board, to Robert S. Boyer and J Strother Moore for their idea, explained in [Boyer and Moore, 1971], of representing clauses as their own genesis. The Board declared, on making the announcement of the award, that this idea is '... bloody marvelous'."

2 METHOD OF PROCEDURE AND PRESENTATION

Contrary to the excellent handbook articles [Walther, 1994] and [Bundy, 1999] on the *automation of explicit induction*, our focus in this article is neither on current standards, nor on the engineering and research problems of the field, but on the *history of the automation of mathematical induction*.

The main problem of historians of recent history is that they are in the timeline of the incidents under consideration and strongly influenced by the tradition that the achievements have established, such that they cannot appreciate the incidents with contemporary eyes, but with the view from today back into the past, "knowing" that the solved problems are the easy ones because they are solved.

We try to mitigate this problem by avoiding the standpoint of a disciple of the leading school of explicit induction. Instead, we put the historic achievements into a broad mathematical context and a space of time from the ancient Greeks to a possible future, based on a most general approach to *recursive definition*, and on *descente infinie* as the most general practical approach to mathematical induction. Then we can see the great achievements in the field with the surprise, astonishment, and awe they historically deserve — after all, they succeeded to a large degree in the automation of operations that used to be considered as the summit of human intellectual abilities before, leaving not much superiority to the human race.

As a historiographical text, this article should be accessible to an audience that goes beyond the technical experts and programmers of the day, should use common mathematical language and representation, focus on the global and eternal ideas and their developments, and paradigmatically display the *historically most significant achievements*.

Because these achievements in the automation of inductive theorem proving manifest themselves mainly in the line of the Boyer–Moore theorem provers, we cannot avoid the confrontation of the reader with some more ephemeral forms of representation found in these software systems. In particular, we cannot avoid some small expressions in the list programming language LISP,[7] simply because the Boyer–Moore theorem provers we discuss in this article, namely the PURE LISP THEOREM PROVER, THM, NQTHM, and ACL2, all have *logics* based on a subset of LISP. Here we do not necessarily refer to the implementation language of these software systems, but actually to the logic language used both for representation of formulas and for communication with the user.

For the first system in this line of development, Boyer and Moore had the free choice, but — in 1972 — there was actually no alternative to a logic based on LISP, because inductive theorem proving with recursively defined functions requires a logic in which

> a *method of symbolic evaluation* can be obtained from an interpretation procedure by generalizing the ground terms of computation to terms with free variables that are implicitly universally quantified.

[7] Cf. [McCarthy *et al.*, 1965].

Beside LISP, a candidate to be considered today is the functional programming language HASKELL.[8] HASKELL, however, was not available in 1972. And still today, LISP is to be preferred to HASKELL as the logic of an inductive theorem prover because of LISP's innermost evaluation strategy, which gives preference to the constructor terms that represent the constructor-based data types, which again establish the most interesting domains in hard- and software verification and the major elements of mathematical induction.

Yet another candidate today would be the rewrite systems of [Wirth and Gramlich, 1994a] and [Wirth, 1991; 2009] with their *constructor variables*[9] and their *positive/negative-conditional equations*, designed and developed for the specification, interpretation, and symbolic evaluation of recursive functions in the context of inductive theorem proving in the domain of constructor-based data types. Neither this tailor-made theory, nor even the general theory of rewrite systems in which its development is rooted,[10] were available in 1972. And still today, the applicative subset of COMMON LISP that provides the logic language for ACL2 $(= (\mathrm{ACL})^2 =$ A Computational Logic for Applicative Common LISP) is again to preferred to these positive/negative-conditional rewrite systems under the aspect of efficiency: The applications of ACL2 in hardware verification and testing require a performance that is still at the very limits of today's computing technology. This challenging efficiency demand requires, among other aspects, that the logic of the theorem prover is so close to its own programming language that — after certain side conditions have been checked successfully — the theorem prover can defer the interpretation of ground terms to the analogous interpretation in its own programming language.

For many of our illustrative examples in this article, however, we will use the higher flexibility and conceptual adequacy of positive/negative-conditional rewrite systems. They are so close to standard logic that we can dispense their semantics to the readers intuition,[11] and they can immediately serve as an intuitively clear replacement of the Boyer–Moore *machines*.[12]

Moreover, the typed (many-sorted) approach of the positive/negative-conditional equations admits to present the formulas in a form that is much easier to

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[8] Cf. e.g. [Hudlak *et al.*, 1999].

[9] See § 4.3 of this article.

[10] The general theory on which the rewrite systems of [Wirth and Gramlich, 1994a] and [Wirth, 1991; 2009] is rooted is documented in [Dershowitz and Jouannaud, 1990]. One may try to argue that the paper that launched the whole field of rewrite systems, [Knuth and Bendix, 1970], was already out in 1972, but the relevant parts of rewrite theory for unconditional equations were developed only in the late 1970s and the 1980s. Especially relevant in the given context are [Huet, 1980] and [Toyama, 1988]. The rewrite theory of *positive/negative-conditional* equations, however, started to become an intensive area of research only at the breath-taking 1st Int. Workshop on Conditional Term Rewriting Systems (CTRS), Orsay (France), 1987; cf. [Kaplan and Jouannaud, 1988].

[11] The readers interested into the precise details are referred to [Wirth, 2009].

[12] Cf. [Boyer and Moore, 1979, p. 165f.].

grasp for human readers than the corresponding sugar-free LISP notation with its overhead of explicit type restrictions.

Another good reason for avoiding LISP notation is the following: We want to make it most obvious that the achievements of the Boyer–Moore theorem provers are not limited to their LISP logic and would well survive in a better world, where — based on the computing technology of the future — positive/negative-conditional rewriting is not too inefficient for the relevant applications anymore.

For the same reason, we also prefer examples from arithmetic to examples from list theory, which might be considered to be especially supported by the LISP logic. The reader can find the famous examples from list theory in almost any other publication on the subject.[13]

In general, we tend to present the challenges and their historical solutions with the help of small intuitive examples and refer the readers interested in the very details of the implementations of the theorem provers to the published and easily accessible documents on which our description is mostly based.

Nevertheless, small LISP expression cannot be completely avoided because we have to describe the crucial parts of the historically most significant implementations and ought to show some of the advantages of LISP's untypedness.[14] The readers interested in these aspects, however, do not have to know more about LISP than the following: A LISP term is either a variable symbol, or a function call of the form `(`$f\ t_1\ \cdots\ t_n$`)`, where f is a function symbol and $t_1, \ldots, t_n$ are LISP terms.

2.1 Organization of This Article

This article is further organized as follows: §§ 3 and 4 are a self-contained reference for the readers who are not acquainted with the field of automated induction. In § 3 we introduce the essentials of mathematical induction. In § 4 we have to become more formal regarding recursive function definitions, their consistency, termination, and induction templates and schemes.

After this self-contained introduction, In § 5 we come to the presentation of the historically most important systems in automated induction, and discuss the details of software systems for explicit induction, with a focus on the 1970s.

The approaches to the automation of induction that do not follow the paradigm of explicit induction are discussed in § 6. After looking beyond induction in § 7, we conclude with § 8.

This should be augmented with more details at the very end of our writing. Note that the handbook style does not admit us to have a table contents.

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[13] Cf. e.g. [Moore, 1973], [Boyer and Moore, 1979; 1988b; 1998], [Walther, 1994], [Bundy, 1999], [Kaufmann *et al.*, 2000a; 2000b].

[14] See e.g. of the advantages of the untyped and type-restriction-free declaration of the shell `CONS` in § 5.3.

3 MATHEMATICAL INDUCTION

In this section, we introduce mathematical induction with its rich history since the 6th century B.C., and clarify the difference between *structural induction* and *Noetherian induction* with its traditional variant called *descente infinie*.

According to Aristotle, *induction* means to go from the special to the general, in particular to obtain *general laws* from special cases, which plays a major rôle in the generation of conjectures in mathematics and the natural sciences. Modern scientists design experiments to falsify such a law of nature, and they accept the law as a scientific fact only after many trials have all failed to falsify it. In the tradition of Euclid of Alexandria, mathematicians accept a conjectured mathematical law as a theorem only after a rigorous proof has been provided. According to Kant, induction is *synthetic* in the sense that it properly extends what we think to know — in opposition to *deduction*, which is *analytic* in the sense that all information we can obtain by it, is implicitly contained in the initial judgments, though we can hardly be aware of all deducible consequences in advance.

Surprisingly, in this well-established and time-honored terminology, *mathematical induction* is not induction, but a special form of deduction for which — in the 19th century[15] — the term "induction" was introduced and became standard in English and German mathematics. In spite of this misnomer, for the sake of brevity, the term "induction" will always refer to mathematical induction in what follows.

Although it received its current name only in 19th century, mathematical induction has been a standard method of every working mathematician at all times. Hippasus of Metapontum (Italy) (ca. 550 B.C.) is reported[16] to have proved the irrationality of the golden number by a form of mathematical induction, which later was named *descente infinie (ou indéfinie)* by Fermat. We find another form of induction, nowadays called *structural induction*, in a text of Plato (427–347 B.C.).[17] In the famous "Elements" of Euclid [ca. 300 B.C.], we find several applications of *descente infinie* and structural induction.[18] Structural induction was known to the Muslim mathematicians around the year 1000, and occurs in a Hebrew book of Levi ben Gerson (Orange and Avignon) (1288–1344).[19] Furthermore, structural induction was used by Francesco Maurolico (Messina) (1494–1575),[20] and by Blaise Pascal (1623–1662).[21] After an absence of more than one millennium, *descente infinie* was reinvented by Pierre Fermat (1607?–1665).[22]

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[15] Cf. [Cajori, 1918].

[16] Cf. [Fritz, 1945].

[17] Cf. [Acerbi, 2000].

[18] An example for *descente infinie* is Proposition 31 of Vol. VII of the Elements, and an example for structural induction is Proposition 8 of Vol. IX, cf. [Wirth, 2010b, § 2.4].

[19] Cf. [Katz, 1998].

[20] Cf. [Bussey, 1917].

[21] Cf. [Pascal, 1954, p. 103].

[22] See [Barner, 2001] for the correction on Fermat's year of birth as compared to the wrong date in the title of [Mahoney, 1994]. The best-documented example of a proof by *descente infinie* of one of Fermat's outstanding results in number theory is the proof of the following theorem: *The area of a Pythagorean triangle with positive integer side lengths is not the square of an integer*; cf. [Wirth, 2010b].

3.1 Well-foundedness and Termination

A relation $<$ is *well-founded* if each proposition $Q(w)$ that is not constantly false holds for a $<$-minimal m, i.e. there is an m with $Q(m)$, for which there is no $v < m$ with $Q(v)$. Writing "$\mathsf{Wellf}(<)$" for "$<$ is well-founded", we can formalize this *definition* as follows:

$$(\mathsf{Wellf}(<)) \qquad \forall Q.\ \Big(\ \exists w.\ Q(w) \quad \Rightarrow \quad \exists m.\ \big(\ Q(m) \wedge \neg\exists u{<}m.\ Q(u)\ \big)\ \Big)$$

Moreover, $<$ is an (irreflexive) *ordering* if it is irreflexive and transitive. There is not much difference between a well-founded relation and a well-founded ordering:[23]

LEMMA 1. *A relation is well-founded if and only if its transitive closure is a well-founded ordering.*

Closely related to the well-foundedness of a relation $<$ is the termination of its *reverse relation* $>$, given as $<^{-1} := \{\ (u,v) \mid (v,u) \in <\ \}$.

A relation $>$ is *terminating* if it has no non-terminating sequences, i.e. if there is no infinite sequence of the form $x_0 > x_1 > x_2 > x_3 \ldots$.

If $>$ has a non-terminating sequence, then this sequence, taken as a set, is a witness for the non-well-foundedness of $<$. The converse implication, however, is a weak form of the Axiom of Choice;[24] indeed, it admits us to pick a non-terminating sequence for $>$ from the set witnessing the non-well-foundedness of $<$.

So well-foundedness is slightly stronger than termination of the reverse relation, and this difference is relevant in our context here, where we cannot take the Axiom of Choice for granted because it can be seen as a very strong induction axiom.[25]

3.2 The Theorem of Noetherian Induction

In its modern standard meaning, the method of mathematical induction is easily seen to be a form of deduction, simply because it can be formalized as the application of the *Theorem of Noetherian Induction*:

> A proposition $P(w)$ can be shown to hold (for all w) by *Noetherian induction* over a well-founded relation $<$ as follows: *Show (for every v) that $P(v)$ follows from the assumption that $P(u)$ holds for all $u < v$.*

Again writing "$\mathsf{Wellf}(<)$" for "$<$ is well-founded", we can formalize the *Theorem of Noetherian Induction* as follows:[26]

$$(\mathsf{N}) \qquad \forall P.\ \left(\ \forall w.\ P(w) \quad \Leftarrow \quad \exists <.\ \left(\begin{array}{l} \quad \forall v.\big(\,P(v) \ \Leftarrow\ \forall u{<}v.\ P(u)\,\big) \\ \wedge \quad \mathsf{Wellf}(<) \end{array}\right)\right)$$

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[23] Cf. Lemma 2.1 of [Wirth, 2004, § 2.1.1].

[24] See [Wirth, 2004, § 2.1.2, p. 18] for the equivalence to the Principle of Dependent Choice, which is found in [Rubin and Rubin, 1985, p. 19] and analyzed in detail as Form 43 (p. 30) in [Howard and Rubin, 1998].

[25] Cf. [Geser, 1995].

[26] When we write an implication $A \Rightarrow B$ in the reverse form of $B \Leftarrow A$, we do this to indicate that a proof attempt will typically start from B and try to reduce it to A.

The term "Noetherian induction" is a tribute to the famous German female mathematician Emmy Noether (1882–1935). It occurs as the "Generalized principle of induction (Noetherian induction)" in [Cohn, 1965, p. 20]. It also occurs in Proposition 7 ("Principle of Noetherian Induction") (p. 190) of § 6.5 in [Bourbaki, 1968a, Chapter III], which is a translation of the French original in its second edition [Bourbaki, 1967], where it occurs in Proposition 7 ("principe de récurrence nœthérienne")[27] of § 6.5. We do not know whether the today most common term "Noetherian Induction" occurred already before 1965;[28] in particular, it does not occur in the first French edition [Bourbaki, 1956] of [Bourbaki, 1967].[29]

3.3 The Natural Numbers

The field of application of mathematical induction most familiar in mathematics is the domain of the natural numbers 0, 1, 2, ... Let us formalize the natural numbers with the help of two constructors: the constant symbol

$$\mathsf{0} : \mathsf{nat}$$

for zero, and the function symbol

$$\mathsf{s} : \mathsf{nat} \to \mathsf{nat}$$

for the direct successor of a natural number. Moreover, let us assume in this article that the variables x, y always range over the natural numbers, and that free variables in formulas are implicitly universally quantified (as standard in mathematics), such that, for example, a formula with the free variable x can be seen as having the implicit outermost quantifier $\forall x : \mathsf{nat}$.

After the definition ($\mathsf{Wellf}(<)$) and the theorem (N), let us now consider some standard *axioms* for specifying the natural numbers, namely that a natural number is either zero or a direct successor of another natural number ($\mathsf{nat1}$), that zero is not a successor ($\mathsf{nat2}$), that the successor function is injective ($\mathsf{nat3}$), and that the so-called *Axiom of Structural Induction over* $\mathsf{0}$ *and* s holds; formally:

$$(\mathsf{nat1}) \qquad x = \mathsf{0} \quad \vee \quad \exists y.\ \big(\ x = \mathsf{s}(y)\ \big)$$

$$(\mathsf{nat2}) \qquad \mathsf{s}(x) \neq \mathsf{0}$$

$$(\mathsf{nat3}) \qquad \mathsf{s}(x) = \mathsf{s}(y) \quad \Rightarrow \quad x = y$$

$$(\mathsf{S}) \qquad \forall P.\ \Big(\ \forall x.\ P(x) \quad \Leftarrow \quad P(\mathsf{0})\ \wedge\ \forall y.\ \big(\ P(\mathsf{s}(y)) \Leftarrow P(y)\ \big)\ \Big)$$

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[27] The peculiar French spelling "Nœthér" in "nœthérienne" tries to imitate the German pronunciation of "Noether", where the "oe" is to be pronounced neither as a long "o" (which would be the default, as in "Itzehoe"), nor as two separate vowels as indicated by the diaeresis in "oë", but as an umlaut, typically written in German as the ligature "ö". Neither Emmy Noether nor her father, the mathematics professor Max Noether (1844–1921), ligated the "oe" in their name; the ligature occurs, however, in some of their official German documents.

[28] Even in 1967, "Noetherian Induction" was not generally used as a name for the Theorem of Noetherian Induction: For instance, in [Schoenfield, 1967, p. 205], this theorem (instantiated with the ordering of the natural numbers) is called the *principle of complete induction*, which is a most confusing name that should be avoided because "complete induction" looks like the straightforward translation of the German technical term "vollständige Induction", which traditionally means structural induction (cf. Note 30), and which already in the 1920s had become a very vague notion that is best translated as "mathematical induction", as done already in [Heijenoort, 1971, p.130] and as it is standard today, cf. [Hilbert and Bernays, 2011, Note 23.4].

Richard Dedekind (1831–1916) proved the Axiom of Structural Induction (S) for his model of the natural numbers in [Dedekind, 1888], where he states that the proof method resulting from the application of this axiom is known under the name "vollständige Induction" ("complete induction").[30]

Now we can go on by defining — in two equivalent ways[31] — the destructor function $\mathsf{p} : \mathsf{nat} \rightarrow \mathsf{nat}$, returning the predecessor of a positive natural number:

1. In (p1) in *constructor style*, where constructor terms may occur on the left-hand side of the positive/negative-conditional equation as arguments of the function being defined.

2. In (p1′) in *destructor style*, where only variables may occur as arguments on the right-hand side.

For both definition styles, the term on the left-hand side must be linear (i.e. all its variable occurrences must be distinct variables) and have the function symbol to be defined as the top symbol.

(p1) $\mathsf{p}(\mathsf{s}(x)) = x$

(p1′) $\mathsf{p}(x') = x \;\Leftarrow\; \mathsf{s}(x) = x'$

Let us define some recursive functions over the natural numbers, such as addition and multiplication $+, * : \mathsf{nat}, \mathsf{nat} \rightarrow \mathsf{nat}$, the irreflexive ordering of the natural numbers $\mathsf{lessp} : \mathsf{nat}, \mathsf{nat} \rightarrow \mathsf{bool}$ (see § 3.4 for the data type bool of Boolean values), and the Ackermann function $\mathsf{ack} : \mathsf{nat}, \mathsf{nat} \rightarrow \mathsf{nat}$:[32]

(+1)	$\mathsf{0} + y = y$		(∗1)	$\mathsf{0} * y = \mathsf{0}$
(+2)	$\mathsf{s}(x) + y = \mathsf{s}(x + y)$		(∗2)	$\mathsf{s}(x) * y = y + (x * y)$

(lessp1) $\mathsf{lessp}(x, \mathsf{0}) = \mathsf{false}$

(lessp2) $\mathsf{lessp}(\mathsf{0}, \mathsf{s}(y)) = \mathsf{true}$

(lessp3) $\mathsf{lessp}(\mathsf{s}(x), \mathsf{s}(y)) = \mathsf{lessp}(x, y)$

(ack1) $\mathsf{ack}(\mathsf{0}, y) = \mathsf{s}(y)$

(ack2) $\mathsf{ack}(\mathsf{s}(x), \mathsf{0}) = \mathsf{ack}(x, \mathsf{s}(\mathsf{0}))$

(ack3) $\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)) = \mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y))$

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[29] Indeed, the main text of § 6.5 in the 1st edition [Bourbaki, 1956] ends (on Page 98) three lines before the text of Proposition 7 begins in the 2nd edition [Bourbaki, 1967] (on Page 76 of § 6.5).

[30] The first occurrence of the name "vollständige Induction" with the meaning of mathematical induction seems to be on Page 46f. in [Fries, 1822]. See also Note 28.

[31] For the equivalence transformation between constructor and destructor style see Example 15 in § 5.3.2.

[32] Rósza Péter (1905–1977), a female mathematician in Budapest of Jewish parentage, published a simplified version [1951] of the first recursive, but not primitive recursive function developed by Wilhelm Ackermann (1896–1962) [Ackermann, 1928]. What is simply called "the Ackermann function" today is actually Péter's version.

The relation from a natural number to its direct successor can be formalized by the binary relation $\lambda x, y.\ (\mathsf{s}(x)=y)$. Then $\mathsf{Wellf}(\lambda x, y.\ (\mathsf{s}(x)=y))$ states the well-foundedness of this relation, which means according to Lemma 1 that its transitive closure — i.e. the irreflexive ordering of the natural numbers — is a well-founded ordering; so, in particular, we have $\mathsf{Wellf}(\lambda x, y.\ (\mathsf{lessp}(x,y)=\mathsf{true}))$.

Now the natural numbers can be specified up to isomorphism either by[33]

- (nat2), (nat3), and (S) — following Guiseppe Peano (1858–1932),

or else by

- (nat1) and $\mathsf{Wellf}(\lambda x, y.\ (\mathsf{s}(x)=y))$ — following Mario Pieri (1860–1913).[34]

Immediate consequences of the axiom (nat1) and the definition (p1) are the lemma (s1) and its flattened version (s1′):

(s1) $\mathsf{s}(\mathsf{p}(x')) = x' \quad \Leftarrow \quad x' \neq \mathsf{0}$

(s1′) $\mathsf{s}(x) = x' \quad \Leftarrow \quad x' \neq \mathsf{0} \ \wedge \ x = \mathsf{p}(x')$

Moreover, on the basis of the given axioms we can most easily show

(lessp4) $\mathsf{lessp}(x, \mathsf{s}(x)) = \mathsf{true}$

(lessp5) $\mathsf{lessp}(x, \mathsf{s}(x+y)) = \mathsf{true}$

by *structural induction on* x, i.e. by taking the predicate variable P in the Axiom of Structural Induction (S) to be $\lambda x.\ (\mathsf{lessp}(x, \mathsf{s}(x))=\mathsf{true})$ in case of (lessp4), and $\lambda x.\ (\mathsf{lessp}(x, \mathsf{s}(x+y))=\mathsf{true})$ in case of (lessp5).

Moreover — to see the necessity of doing induction on several variables in parallel — we will do[35] the more complicated proof of the strengthened transitivity of the irreflexive ordering of the natural numbers, i.e. of

(lessp7) $\mathsf{lessp}(\mathsf{s}(x), z)=\mathsf{true} \ \Leftarrow \ \big(\ \mathsf{lessp}(x,y)=\mathsf{true} \ \wedge \ \mathsf{lessp}(y,z)=\mathsf{true}\ \big)$

We will also prove the commutativity lemma (+3)[36] and the simple lemma (ack4) about the Ackermann function:[37]

(+3) $x+y = y+x$,

(ack4) $\mathsf{lessp}(y, \mathsf{ack}(x,y)) = \mathsf{true}$

[33] Cf. [Wirth, 2004, § 1.1.2].

[34] Pieri [1908] stated these axioms informal and showed their equivalence to the version of the Peano axioms of Alessandro Padoa (1868–1937). For a discussion and an English translation see [Marchisotto and Smith, 2007]. Pieri [1908] has also a version where, instead of the symbol 0, there is only the statement that there is a natural number, and where (nat1) is replaced with the weaker statement that there is at most one s-minimal element:

$$\neg\exists y_0.\ (x_0=\mathsf{s}(y_0)) \wedge \neg\exists y_1.\ (x_1=\mathsf{s}(y_1)) \ \Rightarrow \ x_0=x_1.$$

That non-standard natural numbers cannot exist in Pieri's specification is easily shown as follows: For every natural number x we can form the set of all elements that can be reached from x by the inverse of the successor relation; by well-foundedness of s, this set contains the unique s-minimal element (0); thus, we have $x=\mathsf{s}^n(\mathsf{0})$ for some standard meta-level natural number n.

[35] We will prove (lessp7) twice: once in Example 3 in § 3.6, and again in Example 12 in § 5.2.6.

[36] We will prove (+3) twice: once in Example 2 in § 3.6, and again in Example 4 in § 3.7.1.

[37] We will prove (ack4) in Example 5 in § 3.8.

3.4 Standard Data Types

As we are interested in the verification of hard- and software, more important for us than natural numbers are standard data types, such as those well-known today from their occurrence in higher-level programming languages.

To clarify the inductive character of data types defined by constructors, and to show the additional complications arising from constructors with no or more than one argument, let us present the data types bool (of Boolean values) and $\mathsf{list(nat)}$ (of lists over natural numbers), which we need for our further examples as well.

A special case is the data type bool of the Boolean values given by the two constructors $\mathsf{true}, \mathsf{false} : \mathsf{bool}$ without any arguments, for which we get only the two following axioms by analogy to the axioms for the natural numbers. We globally declare the variable $b : \mathsf{bool}$, such that it always ranges over the Boolean values.

(bool1) $\quad b = \mathsf{true} \quad \vee \quad b = \mathsf{false}$

(bool2) $\quad \mathsf{true} \neq \mathsf{false}$

Note that the analogy of the axioms of Boolean values to the axioms of the natural numbers (cf. § 3.3) is not perfect: An axiom (bool3) analogous to (nat3) cannot exist because there are no constructors for bool that take arguments. Moreover, an axiom analogous to (S) is superfluous because it is implied by (bool1).

Furthermore, let us define the Boolean function $\mathsf{and} : \mathsf{bool}, \mathsf{bool} \rightarrow \mathsf{bool}$:

(and1) $\quad \mathsf{and}(\mathsf{false}, b) \quad = \mathsf{false}$

(and2) $\quad \mathsf{and}(b, \mathsf{false}) \quad = \mathsf{false}$

(and3) $\quad \mathsf{and}(\mathsf{true}, \mathsf{true}) = \mathsf{true}$

Let us now formalize the data type of the (finite) lists over natural numbers with the help of the following two constructors: the constant symbol

$$\mathsf{nil} : \mathsf{list(nat)}$$

for the empty list, and the function symbol

$$\mathsf{cons} : \mathsf{nat}, \ \mathsf{list(nat)} \rightarrow \mathsf{list(nat)},$$

which takes a natural number and a list of natural numbers, and returns the list where the number has been added to the input list as a new first element. We globally declare the variables $k, l : \mathsf{list(nat)}$.

In analogy to natural numbers, the axioms of this data type are the following:

(list(nat)1) $\quad l = \mathsf{nil} \ \vee \ \exists y, k.\ \big(\ l = \mathsf{cons}(y, k)\ \big)$

(list(nat)2) $\quad \mathsf{cons}(x, l) \neq \mathsf{nil}$

(list(nat)3_1) $\quad \mathsf{cons}(x, l) = \mathsf{cons}(y, k) \quad \Rightarrow \quad x = y$

(list(nat)3_2) $\quad \mathsf{cons}(x, l) = \mathsf{cons}(y, k) \quad \Rightarrow \quad l = k$

(list(nat)S) $\quad \forall P.\ \big(\forall l.\ P(l) \quad \Leftarrow \quad \big(P(\mathsf{nil}) \ \wedge \ \forall x, k.\ \big(P(\mathsf{cons}(x, k)) \Leftarrow P(k)\big)\big)\big)$

Moreover, let us define the recursive functions $\mathsf{length}, \mathsf{count} : \mathsf{list(nat)} \rightarrow \mathsf{nat}$, returning the length and the size of a list:

(length1) $\quad \mathsf{length}(\mathsf{nil}) \qquad = 0$

(length2) $\quad \mathsf{length}(\mathsf{cons}(x, l)) = \mathsf{s}(\mathsf{length}(l))$

(count(list(nat))1) $\quad \mathsf{count}(\mathsf{nil}) \qquad = 0$

(count(list(nat))2) $\quad \mathsf{count}(\mathsf{cons}(x, l)) = \mathsf{s}(x + \mathsf{count}(l))$

Note that the analogy of the axioms of lists to the axioms of the natural numbers is again not perfect:

1. There is an additional axiom $(\mathsf{list(nat)}3_1)$, which has no analog among the axioms of the natural numbers.
2. None of the axioms $(\mathsf{list(nat)}3_1)$ and $(\mathsf{list(nat)}3_2)$ is implied by the axiom $(\mathsf{list(nat)}1)$ together with the axiom
$$\mathsf{Wellf}(\lambda l, k.\ \exists x.\ (\mathsf{cons}(x, l) = k)),$$
which is the analog to Pieri's second axiom for the natural numbers.[38]
3. The latter axiom is weaker than each of the following axioms
$$\mathsf{Wellf}(\lambda l, k.\ (\mathsf{lessp}(\mathsf{length}(l), \mathsf{length}(k)) = \mathsf{true})),$$
$$\mathsf{Wellf}(\lambda l, k.\ (\mathsf{lessp}(\mathsf{count}(l), \mathsf{count}(k)) = \mathsf{true})).$$
which state the well-foundedness of bigger[39] relations. In spite of their relative strength, the well-foundedness of these relations is already implied by the well-foundedness that Pieri used for his specification of the natural numbers.

Therefore, the lists of natural numbers can be specified up to isomorphism by a specification of the natural numbers up to isomorphism (see § 3.3), plus the axioms $(\mathsf{list(nat)}3_1)$ and $(\mathsf{list(nat)}3_2)$, plus one of the following sets of axioms:

- $(\mathsf{list(nat)}2)$, $(\mathsf{list(nat)S})$ — in the style of Peano,
- $(\mathsf{list(nat)}1)$, $\mathsf{Wellf}(\lambda l, k.\ \exists x.\ (\mathsf{cons}(x, l) = k))$ — in the style of Pieri,[40]
- $(\mathsf{list(nat)}1)$, $(\mathsf{length}1\text{–}2)$ — refining the style of Pieri.[41]

Today it is standard to avoid higher-order axioms in the way exemplified in the last of these three items,[42] and to get along with one second-order axiom for the natural numbers, or even with the first-order instances of the axiom.

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[38] See § 3.3 for Pieri's specification of the natural numbers. The axioms $(\mathsf{list(nat)}3_1)$ and $(\mathsf{list(nat)}3_2)$ are not implied because all axioms beside $(\mathsf{list(nat)}3_1)$ or $(\mathsf{list(nat)}3_2)$ are satisfied in the structure where both natural numbers and lists are isomorphic to the standard model of the natural numbers, and where lists differ only in their sizes.

[39] Indeed, in case of $\mathsf{cons}(x, l) = k$, we have $\mathsf{lessp}(\mathsf{length}(l), \mathsf{length}(k)) =$ $= \mathsf{lessp}(\mathsf{length}(l), \mathsf{length}(\mathsf{cons}(x, l))) = \mathsf{lessp}(\mathsf{length}(l), \mathsf{s}(\mathsf{length}(l))) = \mathsf{true}$ because of $(\mathsf{lessp}4)$, and we also have $\mathsf{lessp}(\mathsf{count}(l), \mathsf{count}(k)) = \mathsf{lessp}(\mathsf{count}(l), \mathsf{count}(\mathsf{cons}(x, l))) =$ $\mathsf{lessp}(\mathsf{count}(l), \mathsf{s}(x + \mathsf{count}(l))) = \mathsf{true}$ because of $(+3)$ and $(\mathsf{lessp}5)$.

[40] This option is essentially the choice of the "shell principle" of [Boyer and Moore, 1979, p.37ff.]: The one but last axiom of Item (1) of the shell principle means $(\mathsf{list(nat)}2)$ in our formalization, and guarantees that Item (6) implies $\mathsf{Wellf}(\lambda l, k.\ \exists x.\ (\mathsf{cons}(x, l) = k))$.

[41] Although $(\mathsf{list(nat)}2)$ follows from $(\mathsf{length}1\text{–}2)$ and $(\mathsf{nat}2)$, it should be included to this standard specification because of its frequent applications.

[42] For this avoidance, however, we have to admit the additional function length. The same can be achieved with count instead of length, which is only possible, however, for lists over element types that have a mapping into the natural numbers.

Moreover, as some of the most natural functions on lists, let us define the destructors $\mathsf{car}: \mathsf{list(nat)} \to \mathsf{nat}$ and $\mathsf{cdr}: \mathsf{list(nat)} \to \mathsf{list(nat)}$, both in constructor and destructor style. Furthermore, let us define the recursive member predicate $\mathsf{mbp}: \mathsf{nat}, \mathsf{list(nat)} \to \mathsf{bool}$, and $\mathsf{delfirst}: \mathsf{list(nat)} \to \mathsf{list(nat)}$, a recursive function that deletes the first occurrence of a natural number in a list:

(car1) $\mathsf{car}(\mathsf{cons}(x,l)) = x$

(cdr1) $\mathsf{cdr}(\mathsf{cons}(x,l)) = l$

(car1′) $\mathsf{car}(l') = x \ \Leftarrow\ \mathsf{cons}(x,l) = l'$

(cdr1′) $\mathsf{cdr}(l') = l \ \Leftarrow\ \mathsf{cons}(x,l) = l'$

(mbp1) $\mathsf{mbp}(x,\mathsf{nil}) = \mathsf{false}$
(mbp2) $\mathsf{mbp}(x,\mathsf{cons}(y,l)) = \mathsf{true} \ \Leftarrow\ x = y$
(mbp3) $\mathsf{mbp}(x,\mathsf{cons}(y,l)) = \mathsf{mbp}(x,l) \ \Leftarrow\ x \neq y$

(delfirst1) $\mathsf{delfirst}(x,\mathsf{cons}(y,l)) = l \ \Leftarrow\ x = y$
(delfirst2) $\mathsf{delfirst}(x,\mathsf{cons}(y,l)) = \mathsf{cons}(y,\mathsf{delfirst}(x,l)) \ \Leftarrow\ x \neq y$

Immediate consequences of the axiom (list(nat)1) and the definitions (car1) and (cdr1) are the lemma (cons1) and its flattened version (cons1′):

(cons1) $\mathsf{cons}(\mathsf{car}(l'),\mathsf{cdr}(l')) = l' \ \Leftarrow\ l' \neq \mathsf{nil}$

(cons1′) $\mathsf{cons}(x,l) = l' \ \Leftarrow\ l' \neq \mathsf{nil} \ \wedge\ x = \mathsf{car}(l') \ \wedge\ l = \mathsf{cdr}(l')$

Furthermore, let us define the Boolean function $\mathsf{lexless}: \mathsf{list(nat)}, \mathsf{list(nat)} \to \mathsf{bool}$, which lexicographically compares lists according to the ordering of the natural numbers, and $\mathsf{lexlimless}: \mathsf{list(nat)}, \mathsf{list(nat)}, \mathsf{nat} \to \mathsf{bool}$, its restriction to list of a length up to a given natural number:

(lexless1) $\mathsf{lexless}(l,\mathsf{nil}) = \mathsf{false}$
(lexless2) $\mathsf{lexless}(\mathsf{nil},\mathsf{cons}(y,k)) = \mathsf{true}$
(lexless3) $\mathsf{lexless}(\mathsf{cons}(x,l),\mathsf{cons}(y,k)) = \mathsf{lexless}(l,k) \ \Leftarrow\ x = y$
(lexless4) $\mathsf{lexless}(\mathsf{cons}(x,l),\mathsf{cons}(y,k)) = \mathsf{lessp}(x,y) \ \Leftarrow\ x \neq y$

(lexlimless1) $\mathsf{lexlimless}(l,k,x) = \mathsf{and}(\mathsf{lexless}(l,k),\mathsf{lessp}(\mathsf{length}(l),x))$

Such lexicographic combinations play an important rôle in well-foundedness arguments of induction proofs, because they combine given well-founded orderings into new well-founded orderings, provided there is an upper bound for the length of the list:[43]

(lexlimless2) $\mathsf{Wellf}(\lambda l,k.\ (\mathsf{lexlimless}(l,k,x) = \mathsf{true}))$

Finally note that analogous axioms can be used to specify any other data type generated by constructors, such as pairs of natural numbers or binary trees over such pairs.

[43] The length limit is required because otherwise we have the following counterexample to termination: $(\mathsf{s}(0))$, $(0,\mathsf{s}(0))$, $(0,0,\mathsf{s}(0))$, $(0,0,0,\mathsf{s}(0))$, ... Note that the need to compare lists of different lengths typically arises in mutual induction proofs where the two induction hypotheses have a different number of free variables at measured positions. See [Wirth, 2004, § 3.2.2] for a nice example.

3.5 The Standard High-Level Method of Mathematical Induction

A mathematical proof method cannot be completely captured by its non-procedural logic formalization; and so we need effective heuristics for actually finding proofs by induction.

In everyday mathematical practice of an advanced theoretical journal, the common inductive arguments are hardly ever carried out explicitly. Instead, the proof reads something like "by structural induction on n, q.e.d." or "by (Noetherian) induction on (x, y) over $<$, q.e.d.", expecting that the mathematically educated reader could easily expand the proof if in doubt. In contrast, difficult inductive arguments, sometimes covering several pages,[44] require considerable ingenuity and have to be carried out.

In case of a proof on natural numbers, the experienced mathematician engineers his proof roughly according to the following pattern:

> He starts with the conjecture and simplifies it by case analysis, typically based on the axiom (nat1). When he realizes that the current goal becomes similar to an instance of the conjecture, he applies the instantiated conjecture just like a lemma, but keeps in mind that he has actually applied an induction hypothesis. Finally, using the free variables of the conjecture, he constructs some ordering whose well-foundedness follows from the axiom $\mathsf{Wellf}(\lambda x, y : \mathsf{nat}.\ (\mathsf{s}(x) = y))$ and in which all instances of the conjecture applied as induction hypotheses are smaller than the original conjecture.

The hard tasks of a proof by mathematical induction are thus:

(Induction-Hypotheses Task)
to find the numerous induction hypotheses,[45] and

(Induction-Ordering Task)
to construct an *induction ordering* for the proof, i.e. a well-founded ordering that satisfies the ordering constraints of all these induction hypotheses in parallel.[46]

The above induction method can be formalized as an application of the Theorem of Noetherian Induction. For non-trivial proofs, mathematicians indeed prefer the the axioms of Pieri's specification in combination with the Theorem of Noetherian Induction (N) to Peano's alternative with the Axiom of Structural Induction (S), because the instances for P and $<$ in (N) are often still easy to find when the instances for P in (S) are not.

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[44] For example, such difficult inductive arguments are the proofs of Hilbert's *first ε-theorem* [Hilbert and Bernays, 1970], Gentzen's *Hauptsatz* [Gentzen, 1935], and confluence theorems such as the ones in [Gramlich and Wirth, 1996], [Wirth, 2009].

[45] As, e.g., in the proof of Gentzen's Hauptsatz on Cut-elimination.

[46] For instance, this was the hard part in the elimination of the ε-formulas in the proof of the 1st ε-theorem in [Hilbert and Bernays, 1970], and in the proof of the consistency of arithmetic by the ε-substitution method in [Ackermann, 1940].

3.6 Descente Infinie

The soundness of the induction method of § 3.5 is most easily seen when the argument is structured as a proof by contradiction, assuming a counterexample. For Fermat's historic reinvention of the method, it is thus just natural that he developed the method in terms of assumed counterexamples.[47] Here is Fermat's Method of *Descente Infinie* in modern language, very roughly speaking:

> A proposition $P(w)$ can be proved by *descente infinie* as follows: *Show that for each assumed counterexample v of P there is a smaller counterexample u of P w.r.t. a well-founded relation $<$, which does not depend on the counterexamples.*

If this method is executed successfully, we have proved $\forall w.\ P(w)$ because no counterexample can be $<$-minimal and so the well-foundedness of $<$ implies that there are no counterexamples at all.

Nowadays every logician immediately realizes that a formalization of the method of *descente infinie* is obtained from the Theorem of Noetherian Induction (N) (cf. § 3.2) simply by replacing

$$P(v) \ \Leftarrow\ \forall u{<}v.\ P(u)$$

with its contrapositive

$$\neg P(v) \ \Rightarrow\ \exists u{<}v.\ \neg P(u).$$

Although it was still very hard for Fermat to obtain a positive version of his counterexample method,[48] the negation is irrelevant in our context here, which is the one of the 19$^{\text{th}}$ and 20$^{\text{th}}$ centuries and which is based on classical logic. What matters for us is the heuristic task of finding proofs. Therefore, we take *descente infinie* in this article[49] as a synonym for the modern standard high-level method of mathematical induction described in this section.

Let us now prove the lemmas (+3) and (lessp7) of § 3.3 (in the axiomatic context of § 3.3) by *descente infinie*, seen as the standard high-level method of mathematical induction described in § 3.5.

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[47] Cf. [Fermat, 1891ff.], [Mahoney, 1994], [Bussotti, 2006], [Wirth, 2010b].

[48] Fermat reported in his letter for Huygens that he had had problems to apply the Method of *Descente Infinie* to positive mathematical statements. See [Wirth, 2010b, p. 11] and the references there, in particular [Fermat, 1891ff., Vol. II, p. 432].

Moreover, a natural-language presentation via *descente infinie* (such as Fermat's representation in Latin) is often simpler than a presentation via the Theorem of Noetherian Induction, because it is easier to speak of one counterexample v and to find one smaller counterexample u, than to administrate the dependences of universally quantified variables.

[49] In general, in the tradition of [Wirth, 2004], *descente infinie* is nowadays taken as a synonym for the standard high-level method of mathematical induction as described in § 3.5. This way of using the term *"descente infinie"* is found in [Brotherston and Simpson, 2007; 2011], [Voicu and Li, 2009], [Wirth, 2005; 2010a; 2012b; 2013].

If, however, the historical perspective before the 19$^{\text{th}}$ century is taken, then this identification is not appropriate because a more fine-grained differentiation is required, such as found in [Bussotti, 2006], [Wirth, 2010b].

EXAMPLE 2 (Proof of (+3) by *descente infinie*).
By application of the Theorem of Noetherian Induction (N) (cf. § 3.2) with P set to $\lambda x, y.\ (x+y=y+x)$, and the variables v, u renamed to (x,y), (x'',y''), respectively, the conjectured lemma (+3) reduces to

$$\exists <. \left(\begin{array}{ll} & \forall(x,y).\ \big((x+y=y+x) \ \Leftarrow\ \forall(x'',y'')<(x,y).\ (x''+y''=y''+x'')\big) \\ \wedge & \mathsf{Wellf}(<) \end{array} \right).$$

Let us focus on the sub-formula $x+y = y+x$. Based on axiom $(\mathsf{nat1})$ we can reduce this task to the two cases $x=\mathsf{0}$ and $x=\mathsf{s}(x')$ with the two goals

$$\mathsf{0}+y = y+\mathsf{0}; \qquad \mathsf{s}(x')+y = y+\mathsf{s}(x');$$

respectively. They simplify by (+1) and (+2) to

$$y = y+\mathsf{0}; \qquad \mathsf{s}(x'+y) = y+\mathsf{s}(x');$$

respectively. Based on axiom $(\mathsf{nat1})$ we can reduce each of these goals to the two cases $y=\mathsf{0}$ and $y=\mathsf{s}(y')$, with leaves us with the four open goals

$$\mathsf{0} = \mathsf{0}+\mathsf{0}; \qquad \mathsf{s}(x'+\mathsf{0}) = \mathsf{0}+\mathsf{s}(x');$$
$$\mathsf{s}(y') = \mathsf{s}(y')+\mathsf{0}; \qquad \mathsf{s}(x'+\mathsf{s}(y')) = \mathsf{s}(y')+\mathsf{s}(x').$$

They simplify by (+1) and (+2) to

$$\mathsf{0} = \mathsf{0}; \qquad \mathsf{s}(x'+\mathsf{0}) = \mathsf{s}(x');$$
$$\mathsf{s}(y') = \mathsf{s}(y'+\mathsf{0}); \qquad \mathsf{s}(x'+\mathsf{s}(y')) = \mathsf{s}(y'+\mathsf{s}(x'));$$

respectively. Now we can make immediate progress only if we instantiate the induction hypothesis that is available in the context[50] given by our above formula in four different forms, namely we have to instantiate (x'',y'') with $(x',\mathsf{0})$, $(\mathsf{0},y')$, $(x',\mathsf{s}(y'))$, $(\mathsf{s}(x'),y')$, respectively. Rewriting with these instances, the four goals become:

$$\mathsf{0} = \mathsf{0}; \qquad \mathsf{s}(\mathsf{0}+x') = \mathsf{s}(x');$$
$$\mathsf{s}(y') = \mathsf{s}(\mathsf{0}+y'); \qquad \mathsf{s}(\mathsf{s}(y')+x') = \mathsf{s}(\mathsf{s}(x')+y');$$

which simplify by (+1) and (+2) to

$$\mathsf{0} = \mathsf{0}; \qquad \mathsf{s}(x') = \mathsf{s}(x');$$
$$\mathsf{s}(y') = \mathsf{s}(y'); \qquad \mathsf{s}(\mathsf{s}(y'+x')) = \mathsf{s}(\mathsf{s}(x'+y')).$$

Now the first three goals following directly from the reflexivity of equality, whereas the last goal needs also an application of our induction hypothesis: This time we have to instantiate (x'',y'') with (x',y').

Finally, we instantiate our induction ordering $<$ to the lexicographic combination of length less than 3 of the ordering of the natural numbers. If we read our pairs as two element lists, i.e. (x'',y'') as $\mathsf{cons}(x'',\mathsf{cons}(y'',\mathsf{nil}))$, then we can set $<$ to $\lambda l,k.\ (\mathsf{lexlimless}(l,k,\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) = \mathsf{true})$, which is well-founded according to $(\mathsf{lexlimless2})$ (cf. § 3.4). Then it is trivial to show that $(\mathsf{s}(x'),\mathsf{s}(y'))$ is greater than each of $(x',\mathsf{0})$ $(\mathsf{0},y')$, $(x',\mathsf{s}(y'))$, $(\mathsf{s}(x'),y')$, (x',y').

This completes the proof of our conjecture. □

[50] On how this availability can be understood formally, see [Autexier, 2005].

EXAMPLE 3 (Proof of ($\mathsf{lessp7}$) by *descente infinie*).
In the previous proof in Example 2 we wanted to make the application of the Theorem of Noetherian Induction most explicit, and so its presentation was rather formal w.r.t. the underlying logic, beside the intuitive technique of focusing.

Contrary to this, let us now proceed more in the vernacular of a working mathematician. Moreover, instead of $p = \mathsf{true}$, let us just write p.

To prove the strengthened transitivity of lessp as expressed in lemma ($\mathsf{lessp7}$) in the axiomatic context of § 3.3, we then have to show

$$\mathsf{lessp}(\mathsf{s}(x), z) \;\Leftarrow\; \big(\; \mathsf{lessp}(x, y) \;\wedge\; \mathsf{lessp}(y, z) \;\big).$$

If we apply the axiom ($\mathsf{nat1}$) twice to both y and z with the intention to reduce the last literal, then, after reduction with ($\mathsf{lessp1}$), the two base cases have an atom false in their conditions, abbreviating $\mathsf{false} = \mathsf{true}$, which is false according to ($\mathsf{bool2}$), and so the base cases are true (*ex falso quodlibet*). The remaining case, where we have both $y = \mathsf{s}(y')$ and $z = \mathsf{s}(z')$, reduces with ($\mathsf{lessp3}$) to

$$\mathsf{lessp}(x, z') \;\Leftarrow\; \big(\; \mathsf{lessp}(x, \mathsf{s}(y')) \;\wedge\; \mathsf{lessp}(y', z') \;\big)$$

If we apply the induction hypothesis instantiated via $\{y {\mapsto} y',\; z {\mapsto} z'\}$ to match the last atom in the condition, then we obtain the two goals

$$\mathsf{lessp}(x, z') \;\Leftarrow\; \big(\; \mathsf{lessp}(\mathsf{s}(x), z') \;\wedge\; \mathsf{lessp}(x, \mathsf{s}(y')) \;\wedge\; \mathsf{lessp}(y', z') \;\big)$$

$$\mathsf{lessp}(x, y') \;\vee\; \mathsf{lessp}(\mathsf{s}(x), z') \;\vee\; \mathsf{lessp}(x, z') \;\Leftarrow\; \big(\; \mathsf{lessp}(x, \mathsf{s}(y')) \;\wedge\; \mathsf{lessp}(y', z') \;\big)$$

By elimination of irrelevant literals, the first goal can be reduced to the valid conjecture $\mathsf{lessp}(x, z') \Leftarrow \mathsf{lessp}(\mathsf{s}(x), z')$, but we cannot obtain a lemma simpler than our initial conjecture ($\mathsf{lessp7}$) by generalization and elimination of irrelevant literals from the second goal. This means that the application of the given instantiation of the induction hypothesis is useless.

Thus, instead of induction-hypothesis application, we had better apply the axiom ($\mathsf{nat1}$) also to x, obtaining the cases $x = \mathsf{0}$ and $x = \mathsf{s}(x')$ with the two goals — after reduction with ($\mathsf{lessp2}$) and ($\mathsf{lessp3}$) —

$$\mathsf{lessp}(\mathsf{0}, z') \;\Leftarrow\; \mathsf{lessp}(y', z')$$

$$\mathsf{lessp}(\mathsf{s}(x'), z') \;\Leftarrow\; \big(\; \mathsf{lessp}(x', y') \;\wedge\; \mathsf{lessp}(y', z') \;\big),$$

respectively. The first is trivial by ($\mathsf{lessp1}$), ($\mathsf{lessp2}$) after another application of the axiom ($\mathsf{nat1}$) to z'. The second is just an instance of the induction hypothesis via $\{x {\mapsto} x',\; y {\mapsto} y',\; z {\mapsto} z'\}$. As the induction ordering we can take any of the variables of the original conjecture w.r.t. the irreflexive ordering on the natural numbers or the successor relation.

This completes the proof of the conjecture.

Note that we also have shown that the given proof can only be successful with the given induction hypotheses. It is actually possible to show that this simple example — *ceteris paribus* — requires an induction hypothesis resulting from an instance $\{x {\mapsto} x'',\; y {\mapsto} y'',\; z {\mapsto} z''\}$ where, for some standard meta-level natural number n, we have $x = \mathsf{s}^{n+1}(x'') \;\wedge\; y = \mathsf{s}^{n+1}(y'') \;\wedge\; z = \mathsf{s}^{n+1}(z'')$. □

3.7 Explicit Induction

3.7.1 From the Theorem of Noetherian Induction to Explicit Induction

To admit the realization of the standard high-level method of mathematical induction as described in § 3.5, a proof calculus should have an explicit concept of an induction hypothesis. Moreover, it has to cope in some form with the second-order variables P and $<$ in the Theorem of Noetherian Induction (N) (cf. § 3.2), and also with the second-order variable Q in the definition of well-foundedness ($\mathsf{Wellf}(<)$) (cf. § 3.1).

Such an implementation needs special care regarding the calculus level and its heuristics. For example, the best theorem provers for higher-order logic today are still not able to prove standard inductive theorems by just adding the Theorem of Noetherian Induction, which immediately effects an explosion of the search space. It is indeed a main obstacle to practical usefulness of higher-order automated theorem provers today that they are poor in mathematical induction.

Therefore, it is probable that — on the basis of the logic calculi and the computer technology of the 1970s – also Boyer and Moore would have failed to implement induction via these human-oriented and higher-order features, and that they were lucky to find a way to confine the concept of an induction hypothesis to the internals of single reductive inference steps — namely the applications of the so-called *induction rule* — and to restrict all other inference steps to quantifier-free first-order deductive reasoning.

Described in terms of the Theorem of Noetherian Induction, this *induction rule* immediately instantiates the higher-order variables P and $<$ with first-order predicates. This is rather straightforward for the predicate variable P, which simply becomes the (properly simplified and generalized) quantifier-free first-order conjecture that is to be proved by induction, and the tuple of the free first-order variables of this conjecture takes the place of the single argument of P.

The instantiation of the higher-order variable $<$ is more difficult: Instead of a simple instantiation, the whole context of its two occurrences is transformed. For the first occurrence, namely the one in the sub-formula $\forall u{<}v.\ P(u)$, the whole sub-formula is replaced with a conjunction of instances of $P(u)$, for which u is know to be smaller than v in some lexicographic combination of given orderings that are already know to be well-founded. As a consequence, the second occurrence of $<$, i.e. the one in $\mathsf{Wellf}(<)$, simplifies to true, and so we can completely drop the conjunction that contains it.

At a first glance, it seems highly unlikely that there could be any framework of proof-search heuristics in which such an induction rule could succeed in implementing all applications of the Theorem of Noetherian Induction, simply because this rule has to solve the two hard tasks of an induction proof, namely the Induction-Hypotheses Task and the Induction-Ordering Task (cf. § 3.5), right at the beginning of the proof attempt, before the proof has been sufficiently developed to exhibit its structural difficulties.

Most surprisingly, but as a matter of fact, the induction rule has proved to be most successful in realizing all applications of the Theorem of Noetherian Induction required within the proof-search heuristics of the Boyer–Moore waterfall (cf. Figure 1). This has only partly to do with the poor quantifier-free first-order logic, which limits the expressibility of induction hypotheses to a certain degree and often requires the definition of new recursive functions, which then again provide some guidance for the induction rule. Crucial is the possibility of having additional lemmas that can then be proved again by induction. This is best shown by an example.

EXAMPLE 4 (Proof of (+3) by explicit induction).
Let us prove (+3) in the context of § 3.3, just as we have done already in Example 2 (cf. § 3.6), but now with the induction rule as the only way to apply the Theorem of Noetherian Induction.

As the conjecture is already properly simplified and concise, we instantiate $P(w)$ in the Theorem of Noetherian Induction again to the whole conjecture and reduce this conjecture by application of the Theorem of Noetherian Induction to

$$\exists{<}.\left(\begin{array}{ll} & \forall(x,y).\ \big((x+y=y+x)\ \Leftarrow\ \forall(x'',y'')<(x,y).\ (x''+y''=y''+x'')\big) \\ \wedge & \mathsf{Wellf}(<) \end{array}\right).$$

Based, roughly speaking, on a termination analysis for the function $+$, the heuristic of the induction rules of the Boyer–Moore theorem provers suggest to instantiate $<$ to $\lambda x'', y'', x, y.\ (\mathsf{s}(x'')=x)$. As this relation is known to be well-founded, the induction rule reduces the task based on axiom (nat1) to two goals, namely the base case

$$\mathsf{0}+y=y+\mathsf{0};$$

and the step case

$$(\mathsf{s}(x')+y=y+\mathsf{s}(x'))\ \Leftarrow\ (x'+y=y+x').$$

This completes the application of the induction rule, and these two goals must now be shown without the possibility to apply further instances of the induction hypotheses.

The induction rules of the Boyer–Moore theorem provers are not able to find the many instances we applied in the proof of Example 2. This is different for a theoretically more powerful, but heuristically less successful induction rule suggested by Christoph Walther (*1950), which actually admits to execute the proof of Example 2.[51] In general, however, for harder conjectures, a simulation of *descente infinie* by the induction rule of explicit induction would require an arbitrary look-ahead into the proofs, depending on the size of the structure of these proofs; thus, because the induction rule is understood to have a limited look-ahead into the proofs, such a simulation would not fall under the paradigm of explicit induction anymore. Indeed, the look-ahead of induction rules into the proofs is typically not more than a single unfolding of a single occurrence of a recursive function symbol, for each such occurrence in the conjecture.

Note that the two above goals of the base and the step case are exactly the ones

[51] See [Walther, 1993, p. 99f.]. On Page 100, the most interesting step case computed by Walther's induction is (rewritten to constructor-style):

$$\mathsf{s}(x)+\mathsf{s}(y)=\mathsf{s}(y)+\mathsf{s}(x)\ \Leftarrow\ \big(\ x+\mathsf{s}(y)=\mathsf{s}(y)+x\ \wedge\ \forall z.\ (z+y=y+z)\ \big).$$

that would result from the reduction of the input conjecture by an application of the Axiom of Structural Induction over $\mathsf{0}$ and s (cf. axiom (S) of § 3.3). Nevertheless, the induction rule is in general able to produce much more complicated base and step cases than a simple structural induction on x over $\mathsf{0}$ and s.

Now the first goal is simplified again to $y = y + \mathsf{0}$, and then another application of the induction rule results in two goals that can be proved without further induction.

The second goal is simplified to

$$(\mathsf{s}(x' + y) = y + \mathsf{s}(x')) \;\Leftarrow\; (x' + y = y + x').$$

Now we use the condition from *left* to right for rewriting only the *left*-hand side of the conclusion and then we throw away the condition completely, with the intention to obtain a stronger induction hypothesis. This is the famous *"cross-fertilization"* of the Boyer–Moore waterfall (cf. Figure 1). By this, the simplified second goal reduces to

$$\mathsf{s}(y + x') = y + \mathsf{s}(x').$$

Now the induction rule triggers a structural induction on y, which is successful without further induction.

All in all, although the induction rules of the Boyer–Moore theorem provers do not find the more complicated induction hypotheses of the *descente infinie* proof of Example 2, they are well able prove our original conjecture with the help of the additional lemmas $y = y + \mathsf{0}$ and $\mathsf{s}(y + x') = y + \mathsf{s}(x')$. From a logical viewpoint, these lemmas are redundant because they follow from the original conjecture and the definition of $+$. From a heuristic viewpoint, however, they are more useful than the original conjecture, because — oriented for rewriting from right to left — their application tends to terminate in the context of the overall simplification by symbolic evaluation, which constitutes the first stage in the Boyer–Moore waterfall (cf. Figure 1). □

Although the two proofs of the very simple conjecture $(+3)$ given in Examples 2 and 4 can only give a very rough idea on the advantage of *descente infinie* for hard induction proofs,[52] these two proofs nicely demonstrate how the induction rule of explicit induction manages to prove simple theorems very efficiently and with additional benefits for the further performance of the simplification procedure.

Moreover, if the overall waterfall heuristic fails, the user can help any Boyer–

[52] For some of the advantages of *descente infinie*, see our Example 12 in § 5.2.6. For a more difficult higher-order proof by descente infinie see § 3.4 of [Wirth, 2004], where a complete formal proof of M. H. A. Newman's famous lemma is given, i.e. the reverse of a well-founded relation is shown to be confluent in case of local confluence by *induction w.r.t. this well-founded relation itself*, a situation where explicit induction cannot even be applied.

Note that, though confluence is the Church–Rosser property, the Newman Lemma has nothing to do with the Church–Rosser Theorem stating the confluence of the rewrite relation of $\alpha\beta$-reduction in untyped λ-calculus, which has actually been verified with a Boyer–Moore theorem prover in the first half of the 1980s by Shankar [1988], following the short Tait/Martin-Löf proof found e.g. in [Barendregt, 2012, p. 59ff.]. Unlike the Newman Lemma, Shankar's proof proceeds by structural induction on the λ-terms and not by Noetherian induction w.r.t. the reverse of the rewrite relation; indeed, untyped λ-calculus in non-terminating.

Moore theorem prover except the PURE LISP THEOREM PROVER by stating hints and additional lemmas with additional notions, which will finally help the induction rule to prove also very hard theorems.

3.7.2 Theoretical Viewpoint

From a theoretical viewpoint, we should be aware of the possibility that the intended models of specifications in explicit-induction systems, say for the natural numbers, also include non-standard models, where — contrary to the higher-order specifications of Peano and Pieri — there may be **Z**-chains in addition to the natural numbers **N**.[53] These **Z**-chains cannot be excluded because all applications of the Theorem of Noetherian Induction are confined to the induction rule, which does not use any higher-order properties, but only well-founded orderings that are defined in the first-order logic of the explicit-induction system,[54] and because the remaining inference rules realize only first-order deductive reasoning.

3.7.3 Practical Viewpoint

From a practical viewpoint, we have to be aware that application of the induction rule of explicit induction is not implemented via a reference to the Theorem of Noetherian Induction, but directly handles the following practical tasks and their heuristic decisions.

In general, the *induction stage* of the Boyer–Moore waterfall (cf. Figure 1) applies the induction rule once to its input formula, which results in a conjunction — or conjunctive set — of base and step cases to which to the input conjecture reduces, i.e. whose validity implies the validity of the input conjecture.

Therefore, from the viewpoint of a working mathematician, the induction rule of explicit induction has to solve the following tasks:

1. Choose some of the variables in the conjecture as *induction variables*, and split the conjecture into several base and step cases, based on the induction variables' demand on which governing conditions and constructor substitutions[55] have to be added to be able to unfold some of the recursive function calls that contain the induction variables as direct arguments without further case analysis.
2. Eagerly generate the induction hypotheses for the step cases.

As we will see,[56] the actual realization of these tasks in the induction rule is quite different because it focuses on complete step cases including eagerly generated

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[53] Contrary to the **Z**-chains, which are structures similar to the integers **Z**, where every element is greater than every standard natural number, s-circles cannot exist because it is possible to show by structural induction on x the two lemmas $\mathsf{lessp}(x,x)=\mathsf{false}$ and $\mathsf{lessp}(x,\mathsf{s}^{n+1}(x))=\mathsf{true}$ for each standard meta-level natural number n.

[54] See also Note 120.

[55] This adding of constructor substitutions refers to the application of axioms like (nat1) (cf. § 3.3), and is required whenever constructor style either is found in the recursive function definitions or is to be used for the step cases. In the PURE LISP THEOREM PROVER, only the latter is the case. In THM, none of this is the case.

induction hypotheses, and generates the base case to complement the step cases only in the very end; moreover, "induction variables" only play a very minor rôle at the end of the procedure (in the deletion of faulty induction schemes).

3.8 Generalization

Contrary to merely deductive, analytic theorem proving, an input conjecture for a proof by induction is not only a task (as induction conclusion) but also a tool (as induction hypothesis) in the proof attempt. Therefore, a stronger conjecture is often easier to prove because it supplies us with a stronger induction hypothesis during the proof attempt.

Such a step from a weaker to a stronger input conjecture is called *generalization*.

Generalization is to be handled with great care because it is an *unsafe* reduction step in the sense that it may reduce a valid conjecture to an invalid one; such a reduction is called *over-generalization*.

Generalization is hardly needed when input conjectures are supplied by humans. As we have seen in Example 4 of § 3.7.1, however, explicit induction often has to start another induction during the proof, and then the secondary, machine-generated input conjecture often requires generalization for its proof attempt to be successful.

Syntactically, there are two kinds of generalization, namely the replacement of terms with universal variables and the removal of irrelevant side conditions.

In the vernacular of Boyer–Moore theorem provers, the first is simply called "generalization" and the second is called "elimination of irrelevance". They are dealt with in two consecutive stages of these names in the Boyer–Moore waterfall, which come right before the induction stage.

We will use the technical term "generalization" in this article from now on only in the narrower sense that is standard in explicit induction.

The removal of irrelevant side conditions is intuitively clear. For formulas in clausal form, it simply means to remove irrelevant literals. More interesting are the heuristics of its realization, which we discuss in § 5.3.5.

The less clear process of generalization typically proceeds by the replacement of all occurrences of a non-variable term with a fresh variable.

This is especially promising for a subsequent induction if the same non-variable term t has multiple occurrences in the conjecture, and becomes even more promising if these occurrences are found on both sides of the same positive equation or in literals of different polarity, say in a conclusion and a condition of an implication.

To avoid *over-generalization*, sub-terms are to be preferred to their super-terms, and one should never generalize if t is of any of the following forms: a constructor term, a top level term, a term with a logical operator (such as implication or equality) as top symbol, a direct argument of a logical operator, or the first argument of a conditional (`IF`). In any of these cases, the information loss by generalization

[56] See, e.g., Example 10 of § 4.5.

is typically so high that it probably results in an invalid conjecture.

How powerful generalization can be is best seen by the multitude of its successful applications, which often surprise the human users. Here is one of these:

EXAMPLE 5 (Proof of (ack4) by Explicit Induction and Generalization).
Let us prove (ack4) in the context of § 3.3 by explicit induction. It is obvious that such a proof has to follow the definition of ack in the three cases (ack1), (ack2), (ack3), using the termination ordering of ack, which is just the lexicographic combination of its arguments. So the induction rule of all Boyer–Moore theorem provers except the Pure LISP Theorem Prover reduces the input formula (ack4) to the following goals:[57]

$$\mathsf{lessp}(y, \mathsf{ack}(0, y)) = \mathsf{true};$$
$$\mathsf{lessp}(0, \mathsf{ack}(\mathsf{s}(x'), 0)) = \mathsf{true} \;\Leftarrow\; \mathsf{lessp}(\mathsf{s}(0), \mathsf{ack}(x', \mathsf{s}(0))) = \mathsf{true};$$
$$\mathsf{lessp}(\mathsf{s}(y'), \mathsf{ack}(\mathsf{s}(x'), \mathsf{s}(y'))) = \mathsf{true} \;\Leftarrow\; \left(\begin{array}{ll} & \mathsf{lessp}(y', \mathsf{ack}(\mathsf{s}(x'), y')) = \mathsf{true} \\ \wedge & \mathsf{lessp}(\mathsf{ack}(\mathsf{s}(x'), y'), \mathsf{ack}(x', \mathsf{ack}(\mathsf{s}(x'), y'))) = \mathsf{true} \end{array}\right).$$

After simplifying with (ack1), (ack2), (ack3), respectively, we obtain:

$$\mathsf{lessp}(y, \mathsf{s}(y)) = \mathsf{true};$$
$$\mathsf{lessp}(0, \mathsf{ack}(x', \mathsf{s}(0))) = \mathsf{true} \;\Leftarrow\; \mathsf{lessp}(\mathsf{s}(0), \mathsf{ack}(x', \mathsf{s}(0))) = \mathsf{true};$$
$$\mathsf{lessp}(\mathsf{s}(y'), \mathsf{ack}(x', \mathsf{ack}(\mathsf{s}(x'), y'))) = \mathsf{true} \;\Leftarrow\; \left(\begin{array}{ll} & \mathsf{lessp}(y', \mathsf{ack}(\mathsf{s}(x'), y')) = \mathsf{true} \\ \wedge & \mathsf{lessp}(\mathsf{ack}(\mathsf{s}(x'), y'), \mathsf{ack}(x', \mathsf{ack}(\mathsf{s}(x'), y'))) = \mathsf{true} \end{array}\right).$$

Now the base case is simply an instance of our lemma (lessp4). Let us simplify the two step cases by introducing variables for their common subterms:

$$\mathsf{lessp}(0, z) = \mathsf{true} \;\Leftarrow\; \left(\; \mathsf{lessp}(\mathsf{s}(0), z) = \mathsf{true} \;\wedge\; z = \mathsf{ack}(x', \mathsf{s}(0)) \;\right);$$
$$\mathsf{lessp}(\mathsf{s}(y'), z_2) = \mathsf{true} \;\Leftarrow\; \left(\begin{array}{ll} & \mathsf{lessp}(y', z_1) = \mathsf{true} \;\wedge\; \mathsf{lessp}(z_1, z_2) = \mathsf{true} \\ \wedge & z_1 = \mathsf{ack}(\mathsf{s}(x'), y') \;\wedge\; z_2 = \mathsf{ack}(x', z_1) \end{array}\right).$$

Now the first follows from applying (nat1) to z. Before we can prove the second by another induction, however, we have to generalize it to the lemma (lessp7) of § 3.3 by deleting the last two literals from the condition. □

In combination with explicit induction, generalization becomes especially powerful in the invention of new lemmas of general interest, because the step cases of explicit induction tend to have common occurrences of the same term in their conclusion and their condition. Indeed, the lemma (lessp7), which we have just discovered in Example 5, is one of the most useful lemmas in the theory of natural numbers.

It should be noted that all Boyer–Moore theorem provers except the Pure LISP Theorem Prover are able to do this whole proof completely automatically and invent the lemma (lessp7) by generalization of the the second step case; and they do this even when they work with an arithmetic theory that was redefined, so that no decision procedures or other special knowledge on the natural numbers can be used by the system. Moreover, as shown in § 3.3 of [Wirth, 2004], in a slightly richer logic, these heuristics would additionally admit to synthesize the lower bound in

[57] See Example 10 of § 4.5 on how these step cases are actually found in explicit induction.

the first argument of lessp from the input conjecture $\exists z.\ (\mathsf{lessp}(z, \mathsf{ack}(x, y)) = \mathsf{true})$, simply because lessp does not contribute to the choice of the base and step cases.

3.9 Proof-Theoretical Peculiarities of Mathematical Induction

The following two proof-theoretical peculiarities of induction compared to first-order deduction may be considered noteworthy:[58]

- A calculus for arithmetic cannot be complete, simply because the theory of the arithmetic of natural numbers is not enumerable.[59]
- According to Gentzen's Hauptsatz,[60] a proof of a first-order theorem can always be restricted to the "sub"-formulas of this theorem. In contrast to lemma application in a deductive proof tree, however, the application of induction hypotheses and lemmas inside an inductive reasoning cycle cannot generally be eliminated in the sense that the "sub"-formula property could be obtained.[61] As a consequence, in first-order inductive theorem proving, "creativity" cannot be restricted to finding just the proper instances, but may require the invention of new lemmas and notions.[62]

3.10 Conclusion

In this section, after briefly presenting the induction method in its rich historical context, we have offered a formalization and a first practical description. Moreover, we have explained why we can take Fermat's term *"descente infinie"* in our modern context as a synonym for the standard high-level method of mathematical induction. Finally, we have introduced to explicit induction and generalization.

Noetherian induction requires domains for its well-founded orderings; and these domains are typically built-up by constructors. Therefore, the discussion of the method of induction required the introduction of some paradigmatic data types, such as natural numbers and lists.

To express the relevant notions in these data types, we need *recursion*, a method of definition, which we have often used in this section intuitively. We did not discuss its formal admissibility requirements, however, which we will do in § 4, with a focus on modes of recursion that admit an effective consistency test, including termination aspects such as induction templates and schemes.

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[58] Note, however, that these peculiarities of induction do not make a difference to first-order deductive theorem proving *in practice*. See Notes 59 and 62.

[59] This theoretical result is Gödel's first incompleteness theorem [1931]. In practice, however, it does not matter whether our proof attempt fails because our theorem will not be enumerated ever, or will not be enumerated before doomsday.

[60] Cf. [Gentzen, 1935].

[61] Cf. [Kreisel, 1965].

[62] In practice, however, we have to extend our proof search to additional lemmas and notions anyway, and it does not really matter whether we have to do this for principled reasons (as in induction) or for tractability (as required in first-order deductive theorem proving, cf. [Baaz and Leitsch, 1995]).

4 RECURSION AND TERMINATION

Recursion is a form of programming or definition where a newly defined notion may even occur in its *definientia*. Contrary to *explicit* definitions, where we can always get rid of the new notions by reduction (i.e. by rewriting the *definienda* (*left-hand sides* of the defining equations) to the *definientia* (*right-hand sides*)), reduction with *recursive* definitions may run forever.

We have already seen some recursive function definitions in §§ 3.3 and 3.4, such as the ones of $+$, lessp, length, and count, where these function symbols occurred in some of the right-hand sides of the equations of their own definitions; for instance, the function symbol $+$ occurs in the right-hand side of (+2) in § 3.3.

4.1 Confluence

The restriction that is to be required for every recursive function definition is the *confluence*[63] of the *rewrite relation* that results from reading the defining equations as reduction rules, in the sense that they admit us to replace occurrences of left-hand sides of instantiated equations with their respective right-hand sides, provided that their conditions are fulfilled.[64]

The confluence restriction guarantees that no distinct objects of the data types can be equated by the recursive function definitions.[65] If we assume axioms such as (nat2–3) (cf. § 3.3) or ($\mathsf{list}(\mathsf{nat})$2–3) (cf. § 3.4), then this restriction is essential for consistency.

Indeed, without confluence, a definition of a recursive function could destroy the data type in the sense that the specification has no model anymore; for example, if we added $\mathsf{p}(x) = \mathsf{0}$ as a further defining equation to (p1), then we would get $\mathsf{s}(\mathsf{0}) = \mathsf{p}(\mathsf{s}(\mathsf{s}(\mathsf{0}))) = \mathsf{0}$, in contradiction to the axiom (nat2) of § 3.3.

For the recursive function definitions admissible in the Boyer–Moore theorem provers, confluence results from the restrictions that there is only one (unconditional) defining equation for each new function symbol,[66] and that all variables occurring on the right-hand side of the definition also occur on the left-hand side of the defining equation.[67]

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[63] A relation $\longrightarrow$ is *confluent* (or has the "Church–Rosser property") if two sequences of steps with $\longrightarrow$, starting from the same element, can always be joined by an arbitrary number of further steps on each side; formally: $\overset{+}{\longleftarrow} \circ \overset{+}{\longrightarrow} \ \subseteq\ \overset{*}{\longrightarrow} \circ \overset{*}{\longleftarrow}$. Here $\longleftarrow\ =\ \longrightarrow^{-1}$ is the reverse relation of $\longrightarrow$ (cf. § 3.1). $\overset{+}{\longrightarrow}$ is the transitive closure of $\longrightarrow$, which admits an arbitrary positive number of steps with $\longrightarrow$; and $\overset{*}{\longrightarrow}$ is the reflexive closure of $\overset{+}{\longrightarrow}$, which additionally admits us to do no step at all. Finally, $\circ$ is the concatenation of binary relations.

[64] For the technical meaning of *fulfilledness* in the recursive definition of the rewrite relation see [Wirth, 2009], where it is also explained why the rewrite relation respects the straightforward purely logical semantics of positive/negative-conditional equation equations.

[65] As constructor terms are irreducible w.r.t. this rewrite relation, if the application of a defined function symbol rewrites to two constructor terms, these constructor terms must be identical in case of confluence.

[66] Cf. Item (a) of the "definition principle" of [Boyer and Moore, 1979, p. 44f.]. Confluence is also discussed under the label "uniqueness" on Page 87ff. of [Moore, 1973].

These two restrictions are an immediate consequence of the general definition style of list programming language LISP. More precisely, recursive functions are to be defined in all Boyer–Moore theorem provers in the more restrictive style of *applicative* LISP.[68]

EXAMPLE 6 (A Recursive Function Definition in Applicative LISP).
To avoid the association of routine knowledge, let us not consider a function definition over lists (as standard in LISP), but over the natural numbers. For example, instead of our two equations (+1), (+2) for +, we find the following single equation on Page 53 in the standard reference for the Boyer–Moore heuristics [Boyer and Moore, 1979]:

```
(PLUS X Y) = (IF (ZEROP X)
                 (FIX Y)
                 (ADD1 (PLUS (SUB1 X) Y)))
```

Note that `(IF` x y z`)` is nothing but the conditional "`IF` z then y else z", that `ZEROP` is a Boolean function checking for being zero, that `(FIX Y)` returns `Y` if `Y` is a natural number, and that `ADD1` is the successor function s.

The primary difference to (+1), (+2) is that `PLUS` is defined in *destructor style* instead of the *constructor style* of our equations (+1), (+2) in § 3.3. As there is no essential semantical difference between these two styles, let us transform our definition of + from (+1), (+2) into destructor style.

In place of the untyped destructor `SUB1`, let us use the typed destructor p defined by either by $(\mathsf{p}1)$ or by $(\mathsf{p}1')$ of § 3.3, which — just as `SUB1` — returns the predecessor of a positive natural number. Now our destructor-style definition of + consists of the following two positive/negative-conditional equations:

$(+1')$ $\quad x+y = y \quad \Leftarrow \quad x = \mathsf{0}$

$(+2')$ $\quad x+y = \mathsf{s}(\mathsf{p}(x)+y) \quad \Leftarrow \quad x \neq \mathsf{0}$

If we compare this definition of + to the one via the equations (+1), (+2), then we find that the constructors $\mathsf{0}$ and s have been removed from the left-hand sides of the defining equations; they are replaced with the destructor p on the right-hand side and with some conditions.

Now it is easy to see that $(+1')$, $(+2')$ represent the above definition of `PLUS` in positive/negative-conditional equations, provided that we ignore that Boyer–Moore theorem provers have no types and no typed variables. □

If we considered the recursive equation (+2) together with the alternative recursive equation $(+2')$, then we could rewrite $\mathsf{s}(x)+y$ on the one hand with (+2) into $\mathsf{s}(x+y)$, and, on the other hand, with $(+2')$ into $\mathsf{s}(\mathsf{p}(\mathsf{s}(x))+y)$. This does not seem to be problematic, because the latter result can be rewritten to the former

[67] Cf. Item (c) of the "definition principle" of [Boyer and Moore, 1979, p. 44f.].

[68] Cf. [McCarthy *et al.*, 1965] for the definition of LISP. The "'applicative" subset of LISP lacks the imperative commands of LISP, such as variants of `PROG`, `SET`, `GO`, and `RETURN`, as well as all functions or special forms that depend on the concrete allocation on the system heap, such as `EQ`, `RPLACA`, and `RPLACD`, which can be used in LISP to realize circular structures or to save space on the system heap.

one by (p1). In general, however, confluence is undecidable and criteria sufficient for confluence are extremely hard to develop.

The only decidable criterion that is sufficient for confluence of positive/negative-conditional equations and applies to all our example specifications, but does not require termination, is found in [Wirth, 2009]. It can be more easily tested than the admissibility conditions of the Boyer–Moore theorem provers and avoids divergence even in case of non-termination; only the proof that it indeed guarantees confluence is very involved.

4.2 Termination and Reducibility

There are two restrictions that are additionally required for any function definition in the Boyer–Moore theorem provers, namely *termination* of the rewrite relation and *reducibility* of all ground terms w.r.t. the rewrite relation.

The requirement of termination should be intuitively clear; we will further discuss it in § 4.4.

To understand the requirement of reducibility, note that it is not only so that we can check the soundness of $(+1')$ and $(+2')$ independently from each other, we can even omit one of the equations, resulting in a partial definition of the function $+$. Indeed, for the function p we did not specify any value for p(0); so p(0) is not reducible in the rewrite relation that results from reading the specifying equations as reduction rules.

A function defined in a Boyer–Moore theorem prover, however, must always be specified completely, in the sense that every application of such a function to (constructor) ground terms must be reducible. This reducibility immediately results from the LISP definition style, which requires all arguments of the function symbol on the left-hand side of its defining equation to be distinct variables.[69]

4.3 Constructor Variables

The two further restrictions of the Boyer–Moore theorem provers, namely reducibility and termination of the rewrite relation that results from reading the specifying equations as reduction rules, are not essential, neither for the semantics of recursive function definitions with data types given by constructors,[70] nor for confluence and consistency.[71]

Note that these two additional restrictions imply that only *total recursive* functions[72] are admissible in the Boyer–Moore theorem provers.

As termination, the second of these restrictions, is not in the spirit of the LISP logic of the Boyer–Moore theorem provers, we have to ask why Boyer and Moore brought up these two additional restrictions.

[69] Cf. Item (*b*) of the "definition principle" of [Boyer and Moore, 1979, p. 44f.].

[70] Cf. [Wirth and Gramlich, 1994b].

[71] Cf. [Wirth, 2009].

[72] You may follow the explicit reference to [Schoenfield, 1967] as the basis for the logic of the PURE LISP THEOREM PROVER on Page 93 of [Moore, 1973].

Of course, when these restrictions are both satisfied, then — similar to the classical case of explicitly defined notions — we can get rid of all recursively defined function symbols by rewriting, but in general only in terms without variables.

A better potential answer is found on Page 87ff. of [Moore, 1973], where confluence of the rewrite relation is discussed and a reference to Russell's Paradox serves as an argument that confluence alone would not be sufficient for consistency. The argumentation is essentially the following: First, a Boolean function russell is recursively defined by

(russell1) $\quad \mathsf{russell}(b) = \mathsf{false} \;\Leftarrow\; \mathsf{russell}(b) = \mathsf{true}$

(russell2) $\quad \mathsf{russell}(b) = \mathsf{true} \;\Leftarrow\; \mathsf{russell}(b) = \mathsf{false}$

Then it is claimed that this function definition would result in an inconsistent specification on the basis of the axioms (bool1–2) of § 3.4.

This inconsistency, however, arises only if the variable b of the axiom (bool1) can be instantiated with the term $\mathsf{russell}(b)$, which is actually not our intention and which we do not have to permit: If all variables we have introduced so far are *constructor variables*[73] in the sense that they can only be instantiated with terms formed from constructor function symbols (incl. constructor constants) and constructor variables, then irreducible terms such as $\mathsf{russell}(b)$ can denote *junk objects* different from true and false, and no inconsistency arises.[74]

Note that these constructor variables are implicitly part of the LISP semantics with its innermost evaluation strategy. For instance, in Example 6 of § 4.1, neither the LISP definition of PLUS nor its representation via the positive/negative-conditional equations $(+1')$, $(+2')$ is intended to be applied to a non-constructor term in the sense that X or x should be instantiated to a term that is a function call of a (partially) defined function symbol that may denote a junk object.

Moreover, there is evidence that Moore considered the variables already in 1973 as constructor variables: On Page 87 in [Moore, 1973], we find formulas on definedness and confluence, which make sense only for constructor variables; the one on definedness of the Boolean function AND reads

```
∃Z (COND X (COND Y T NIL) NIL) = Z,
```

which is trivial for a general variable Z and makes sense only if Z is taken to be a constructor variable.

Finally, the termination as it is established via induction templates in the Boyer–Moore theorem provers except the PURE LISP THEOREM PROVER, and as we will describe it in § 4.4, holds for the rewrite relation of the defining equations only if we consider the variables of these equations to be constructor variables (or if we restrict the termination result to an innermost rewriting strategy and require that all function definitions are total).

---

[73] Such *constructor variables* were formally introduced in [Wirth *et al.*, 1993] and became an essential part of the frameworks found in [Wirth and Gramlich, 1994a; 1994b], [Kühler and Wirth, 1996; 1997], [Wirth, 1997; 2009] [Kühler, 2000], [Avenhaus *et al.*, 2003], and in [Schmidt-Samoa, 2006a; 2006b; 2006c].

[74] For the appropriate semantics see in particular [Wirth and Gramlich, 1994b], and [Kühler and Wirth, 1997].

4.4 Termination and General Induction Templates

In addition to LISP definition style, the theorem provers for explicit induction require termination of the rewrite relation that results from reading the specifying equations as reduction rules. More precisely, in all Boyer–Moore theorem provers except the PURE LISP THEOREM PROVER,[75] *before* a new function symbol f_k is admitted to the specification, a "valid induction template" — which immediately implies termination — has to be constructed from the defining equation of f_k.[76]

Induction templates were first used in THM and received their name when they were first described in [Boyer and Moore, 1979].

Every time a new recursive function f_k is defined, systems for explicit reduction immediately try to construct *valid induction templates*; if the system does not find any, then the new function symbol is rejected w.r.t. the given definition; otherwise the system stores a link from the function name to its valid induction templates.

The induction templates actually serve two purposes: as witnesses for termination and as the basic tools of the induction rule of explicit induction for generating the step cases.

For a finite number of mutually recursive functions f_k with arity n_k $(k \in K)$, an induction template in the most general form consists of the following:

1. A *relational description*[77] of the changes in the argument pattern of these recursive functions as found in their recursive defining equations:

   For each $k \in K$ and for each positive/negative-conditional equation with left-hand side $f_k(t_1, \ldots, t_{n_k})$, we take the set R of recursive function calls of the $f_{k'}$ $(k' \in K)$ occurring in the right-hand side or the condition, and some case condition C, which must be implied by the condition of the defining equation. Typically, C is empty (i.e. always true) (in case of constructor style definitions) or just a sub-conjunction of the condition of the equation, which is sufficient to admit a proper destructor application.

   Together they form the triple $(f_k(t_1, \ldots, t_{n_k}), R, C)$, and a set containing such a triple for each such defining equation forms the relational description.

   For our definition of + with (+1), (+2) in § 3.3, there is only one recursive equation and only one relevant relational description, namely the following one with an empty case condition:
   $$\{\ (\ \mathsf{s}(x)+y,\ \ \{x+y\},\ \ \emptyset\ )\ \}.$$

   Also for our definition of + with (+1′), (+2′) in Example 6, there is only one recursive equation and only one relevant relational description, namely
   $$\{\ (\ x+y,\ \ \{\mathsf{p}(x)+y\},\ \ x \neq \mathsf{0}\ )\ \}.$$

---

[75] Note that termination is not proved in the PURE LISP THEOREM PROVER; instead, the soundness of the induction proofs comes with the *proviso* that all the rewrite relation of all defined function symbols terminate.

[76] See also Item (d) of the "definition principle" of [Boyer and Moore, 1979, p. 44f.] for a formulation that avoids the technical term "induction template".

[77] The name "relational description" comes from [Walther, 1992; 1993].

2. For each $k \in K$, a variable-free weight term w_{f_k} containing the position numbers $\{(1),\ldots,(n_k)\}$ instead of variables. The position numbers actually occurring in the term are called the *measured positions*.

   For our two relational descriptions, only the weight term (1) (consisting just of a position number) makes sense as w_+, resulting in the set of measured positions $\{1\}$. Indeed, + terminates in both definitions because the argument in the first position gets smaller.

3. A binary predicate $<$ that is known to represent a well-founded relation.

   For our two relational descriptions, the predicate $\lambda x, y.\ (\mathsf{lessp}(x,y) = \mathsf{true})$, is appropriate.

Now, an induction template is *valid* if for each element of the relational description as given above, and for each $f_{k'}(t'_1,\ldots,t'_{n_{k'}}) \in R$, the following conjecture is valid:

$$w_{f_{k'}}\{(1)\mapsto t'_1,\ldots,(n_{k'})\mapsto t'_{n_{k'}}\} \ < \ w_{f_k}\{(1)\mapsto t_1,\ldots,(n_k)\mapsto t_{n_k}\} \quad \Leftarrow \quad C.$$

For our two relational descriptions, this amounts to showing $\mathsf{lessp}(x,\mathsf{s}(x))$ and $\mathsf{lessp}(\mathsf{p}(x),x) \Leftarrow x \neq \mathsf{0}$, respectively; so their templates are both valid.

EXAMPLE 7 (Two Induction Templates with different Measured Positions).
For the ordering predicate lessp as defined by (lessp1–3) of § 3.3, we get two appropriate induction templates with the sets of measured positions $\{1\}$ and $\{2\}$, respectively, both with the relational description

$$\{\ (\ \mathsf{lessp}(\mathsf{s}(x),\mathsf{s}(y)),\ \ \{\mathsf{lessp}(x,y)\},\ \ \emptyset\ )\ \},$$

and both with the well-founded ordering $\lambda x, y.\ (\mathsf{lessp}(x,y) = \mathsf{true})$. The first template has the weight term (1) and the second one has the weight term (2). The validity of both templates is given by lemma (lessp4) of § 3.3. □

EXAMPLE 8 (One Induction Template with Two Measured Positions).
For the Ackermann function ack as defined by (ack1–3) of § 3.3, we get only one appropriate induction template. The set of its measured positions is $\{1,2\}$, because of the weight function $\mathsf{cons}((1),\mathsf{cons}((2),\mathsf{nil}))$, which we will abbreviate in the following with $[(1),(2)]$. The well-founded relation is the lexicographic ordering $\lambda l, k.\ (\mathsf{lexlimless}(l,k,\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) = \mathsf{true})$. The relational description has two elements: For the equation (ack2) we get

$$(\ \mathsf{ack}(\mathsf{s}(x),\mathsf{0}),\ \ \{\mathsf{ack}(x,\mathsf{s}(\mathsf{0}))\},\ \ \emptyset\ ),$$

and for the equation (ack3) we get

$$(\ \mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)),\ \ \{\mathsf{ack}(\mathsf{s}(x),y),\ \ \mathsf{ack}(x,\mathsf{ack}(\mathsf{s}(x),y))\},\ \ \emptyset\ ).$$

The validity of the template is expressed in the three equations

$$\begin{array}{llll} \mathsf{lexlimless}([x,\mathsf{s}(\mathsf{0})], & [\mathsf{s}(x),\mathsf{0}], \ \ \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) & = & \mathsf{true}; \\ \mathsf{lexlimless}([\mathsf{s}(x),y], & [\mathsf{s}(x),\mathsf{s}(y)], \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) & = & \mathsf{true}; \\ \mathsf{lexlimless}([x,\mathsf{ack}(\mathsf{s}(x),y)], & [\mathsf{s}(x),\mathsf{s}(y)], \mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{0})))) & = & \mathsf{true}; \end{array}$$

which follow deductively from (lessp4), ($\mathsf{lexlimless}$1), ($\mathsf{lexless}$2–4), (length1–2). □

For valid induction templates of destructor-style definitions see Examples 18 and 19 in § 5.3.7.

That the existence a valid induction template for a new set of recursive functions f_k $(k \in K)$ actually implies termination of the rewrite relation on ground terms given after addition of the new equations for the f_k can be executed in any model of the old specification as follows:

For a *reductio ad absurdum*, suppose that there is an infinite sequence of rewrite steps on ground terms. Let us now consider each term to be replaced with the multiset of the weight terms for its function calls for f_k with $k \in K$. Then the rewrite steps with the *old* equations of previous function definitions (of symbols not among the f_k) can only change the multiset by deleting some elements for the following two reasons:

1. The new function symbols do not occur in the old equations.
2. We consider all our variables to be constructor variables as explained in § 4.3.

Moreover, a rewrite step with a new equation reduces the multiset in the well-founded relation given by the multiset extension of the well-founded relation of the template in the assumed model of the old specification, because of the fulfilledness of the conditions of the equation and the validity of the template. Thus, in each rewrite step, the the multiset gets smaller in a well-founded ordering or does not change. Moreover, if we assume that rewriting with the old equations terminates, then the new equations must be applied infinitely often in this sequence, and so the multiset gets smaller in infinitely many steps, which is impossible in a well-founded ordering.

4.5 Induction Templates for Explicit Induction

We restrict the discussion in this section to recursive functions that are not mutually recursive, partly for simplicity and partly because explicit induction is hardly helpful for finding proofs involving mutually recursive functions.

Thus, all the f_k with arity n_k of § 4.4 simplify to one symbol f with arity n. Moreover, under this restriction it is easy to partition the measured positions of a template into "changeable" and "unchangeable" ones.[78]

Changeable are those measured positions i of the template which sometimes change in the recursion, i.e. for which there is a triple $(f(t_1, \ldots, t_n), R, C)$ in the relational description of the template, and an $f(t'_1, \ldots, t'_n) \in R$ such that $t'_i \neq t_i$. The remaining measured positions of the template are called *unchangeable*. Unchangeable positions typically result from the inclusion of a global variable into the argument list of a function for observing applicative programming style.

To improve the applicability of the induction hypotheses of the step cases produced by the induction rule, these induction hypotheses should mirror the recursive calls of the unfolding of the definition of a function f occurring in the induction rule's input formula, say

$$A[f(t''_1, \ldots, t''_n)].$$

[78] This partition into changeable and unchangeable positions (actually: variables) originates in [Boyer and Moore, 1979, p. 185f.].

An induction template is *applicable* to the indicated occurrence of its function symbol f if the terms t''_i at the changeable positions i of the template are *distinct variables* and none of these variables occurs in the terms $t''_{i'}$ that fill the unchangeable positions i' of the template.[79] For templates of constructor-style equations we additionally have to require here that the first element $f(t_1,\ldots,t_n)$ of each triple of the relational description of the template matches $(f(t''_1,\ldots,t''_n))\xi$ for some *constructor substitution* ξ that may replace the variables of $f(t''_1,\ldots,t''_n)$ with constructor terms, i.e. terms consisting of constructor symbols and variables, such that $t''_i\xi = t''_i$ for each unchangeable position i of the template.

EXAMPLE 9 (Applicable Induction Templates).
Let us consider the conjecture $(\mathsf{ack4})$ from § 3.3. From the three induction templates of Examples 7 and 8, only the one of Example 8 is applicable. The two of Example 7 are not applicable because $\mathsf{lessp}(\mathsf{s}(x),\mathsf{s}(y))$ cannot be matched to $(\mathsf{lessp}(y,\mathsf{ack}(x,y)))\xi$ for any constructor substitution ξ. □

For every recursive call $f(t'_{j',1},\ldots,t'_{j',n})$ in a positive/negative-conditional equation with left-hand side $f(t_1,\ldots,t_n)$, the relational description of an induction template for f contains a triple $\left(\ f(t_1,\ldots,t_n),\ \ \{\ f(t'_{j,1},\ldots,t'_{j,n}) \mid j\in J\ \},\ \ C\ \right)$, such that $j'\in J$.

Let us assume that the induction template is valid and applicable to the occurrence indicated in the input formula. Let σ be the substitution whose domain are the variables of $f(t_1,\ldots,t_n)$ and which matches the first element $f(t_1,\ldots,t_n)$ of the triple to $(f(t''_1,\ldots,t''_n))\xi$ for some constructor substitution ξ whose domain are the variables of $f(t''_1,\ldots,t''_n)$, such that $t''_i\xi = t''_i$ for each unchangeable position i of the template. Then we have $t_i\sigma = t''_i\xi$ for $i\in\{1,\ldots,n\}$.

Now, for the well-foundedness of the step case

$$\left(\ (A[f(t''_1,\ldots,t''_n)])\xi\ \Leftarrow\ \bigwedge_{j\in J}(A[f(t''_1,\ldots,t''_n)])\mu_j\ \right)\ \Leftarrow\ C\sigma,$$

to be implied by the validity of the induction template, we have to find substitutions μ_j whose domain $\mathrm{dom}(\mu_j)$ is the set of variables of $f(t''_1,\ldots,t''_n)$, such that the constraint $t''_i\mu_j = t'_{j,i}\sigma$ is satisfied for each measured position i of the template and $j\in J$.

If i is an unchangeable position of the template, then we have $t_i = t'_{j,i}$ and $t''_i\xi = t''_i$. Therefore, we can satisfy the constraint by requiring μ_j to be the identity on the variables of t''_i, simply because then we have $t''_i\mu_j = t''_i = t''_i\xi = t_i\sigma = t'_{j,i}\sigma$.

If i is a changeable position, then we know by the applicability of the template that t''_i is a variable not occurring in another changeable or unchangeable position in $f(t''_1,\ldots,t''_n)$, and we can satisfy the constraint by defining $t''_i\mu_j := t'_{j,i}\sigma$.

On the remaining variables of $f(t''_1,\ldots,t''_n)$, we define μ_j in a way that we get $t''_i\mu_j = t'_{j,i}\sigma$ for as many unmeasured positions i as possible, and otherwise as the identity. This is not required for soundness but it improves the likeliness of applicability of the induction hypothesis $(A[f(t''_1,\ldots,t''_n)])\mu_j$ after unfolding $f(t''_1,\ldots,t''_n)\xi$ in $(A[f(t''_1,\ldots,t''_n)])\xi$. Note that such an eager instantiation is

[79] This definition of applicability originates in [Boyer and Moore, 1979, p. 185f.].

required in explicit induction unless the logic admits one of the following: existential quantification, existential variables,[80] lazy induction-hypothesis generation.

An *induction scheme* for the given input formula is now obtained from the given information as follows: Each triple in the relational description of the considered form is replaced with the triple $(\ \xi,\ \ \{\ \mu_j \mid j \in J\ \},\ \ C\sigma\ )$, the weight term is replaced with the set of *induction variables*, which are the variables at the the changeable positions in $f(t''_1, \ldots, t''_n)$. The well-founded relation is dropped. Moreover, we add a set containing the position of $f(t''_1, \ldots, t''_n)$ in $A[f(t''_1, \ldots, t''_n)]$. Finally, we add the hitting ratio of all substitutions μ_j with $j \in J$:

$$\frac{|\{\ (j,i) \in J{\times}\{1,\ldots,n\} \mid t'_i\mu_j = t_{j,i}\sigma\ \}|}{|J{\times}\{1,\ldots,n\}|},$$

where J actually has to be the disjoint sum over all the J occurring as index sets of second elements of triples like the one above.

EXAMPLE 10 (Induction Schemes).
The template for ack of Example 8 is the only one that is applicable to ($\mathsf{ack}4$) according to Example 9 and gives rise to the following induction scheme. The set containing the position of occurrence is $\{1.2\}$ (left-hand side, second subterm) and the set of induction variables is $\{x, y\}$, because both positions are changeable. The relational description is replaced with $\{\ \ (\xi_1, \{\mu_1\}, \emptyset),\ \ \ (\xi_2, \{\mu_{2,1}, \mu_{2,2}\}, \emptyset)\ \ \}$. From matching the first element of the first triple of the relational description to the position in the input formula we get $(\mathsf{ack}(\mathsf{s}(x), 0))\sigma_1 = (\mathsf{ack}(x, y))\xi_1$, i.e. $\xi_1 = \{x{\mapsto}\mathsf{s}(x'),\ y{\mapsto}0\}$ and $\sigma_1 = \{x{\mapsto}x'\}$ as both positions of the template are changeable, and thus we get the constraint $(\mathsf{ack}(x, y))\mu_1 = (\mathsf{ack}(x, \mathsf{s}(0)))\sigma_1$, i.e. $\mu_1 = \{x{\mapsto}x',\ y{\mapsto}\mathsf{s}(0)\}$. This results in the step case

$$\mathsf{lessp}(0, \mathsf{ack}(\mathsf{s}(x'), 0)) = \mathsf{true} \ \Leftarrow\ \mathsf{lessp}(\mathsf{s}(0), \mathsf{ack}(x', \mathsf{s}(0))) = \mathsf{true}.$$

From matching the first element of the second triple of the relational description to the position in the input formula we get $(\mathsf{ack}(\mathsf{s}(x), \mathsf{s}(y)))\sigma_2 = (\mathsf{ack}(x, y))\xi_2$, i.e. $\xi_2 = \{x{\mapsto}\mathsf{s}(x'),\ y{\mapsto}\mathsf{s}(y')\}$ and $\sigma_2 = \{x{\mapsto}x',\ y{\mapsto}y'\}$. Moreover, we get the constraints

$$\begin{array}{rcl}(\mathsf{ack}(x, y))\mu_{2,1} &=& (\mathsf{ack}(\mathsf{s}(x), y))\sigma_2;\\ (\mathsf{ack}(x, y))\mu_{2,2} &=& (\mathsf{ack}(x, \mathsf{ack}(\mathsf{s}(x), y)))\sigma_2;\end{array}$$

i.e. $\mu_{2,1} = \{x{\mapsto}\mathsf{s}(x'),\ y{\mapsto}y'\}$ and $\mu_{2,2} = \{x{\mapsto}x',\ y{\mapsto}\mathsf{ack}(\mathsf{s}(x'), y')\}$. This results in the step case

$$\begin{array}{l}\mathsf{lessp}(\mathsf{s}(y'), \mathsf{ack}(\mathsf{s}(x'), \mathsf{s}(y'))) = \mathsf{true}\\ \qquad\Leftarrow \left(\begin{array}{rl} & \mathsf{lessp}(y', \mathsf{ack}(\mathsf{s}(x'), y')) = \mathsf{true}\\ \wedge & \mathsf{lessp}(\mathsf{ack}(\mathsf{s}(x'), y'), \mathsf{ack}(x', \mathsf{ack}(\mathsf{s}(x'), y'))) = \mathsf{true}\end{array}\right).\end{array}$$

The hitting ratio is $\frac{6}{6} = 1$. □

[80] Existential variables are called “free variables” in modern tableau systems (see the 2nd rev. edn. [Fitting, 1996], but not its 1st edn. [Fitting, 1990]) and occur with extended functionality under different names in the inference systems of [Wirth, 2004; 2012b; 2012c].

5 AUTOMATED EXPLICIT INDUCTION

5.1 The Application Context of Automated Explicit Induction

Since the upcoming of programmable computing machinery in the middle of the 20th century, a major problem of hard- and software has been and still is the uncertainty whether they actually always do what they should do.

The only viable solution to this problem seems to be the following verification approach:

> Specify the intended functionality in a language of formal logic, and then supply a formal proof that the program actually satisfies the specification!

Such an approach requires a formal specification of the hardware or of the involved programming languages.[81]

The crucial problem, however, are the costs of the many proofs of the huge amounts of application hard- and software in our market-oriented economies. Thus, for our times and in our economical system, we can expect a verification only in areas where the managers know that mere testing does not suffice and where the bugs in the hard- or software have shown to produce more costs than the verification process. Good candidates for this are central processing units (CPUs) in standard processors.[82]

To reduce the costs of verification, we can hope to automate it with automated theorem-proving systems; this has to include an automation of mathematical induction because most data types applied in programming, such as natural numbers, pairs, arrays, lists, and trees, require induction for the verification of their properties. Up to a completely unexpected breakthrough in the future, however, the verification of a new hard- or software system will always require human users who help the theorem-proving systems to explore and develop the notions and theories that properly match the new system or theory. Already today, however, ACL2 achieves a complete automation when re-running the proofs for an already verified CPU after a minor update.

[81] To simplify matters, we assume here that none of the components of the physical hardware is broken and that the computing machinery is situated at a place where we can neglect the perturbation by the environment. Under this assumption, the task of showing that the hard- and software actually implements the intended functionality can be made concrete as we do it here — at least in principle.

To be complete, such an approach would also require a verification that the verification system is sound and that the implementation of the programming languages (via interpreters or compilers) actually follow their specifications. To reduce the complexity of the approach, we could restrict ourselves to a single programming language which is very close to the formal logics of the verification system and admits a simple implementation.

[82] Although the costs for the verification of CPUs are economically maintainable, a company in a market-oriented economy may have problems with these relatively small extra costs, because their competitors may be able to offer slightly lower prices and push that company out of the market. Even the economical ruin of some of these competitors because of the liabilities for their faulty processing units does not guarantee a future success of verification in the business.

5.2 The PURE LISP THEOREM PROVER

Our overall task is to answer — from a historical perspective — the question:

> How could Robert S. Boyer and J Strother Moore — starting virtually from zero[83] in the middle of 1972 — actually invent their outstanding solutions to the hard heuristic problems in the automation of induction and implement them in the sophisticated theorem prover THM as described [Boyer and Moore, 1979]?

As already described in § 1, the breakthrough in the heuristics for automated theorem proving was achieved with the "PURE LISP THEOREM PROVER", developed and implemented by Boyer and Moore. It was presented by Moore at the third IJCAI, which took place in Stanford (CA) in August 1973,[84] but it is best documented in Part II of Moore's PhD thesis [1973], defended in November 1973.

The PURE LISP THEOREM PROVER is given no name in the before-mentioned publications. The only occurrence of the name in publication seems to be in [Moore, 1975, p.1], where it is actually called "the Boyer–Moore PURE LISP THEOREM PROVER".

To understand the achievements a bit better, let us now discuss the material of Part II of Moore's PhD thesis in some detail, because it provides some explanation on how Boyer and Moore could be so surprisingly successful. Especially helpful for understanding the process of creation are those procedures of the PURE LISP THEOREM PROVER that are provisional w.r.t. to their refinement in later Boyer–Moore theorem provers. Indeed, these provisional procedures help to decompose the giant leap from nothing to THM, which has no parallel in the history of automated theorem proving, to steps of a more comprehensible size.

As W. W. Bledsoe (1921–1995) was Boyer's PhD advisor, it is likely that Boyer was building the PURE LISP THEOREM PROVER on the advanced standpoint in the automation of theorem proving that Bledsoe had developed and taught.[85]

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[83] No heuristics at all were explicitly described, for instance, in Burstall's considerations of the year 1968 on program verification by induction over recursive functions in [Burstall, 1969], where the proofs were not even formal, and an implementation seemed to be more or less utopian:

> "The proofs presented will be mathematically rigorous but not formalised to the point where each inference is presented as a mechanical application of elementary rules of symbol manipulation. This is deliberate since I feel that our first aim should be to devise methods of proof which will prove the validity of non-trivial programs in a natural and intelligible manner. Obviously we will wish at some stage to formalise the reasoning to a point where it can be performed by a computer to give a mechanised debugging service." [Burstall, 1969, p. 41]

Still in 1972, all known implementations of automated inductive theorem provers based on recursive functions (we know only of the ones of W. W. Bledsoe and Robert S. Boyer) were just able to apply the Axiom of Structural Induction (S) of § 3.3 to a randomly picked variable of type nat, which is not worth mentioning in comparison with the Boyer–Moore theorem provers.

[84] Cf. [Boyer and Moore, 1973].

[85] On Page 172 of [Moore, 1973] we read on the PURE LISP THEOREM PROVER:

> "The design of the program, especially the straightforward approach of 'hitting' the theorem over and over again with rewrite rules until it can no longer be changed, is largely due to the influence of W. W. Bledsoe."

Boyer and Moore report on the method of for the development of their induction heuristics in late 1972 and early 1973 that they were doing proofs on list data structures on the blackboard and verbalizing to each other the heuristics behind their choices on how to proceed with the proof.[86] This means that, although explicit induction is not the approach humans would choose for non-trivial induction tasks, the heuristics of the PURE LISP THEOREM PROVER are learned from human heuristics after all.

Note that Boyer's and Moore's method of learning computer heuristics from their own human behavior in mathematical logic was a step of two young men against the spirit of the time: The dominance of J. Alan Robinson's resolution method suggested the application of vast amounts of computational power to most elementary forms of "machine-oriented" (i.e. not human-like) first-order reasoning. It may be that the orientation toward human-like or "intelligible" methods and heuristics in the automation of theorem proving had also some tradition in Edinburgh at the time,[87] but here the major influence of Boyer and Moore is again W. W. Bledsoe.[88]

The source code of the PURE LISP THEOREM PROVER was written in the programming language POP–2.[89] Boyer and Moore were the only programmers involved in the implementation. The average time in the central processing unit (CPU) of the ICL–4130 for the proof of a theorem is reported to be about 10 s.[90]

One remarkable omission in the PURE LISP THEOREM PROVER is lemma application. As a consequence, the success of proving a set of theorems cannot depend on the order of their presentation to the theorem prover. Indeed, just as the resolution theorem provers of the time, the PURE LISP THEOREM PROVER starts every proof right from the scratch and does not improve its behavior with the help of previously proved lemmas.

Moreover, all induction orderings in the PURE LISP THEOREM PROVER are recombinations of constructor relations, such that all inductions it can do are structural inductions over combinations of constructors. As a consequence, contrary to later Boyer–Moore theorem provers, the well-foundedness of the induction orderings does not depend on the termination of the recursive function definitions.[91]

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[86] Cf. [Boyer and Moore, 2012].

[87] Cf. e.g. the quotation from [Burstall, 1969] in Note 83.

[88] Cf. e.g. [Bledsoe *et al.*, 1972].

[89] Cf. [Burstall *et al.*, 1971].

[90] This timing result is hard to believe and strongly indicates that Boyer and Moore were as great in coding as they were in creating heuristics for theorem proving. Here is the actual wording of the timing result found on Page 171f. of [Moore, 1973]:

> "Despite theses inefficiencies, the 'typical' theorem proved requires only 8 to 10 seconds of CPU time. For comparison purposes, it should be noted that the time for `CONS` in 4130 POP–2 is 400 microseconds, and `CAR` and `CDR` are about 50 microseconds each. The hardest theorems solved, such as those involving `SORT`, require 40 to 50 seconds each."

[91] Note that the well-foundedness of the constructor relations depends on distinctness of the constructor ground terms in the models, but this does not really depend on the termination of the recursive functions because (as discussed in § 4.1) confluence is sufficient here.

Nevertheless, the soundness of the PURE LISP THEOREM PROVER depends directly on the termination of the recursive function definitions, but only in one aspect: It simplifies and evaluates expressions under the assumption of termination. For instance, both `(IF`[92] a d d`)` and `(CDR (CONS` a d`))` simplify to d, no matter whether a terminates; and it is admitted to rewrite with a recursive function definition even if an argument of the function call does not terminate.

The termination of the recursively defined functions, however, is not at all checked by the PURE LISP THEOREM PROVER, but comes as a *proviso* for its soundness.

The logic of the PURE LISP THEOREM PROVER is an applicative[93] subset of the logic of LISP. The only *destructors* in this logic are `CAR` and `CDR`. They are overspecified on the only *constructors* `NIL` and `CONS` by the equations

`(CAR (CONS` a d`))`	=	a	|	`(CAR NIL)`	=	`NIL`
`(CDR (CONS` a d`))`	=	d	|	`(CDR NIL)`	=	`NIL`

As standard in LISP, every term of the form `(CONS` a d`)` is taken to be true in the logic of the PURE LISP THEOREM PROVER if it occurs at an argument position with Boolean intention. The actual truth values (to be returned by Boolean functions) are `NIL` (representing false) and `T`, which is an abbreviation for `(CONS NIL NIL)` and represents true.[94] Moreover, the logic is *pure LISP* in the sense that the natural number `0` is represented by `NIL` and the successor function $\mathsf{s}(d)$ is represented by `(CONS NIL` d`)`.[95]

Let us now discuss the behavior of the PURE LISP THEOREM PROVER by describing the instances of the stages of the Boyer–Moore waterfall (cf. Figure 1) as they are described in Moore's PhD thesis.

5.2.1 *Simplification in the* PURE LISP THEOREM PROVER

The first stage of the Boyer–Moore waterfall — "simplification" in Figure 1 — is called "normalation" in the PURE LISP THEOREM PROVER. It applies the following simplification procedures to LISP expressions until the result does not change anymore: "evaluation", "normalization", and "reduction".

"Normalization" tries find sufficient conditions for a given expression to have the soft type "Boolean" and to normalize logical expressions. Contrary to clausal logic over equational atoms, LISP admits `EQUAL` and `IF` to appear not only on top level, but in arbitrary mutually nested terms. To free later tests and heuristics from checking for their triggers in every equivalent form, such a normalization w.r.t. propositional logic and equality is part of most theorem provers today.

[92] In the logic of the PURE LISP THEOREM PROVER, the special form `IF` is actually called "`COND`". This is most confusing because `COND` is a standard special form in LISP, different from `IF`. Therefore, we will ignore this peculiarity and write "`IF`" here for every "`COND`" of the PURE LISP THEOREM PROVER.

[93] Cf. Note 68.

[94] Cf. 2nd paragraph of Page 86 of [Moore, 1973].

[95] Cf. 2nd paragraph of Page 87 of [Moore, 1973].

"Reduction" is a form of what today is called *contextual rewriting*. It is based on fact that — in the logic of the PURE LISP THEOREM PROVER — in the conditional expression

`(IF` c p n`)`

we can simplify occurrences of c in p to `(CONS (CAR` c`) (CDR` c`))`, and in n to `NIL`. The replacement with `(CONS (CAR` c`) (CDR` c`))` is executed only at positions with Boolean intention and can be improved in the following two special cases:

1. If we know that c is of soft type "Boolean", then we rewrite all occurrences of c in p actually to `T`.
2. If c is of the form `(EQUAL` l r`)`, then we can rewrite occurrences of l in p to r (or vice versa). Note that we have to treat the variables in l and r as constants in this rewriting. The PURE LISP THEOREM PROVER rewrites with this equation here only if one of l, r is a ground term.[96] If this is the case, then the other cannot be a ground term because the equation would otherwise have been simplified to `T` or `NIL` in the previously applied "evaluation". So replacing the latter term with the ground term everywhere in p must terminate, and this is all the contextual rewriting with equalities that the PURE LISP THEOREM PROVER does in "reduction".[97]

"Evaluation" is a procedure that evaluates expressions partly by simplification within the elementary logic as given by Boolean operations and the equality predicate. Moreover, "evaluation" executes some rewrite steps with the equations defining the recursive functions. Thus, "evaluation" can roughly be seen as normalization with the rewrite relation resulting from the elementary logic and from the recursive function definitions. The rewrite relation is applied according to the innermost left-to-right rewriting strategy, which is standard in LISP.

By "evaluation", ground terms are completely evaluated to their normal forms. Terms containing (implicitly universally quantified) variables, however, have to be handled in addition. Surprisingly, the considered rewrite relation is not necessarily terminating on non-ground terms, although the LISP evaluation of ground terms terminates because of the assumed termination of recursive function definitions (cf. § 4.4). The reason for this non-termination is the following: Because of the LISP definition style via *un*conditional equations, the positive/negative conditions are actually part of the *right-hand sides* of the defining equations, such that the rewrite step can be executed even if the conditions evaluate neither to false nor to true. For instance, in Example 6 of § 4.1, a rewrite step with the definition of `PLUS` can always be executed, whereas a rewrite step with $(+1')$ or $(+2')$ requires $x = \mathtt{0}$ to be definitely true or definitely false. This means that non-termination may result from the rewriting of cases that do not occur in the evaluation of any ground instance.[98]

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[96] A *ground* term is a term without variables. Actually, this ground term here is always a *constructor* ground term because the previously applied "evaluation" procedure has reduced any ground term to a constructor ground term, provided that the termination *proviso* is satisfied.

[97] Note, however, that further contextual rewriting with equalities is applied in a later stage of the Boyer–Moore waterfall, named *cross-fertilization*.

As the final aim of the stages of the Boyer–Moore waterfall is a formula that provides sufficiently concise and strong induction hypotheses in the last of these stages, symbolic evaluation must be prevented from unfolding function definitions unless the context admits us to expect an effect of simplification.[99]

Because the main function of "evaluation" — only to be found in the PURE LISP THEOREM PROVER — is to collect data on which base and step cases should be chosen later by the induction rule, the PURE LISP THEOREM PROVER applies a unique procedure to stop the unfolding of recursive function definitions:

A rewrite step with an equation defining a recursive function f is canceled if there is a `CAR` or a `CDR` in an argument to an occurrence of f in the right-hand side of the defining equation that is encountered during the control flow of "evaluation", and if this `CAR` or `CDR` is not removed by the "evaluation" of the arguments of this occurrence of f under the environment updated by matching of the left-hand side of the equation to the redex. For instance, "evaluation" of `(PLUS (CONS NIL X) Y)` returns `(CONS NIL (PLUS X Y))`; whereas "evaluation" of `(PLUS X Y)` returns `(PLUS X Y)` and informs the induction rule that only `(CDR X)` occurred in the recursive call during the trial to rewrite with the definition of `PLUS`. In general, such occurrences indicate which induction hypotheses should be generated by the induction rule.[100] [101]

"Evaluation" provides a link between symbolic evaluation and the induction rule of explicit induction. The question "Which case distinction on which variables should be used for the induction proof and how should the step cases look like?" is reduced to the quite different question "Where do destructors like `CAR` and `CDR` heap up during symbolic evaluation?". This reduction helps to understand by which intermediate steps it was possible to develop the most surprising, sophisticated recursion analysis of later Boyer–Moore theorem provers.

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[98] It becomes clear in the second paragraph on Page 118 of [Moore, 1973] that the code of both the positive and the negative case of a conditional will be evaluated, unless one of them can be canceled by the complete evaluation of the governing condition to true or false. Note that the evaluation of both case is necessary indeed and cannot be avoided in practice.

Moreover, note that a stronger termination requirement that guarantees termination independent of the governing condition is not feasible for recursive function definitions in practice.

Later Boyer–Moore theorem provers also use lemmas for rewriting during symbolic evaluation, which is another source of possible non-termination.

[99] In QUODLIBET this is achieved by *contextual rewriting* where evaluation stops when the governing conditions cannot be established from the context. Cf. [Schmidt-Samoa, 2006b; 2006c].

[100] Actually, "evaluation" also informs which occurrences of `CAR` or `CDR` beside the arguments of recursive occurrences of `PLUS` were permanently introduced during that trial to rewrite. Such occurrences trigger an additional case analysis to be generated by the induction rule, mostly as a compensation for the omission of the stage of "destructor elimination" in the PURE LISP THEOREM PROVER.

[101] The mechanism for partially enforcing termination of "evaluation" according to this procedure is vaguely described in the last paragraph on Page 118 of Moore's PhD thesis. As this kind of "evaluation" is only an intermediate solution on the way to more refined control information for the induction rule in later Boyer–Moore theorem provers, the rough information given here may suffice.

5.2.2 *Destructor Elimination in the* Pure LISP Theorem Prover

There is no such stage in the Pure LISP Theorem Prover.[102]

5.2.3 *(Cross-) Fertilization in the* Pure LISP Theorem Prover

Fertilization is just contextual rewriting with an equality, described before for the "reduction" that is part of the simplification of the Pure LISP Theorem Prover, but now with an equation between *two non-ground* terms.

The most important case of fertilization is called *"cross-fertilization"*. It occurs very often in step cases of induction proofs of equational theorems, and we have seen it already in Example 4 of § 3.7.1.

Neither Boyer nor Moore ever explained why cross-fertilization is cross. Cross-fertilization is actually a term from genetics referring to the alignment of haploid genetic code from male and female to a diploid code in the egg cell.

This image may help to remember that only that side (i.e. left- or right-hand side of the equation) of the induction conclusion which was activated by a successful simplification is further rewritten during cross-fertilization, namely *everywhere where the same side of the induction hypothesis occurs as a redex*, just like two haploid chromosomes have to start at the same (activated) sides for successful recombination.

Furthermore — for getting a sufficiently powerful new induction hypothesis in a follow-up induction — it is crucial to delete the equation used for rewriting (i.e. the old induction hypothesis), which can be memorized by the fact that — in the image — only one (diploid) genetic code remains.

The only noteworthy difference between cross-fertilization in the Pure LISP Theorem Prover and later Boyer–Moore theorem provers is that the generalization that consists in the deletion of the used-up equations is done in a halfhearted way, which admits a later identification of the deleted equation.

5.2.4 *Generalization in the* Pure LISP Theorem Prover

Generalization in the Pure LISP Theorem Prover works as described in § 3.8. The only difference to our presentation there is the following: Instead of just replacing all occurrences of a non-variable subterm t with a new variable z, the definition of the top function symbol of t is used to generate the definition of a new predicate p, such that $p(t)$ holds. Then the generalization of $T[t]$ becomes $T[z] \Leftarrow p(z)$ instead of just $T[z]$. The version of this automated function synthesis actually implemented in the Pure LISP Theorem Prover is just able to generate simple type properties, such as being a number or being a Boolean value.[103]

[102] See, however, Note 100 and the discussion of the Pure LISP Theorem Prover in § 5.3.2.

[103] See § 3.7 of [Moore, 1973]. As explained on Page 156f. of [Moore, 1973], Boyer and Moore failed with the trial to improve the implemented version of the function synthesis, so that it could generate a predicate on a list being ordered from a simple sorting-function.

Note that generalization is essential for the PURE LISP THEOREM PROVER because it does not use lemmas, and so it cannot build up a more and more complex theory successively. It is clear that this limits the complexity of the theorems it can prove, because a proof can only be successful if the implemented non-backtracking heuristics work out all the way from the theorem down to the most elementary theory.

5.2.5 Elimination of Irrelevance in the PURE LISP THEOREM PROVER

There is no such stage in the PURE LISP THEOREM PROVER.

5.2.6 Induction in the PURE LISP THEOREM PROVER

This stage of the PURE LISP THEOREM PROVER applies the induction rule of explicit induction as described in § 3.7. As input it takes a formula and it returns a conjunction of base and step cases to which the input formula reduces. Contrary to later Boyer–Moore theorem provers that gather the relevant information via induction schemes,[104] the induction rule of the PURE LISP THEOREM PROVER is based solely on the information provided by the "evaluation" as described in § 5.2.1.

Instead of trying to describe the general procedure, let us just put the induction rule of the PURE LISP THEOREM PROVER to test with two paradigmatic examples. In these examples we ignore the here irrelevant fact that the PURE LISP THEOREM PROVER actually uses a list representation for the natural numbers. The only effect of this is that the destructor p takes over the rôle of the destructor `CDR`.

EXAMPLE 11 (Induction Rule in the Explicit Induction Proof of ($\mathsf{ack4}$)).
Let us see how the induction rule of the PURE LISP THEOREM PROVER proceeds w.r.t. the proof of ($\mathsf{ack4}$) that we have seen in Example 5 of § 3.8. The substitutions ξ_1, ξ_2 computed as instances for the induction conclusion in Example 10 of § 4.5 suggest an overall case analysis with a base case given by $\{x \mapsto 0\}$, and two step cases given by $\xi_1 = \{x \mapsto \mathsf{s}(x'),\ y \mapsto 0\}$ and $\xi_2 = \{x \mapsto \mathsf{s}(x'),\ y \mapsto \mathsf{s}(y')\}$. The PURE LISP THEOREM PROVER requires the axioms ($\mathsf{ack1}$), ($\mathsf{ack2}$), ($\mathsf{ack3}$) to be in destructor instead of constructor style:

($\mathsf{ack1}'$)	$\mathsf{ack}(x,y) = \mathsf{s}(y)$	$\Leftarrow\ x = 0$
($\mathsf{ack2}'$)	$\mathsf{ack}(x,y) = \mathsf{ack}(\mathsf{p}(x), \mathsf{s}(0))$	$\Leftarrow\ x \neq 0\ \wedge\ y = 0$
($\mathsf{ack3}'$)	$\mathsf{ack}(x,y) = \mathsf{ack}(\mathsf{p}(x), \mathsf{ack}(x, \mathsf{p}(y)))$	$\Leftarrow\ x \neq 0\ \wedge\ y \neq 0$

"Evaluation" does not rewrite the input conjecture with this definition, but writes a "fault description" for the permanent occurrences of p as arguments of the three occurrences of ack on the right-hand sides, essentially consisting of the following three "pockets": $(\mathsf{p}(x))$, $(\mathsf{p}(x), \mathsf{p}(y))$, and $(\mathsf{p}(y))$, respectively. Similarly, the pockets gained from the fault descriptions of rewriting the input conjecture with the definition of lessp essentially consists of the pocket $(\mathsf{p}(y), \mathsf{p}(\mathsf{ack}(x,y)))$. Similar to

[104] Cf. § 4.5.

the non-applicability of the induction template for lessp in Example 9 of § 4.5, this fault description does not suggest any induction because one of the arguments of p in one of the pockets is not a variable. As this is not the case for the previous fault description, it suggests the set of all arguments of p in all pockets as induction variables. As this is the only suggestion, no merging of suggested inductions is required here.

So the Pure LISP Theorem Prover picks the right set of induction variables. Nevertheless, it fails already with the generation of the base and step cases, because the overall case analysis has two base cases given by $\{x \mapsto 0\}$ and $\{y \mapsto 0\}$, and a step case given by $\{x \mapsto \mathsf{s}(x'),\ \ y \mapsto \mathsf{s}(y')\}$.[105] This turns the first step case of the proof of Example 5 into a base case. The Pure LISP Theorem Prover finally fails with the step case it actually generates:

$$\mathsf{lessp}(\mathsf{s}(y'), \mathsf{ack}(\mathsf{s}(x'), \mathsf{s}(y'))) = \mathsf{true} \ \Leftarrow\ \mathsf{lessp}(y', \mathsf{ack}(x', y')) = \mathsf{true}.$$

This step case has only one hypothesis, which is none of the two we need. □

EXAMPLE 12 (Proof of (lessp7) by Explicit Induction with Merging).
Let us write $T(x, y, z)$ for (lessp7) of § 3.3. From the proof of (lessp7) in Example 3 of § 3.6 we can learn the following: The proof becomes simpler when we take $T(0, \mathsf{s}(y'), \mathsf{s}(z'))$ as base case (beside say $T(x, y, 0)$ and $T(x, 0, \mathsf{s}(z'))$), instead of any of $T(0, y, \mathsf{s}(z'))$, $T(0, \mathsf{s}(y'), z)$, $T(0, y, z)$. The crucial lesson from Example 3, however, is that the step case of explicit induction has to be

$$T(\mathsf{s}(x'), \mathsf{s}(y'), \mathsf{s}(z')) \Leftarrow T(x', y', z').$$

Note that the induction rule of explicit induction looks ahead only one rewrite step, separately for each occurrence of a recursive function in the conjecture.

This means that there is no way for explicit induction to apply case distinctions on variables step by step, most interesting first, until finally we end up with an instance of the induction hypothesis as in Example 3.

Nevertheless, even the Pure LISP Theorem Prover manages the pretty hard task of suggesting exactly the right step case. It requires the axioms (lessp1), (lessp2), (lessp3) to be in destructor style:

(lessp1′)	$\mathsf{lessp}(x, y) = \mathsf{false}$	$\Leftarrow$	$y = 0$
(lessp2′)	$\mathsf{lessp}(x, y) = \mathsf{true}$	$\Leftarrow$	$y \neq 0 \ \wedge\ x = 0$
(lessp3′)	$\mathsf{lessp}(x, y) = \mathsf{lessp}(\mathsf{p}(x), \mathsf{p}(y))$	$\Leftarrow$	$y \neq 0 \ \wedge\ x \neq 0$

"Evaluation" does not rewrite any of the occurrences of lessp in the input conjecture with this definition, but writes one "fault description" for each of these occurrences about the permanent occurrences of p as argument of the one occurrence of lessp on the right-hand sides, resulting in one "pocket" in each fault description, which essentially consist of $((\mathsf{p}(z)))$, $((\mathsf{p}(x), \mathsf{p}(y)))$, and $((\mathsf{p}(y), \mathsf{p}(z)))$, respectively. The Pure LISP Theorem Prover merges these three fault descriptions to the single one $((\mathsf{p}(x), \mathsf{p}(y), \mathsf{p}(z)))$, and so suggests the proper step case indeed, although it suggests the base case $T(0, y, z)$ instead of $T(0, \mathsf{s}(y'), \mathsf{s}(z'))$, which requires some considerable extra work, but does not result in a failure. □

[105] We can see this from a similar case on Page 164 and from the explicit description on the bottom of Page 166 in [Moore, 1973].

5.2.7 Conclusion on the PURE LISP THEOREM PROVER

The PURE LISP THEOREM PROVER establishes the historic breakthrough regarding the heuristic automation of inductive theorem proving in theories specified by recursive function definitions.

Moreover, it is the first implementation of a prover for explicit induction going beyond most simple structural inductions.

Furthermore, the PURE LISP THEOREM PROVER has already most of the stages of the Boyer–Moore waterfall (cf. Figure 1), and these stages occur in the final order and with the final overall behavior of throwing the formulas back to the center pool after a stage was successful in changing them.

As we have seen in Example 11 of § 5.2.6, the main weakness of the PURE LISP THEOREM PROVER is the realization of its induction rule, which ignores most of the structure of the recursive calls in the right-hand sides of recursive function definitions.[106] In the PURE LISP THEOREM PROVER, all information on this structure taken into account by the induction rule comes from the fault descriptions of previous applications of "evaluation", which drop a lot of information that is actually required for finding the proper instances for the eager instantiation of induction hypotheses required in explicit induction.

As a consequence, all induction hypotheses and conclusions of the PURE LISP THEOREM PROVER are instantiations of the input formula with mere constructor terms. Nevertheless, the PURE LISP THEOREM PROVER can generate multiple hypotheses for astonishingly complicated step cases, which go far beyond the simple ones typical for structural induction.

Although the induction stage of the PURE LISP THEOREM PROVER is pretty underdeveloped compared to the sophisticated *recursion analysis* of the later Boyer–Moore theorem provers, it somehow contains all essential later ideas in a rudimentary form, such as recursion analysis and the merging of step cases. As we have seen in Example 12, the merging procedure of the PURE LISP THEOREM PROVER is surprisingly successful.

The PURE LISP THEOREM PROVER cannot succeed, however, in the rare cases where a step case has to follow a destructor different from CAR and CDR (such as delfirst in § 3.4), or in the more general case that the arguments of the recursive calls contain recursively defined functions at the measured positions (such as the Ackermann function in Example 11).

The weaknesses and provisional procedures of the PURE LISP THEOREM PROVER we have documented help to decompose the giant leap form nothing to THM, and so fulfill our historiographic intention expressed at the beginning of § 5.2.

Especially the link between symbolic evaluation and the induction rule of explicit induction described at the end of § 5.2.1 may be considered to be crucial for the success of the entire development of recursion analysis and explicit induction.

[106] There are indications that the induction rule of the PURE LISP THEOREM PROVER had to be implemented in a hurry. For instance, on top of Page 168 of [Moore, 1973], we read on the PURE LISP THEOREM PROVER: "The case for n term induction is much more complicated, and is not handled in its full generality by the program."

5.3 Thm

Boyer and Moore never gave names to their theorem provers.[107] The names "Thm" (for "theorem prover"), "Qthm" (for "quantified Thm"), and "Nqthm" (for "new quantified Thm") were actually the directory names under which the different versions of their theorem provers were developed and maintained.[108] Qthm was never released and its development was discontinued soon after the "quantification" in Nqthm had turned out to be superior; so the name Qthm was never used in public. Until today, it seems that "Thm" appeared in publication only as a mode in Nqthm,[109] which simulates the release previous to the release of Nqthm (i.e. before "quantification" was introduced) with a logic that is a further development of the one described in [Boyer and Moore, 1979]. It was Matt Kaufmann (*1952) who started calling the prover "Nqthm", in the second half of the 1980s.[110] The name "Nqthm" appeared in publication first in [Boyer and Moore, 1988b] as a mode in Nqthm.

In this section we describe the enormous heuristic improvements documented in [Boyer and Moore, 1979] as compared to [Moore, 1973] (cf. § 5.2). In case of the minor differences of the logic described in [Boyer and Moore, 1979] and of the later released version that is simulated by the `THM` mode in Nqthm as documented in [Boyer and Moore, 1988b; 1998], we try to follow the later descriptions, partly because of their elegance, partly because Nqthm is still an available program. For this reason, we have entitled this section "Thm" instead of "The standard reference on the Boyer–Moore heuristics [Boyer and Moore, 1979]".

Note the clearness, precision, and detail of the natural-language descriptions of heuristics in [Boyer and Moore, 1979] is unique and unrivaled.[111] To the best of our knowledge, there is no similarly broad treatment of heuristics in theorem proving.

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[107] The only exception seems to be [Moore, 1975], where the Pure LISP Theorem Prover is called "the Boyer–Moore Pure LISP Theorem Prover", probably because Moore wanted to stress that, though Boyer appears in the references of [Moore, 1975] only in [Boyer and Moore, 1975], Boyer has had an equal share in contributing to the Pure LISP Theorem Prover right from the start.

[108] Cf. [Boyer, 2012].

[109] For the occurrences of "`THM`" in publications, and for the exact differences between the `THM` and `NQTHM` modes and logics, see Pages 256–257 and 308 in [Boyer and Moore, 1988b], as well as Pages 303–305, 326, 357, and 386 in the second edition [Boyer and Moore, 1998].

[110] Cf. [Boyer, 2012].

[111] In [Boyer and Moore, 1988b, p. xi] and [Boyer and Moore, 1998, p. xv] we can read about the book [Boyer and Moore, 1979]:

> "The main purpose of the book was to describe in detail how the theorem prover worked, its organization, proof techniques, heuristics, etc. One measure of the success of the book is that we know of three independent successful efforts to construct the theorem prover from the book."

From 1973 to 1981 Boyer and Moore were researchers at Xerox Palo Alto Research Center (Moore only) and — just a few miles apart — at SRI International in Menlo Park (CA). Since 1981 they were both professors at The University of Texas at Austin and scientists at Computational Logic Inc. in Austin (TX). So they could most easily meet and work together. And — just like the PURE LISP THEOREM PROVER — the provers THM and NQTHM were again developed and implemented exclusively by Boyer and Moore.[112]

The approach for developing the heuristics remained the same as before: Just as in the PURE LISP THEOREM PROVER, the heuristics of THM are still learned from the heuristics of the human mathematicians Boyer and Moore.

The logic of THM has changed a bit compared to the PURE LISP THEOREM PROVER. By means of the new *shell principle*,[113] it is now possible to define new data types by describing the *shell*, a constructor with at least one argument, each of whose arguments may have a simple type restriction, and the optional *base object*, a nullary constructor.[114] Each argument of the shell can be accessed[115] by its destructor, for which a name and a default value (for the sake of totality) have to be given in addition. The user also has to supply a name for the predicate that that recognizes[115] the objects of the new data type (as the logic remains untyped).

`NIL` has lost its elementary status and is now an element of the shell `PACK` of symbols.[116] `T` and `F` now abbreviate the nullary function calls `(TRUE)` and `(FALSE)`, respectively, which are the only Boolean values. Any argument with Boolean intention beside `F` is taken to be `T` (including `NIL`).

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[112] In both [Boyer and Moore, 1988b, p. xv] and [Boyer and Moore, 1998, p. xix] we read:

> "Notwithstanding the contributions of all our friends and supporters, we would like to make clear that ours is a very large and complicated system that was written entirely by the two of us. Not a single line of Lisp in our system was written by a third party. Consequently, every bug in it is ours alone. Soundness is the most important property of a theorem prover, and we urge any user who finds such a bug to report it to us at once."

[113] Cf. [Boyer and Moore, 1979, p. 37ff.].

[114] Note that this restriction to at most two constructors, including exactly one with arguments, is pretty uncomfortable. For instance, it neither admits simple enumeration types, such as the Boolean values, nor record types. Moreover, mutually recursive data types are not possible, such as and-or-trees, where each element is a list of or-and-trees, and vice versa, as given by the following four constructors:

$$\begin{array}{rlrl} \mathsf{empty\text{-}or\text{-}tree}: & \mathsf{or\text{-}tree}; & \mathsf{or}: & \mathsf{and\text{-}tree},\ \mathsf{or\text{-}tree} \rightarrow \mathsf{or\text{-}tree}; \\ \mathsf{empty\text{-}and\text{-}tree}: & \mathsf{and\text{-}tree}; & \mathsf{and}: & \mathsf{or\text{-}tree},\ \mathsf{and\text{-}tree} \rightarrow \mathsf{and\text{-}tree}. \end{array}$$

[115] Actually, in the jargon of [Boyer and Moore, 1979; 1988b; 1998], the destructors are called *accessor functions*, and the type predicates are called *recognizer functions*.

[116] There are the following two different declarations for the shell `PACK`: In [Boyer and Moore, 1979], the shell `CONS` is defined after the shell `PACK` because `NIL` is the default value for the destructors `CAR` and `CDR`; moreover, `NIL` is an abbreviation for `(NIL)`, which is the base object of the shell `PACK`.

In [Boyer and Moore, 1988b; 1998], however, the shell `PACK` is defined after the shell `CONS`, we have `(CAR NIL)` = `0`, the shell `PACK` has no base object, and `NIL` just abbreviates

`(PACK (CONS 78 (CONS 73 (CONS 76 0))))`.

When we discuss the logic of [Boyer and Moore, 1979], we tacitly use the shells `CONS` and `PACK` as described in [Boyer and Moore, 1988b; 1998].

Instead of discussing the shell principle in detail with all its intricacies resulting from the untyped framework, we just present the first two shells:

1. The shell `(ADD1 X1)` of the *natural numbers*, with
   - type restriction `(NUMBERP X1)`,
   - base object `(ZERO)`, abbreviated by `0`,
   - destructor[115] `SUB1` with default value `0`, and
   - type predicate[115] `NUMBERP`.
2. The shell `(CONS X1 X2)` of *pairs*, with
   - destructors `CAR` with default value `0`,
`CDR` with default value `0`, and
   - type predicate `LISTP`.

According to the shell principle, these two shell declarations add axioms to the theory, which are equivalent to the following ones:

#	Axioms Generated by Shell `ADD1`	Axioms Generated by Shell `CONS`
0.1	(NUMBERP X) = T ∨ (NUMBERP X) = F	(LISTP X) = T ∨ (LISTP X) = F
0.2	(NUMBERP (ADD1 X1)) = T	(LISTP (CONS X1 X2)) = T
0.3	(NUMBERP 0) = T	
0.4	(NUMBERP T) = F	(LISTP T) = F
0.5	(NUMBERP F) = F	(LISTP F) = F
0.6		(LISTP X) = F ∨ (NUMBERP X) = F
1	(ADD1 (SUB1 X)) = X ⇐ X ≠ 0 ∧ (NUMBERP X) = T	(CONS (CAR X) (CDR X)) = X ⇐ (LISTP X) = T
2	(ADD1 X1) ≠ 0	
3	(SUB1 (ADD1 X1)) = X1 ⇐ (NUMBERP X1) = T	(CAR (CONS X1 X2)) = X1 (CDR (CONS X1 X2)) = X2
4	(SUB1 0) = 0	
5.1	(SUB1 X) = 0 ⇐ (NUMBERP X) = F	(CAR X) = 0 ⇐ (LISTP X) = F (CDR X) = 0 ⇐ (LISTP X) = F
5.2	(SUB1 (ADD1 X1)) = 0 ⇐ (NUMBERP X1) = F	
L1[117]	(ADD1 X) = (ADD1 0) ⇐ (NUMBERP X) = F	
L2[118]	(NUMBERP (SUB1 X)) = T	

[117] Proof of Lemma L1 from 0.2, 1–2, 5.2: Under the assumption of `(NUMBERP X)` = `F`, we show `(ADD1 X)` = `(ADD1 (SUB1 (ADD1 X)))` = `(ADD1 0)`. The first step is a backward application of the conditional equation 1 via {`X` ↦ `(ADD1 X)`}, where the condition is fulfilled because of 2 and 0.2. The second step is an application of 5.2, where the condition is fulfilled by assumption.

[118] Proof of Lemma L2 from 0.1–0.3, 1–4, 5.1–5.2 by *reductio ad absurdum*:
For a counterexample `X`, we get `(SUB1 X)` ≠ `0` by 0.3, as well as `(NUMBERP (SUB1 X))` = `F` by 0.1. From the first we get `X` ≠ `0` by 4, and `(NUMBERP X)` = `T` by 5.1 and 0.1. Now we get the contradiction `(SUB1 X)` = `(SUB1 (ADD1 (SUB1 X)))` = `(SUB1 (ADD1 0))` = `0`; the first step is a backward application of the conditional equation 1, the second of L1, and the last of 3 (using 0.3).

Note that the two occurrences of "(NUMBERP X1)" in Axioms 3 and 5.2 are exactly the ones that result from the type restriction of ADD1. Moreover, the occurrence of "(NUMBERP X)" in Axiom 0.6 is allocated at the right-hand side because the shell ADD1 is declared *before* the shell CONS.

Let us discuss the axioms generated by declaration of the shell ADD1. Roughly speaking, Axioms 0.1–0.3 are return-type declarations, Axioms 0.4–0.6 are about disjointness of types, Axiom 1 and Lemma L2 imply the axiom (nat1) from § 3.3, Axioms 2 and 3 imply axioms (nat2) and (nat3), respectively. Axioms 4 and 5.1–5.2 overspecify SUB1. Note that Lemma L1 is equivalent to 5.2 under 0.2–0.3 and 1–3.

Analogous to Lemma L1, every shell forces each argument not satisfying its type restriction into behaving like the default object of the argument's destructor.

To the contrary, the arguments of the shell CONS (just as every shell argument without type restriction) are not forced like this, and so even objects of later defined shells (such as PACK) can be properly paired by the shell CONS. For instance, although NIL belongs to the shell PACK defined after the shell CONS (and so (CDR NIL) = 0),[116] we have (CAR (CONS NIL NIL)) = NIL by Axiom 3.

Nevertheless, the the shell principle also admits us to declare a shell

(CONSNAT X1 X2)

of the *lists of natural numbers only* — similar to the ones of § 3.4 — with a type predicate LISTNATP, type restrictions (NUMBERP X1), (LISTNATP X2), base object (NILNAT), and destructors CARNAT, CDRNAT with default values 0, (NILNAT).

Let us come now to the definition of new functions admissible in THM. In § 4 we have already discussed the *definition principle*[119] of THM in detail. The definition of recursive functions has not changed compared to the PURE LISP THEOREM PROVER beside that a function definition is admissible now only after a termination proof, which proceeds as explained in § 4.4. To this end, THM can apply its additional axiom of the well-foundedness of the irreflexive ordering LESSP on the natural numbers,[120] and the theorem of the well-foundedness of the lexicographic combination of two well-founded orderings.

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[119] [Boyer and Moore, 1979, p. 44f.].

[120] See Page 52f. of [Boyer and Moore, 1979] for the informal statement of this axiom on well-foundedness of LESSP.

Because THM is able to prove (LESSP X (ADD1 X)), well-foundedness of LESSP would imply — together with Axiom 1 and Lemma L2 — that THM admits only the standard model of the natural numbers, as explained in Note 34.

Matt Kaufmann, however, was so kind and made clear in a private e-mail communication that non-standard models are not excluded, because the statement "We assume LESSP to be a well-founded relation." of [Boyer and Moore, 1979, p. 53] is actually to be read as the well-foundedness of the formal definition of § 3.1 with the *additional assumption* that the predicate Q must be definable in THM.

Note that in the argument of Note 34, it is not possible to replace the reflexive transitive closure of the successor relation s with the THM-definable predicate

$$\{\ \texttt{Y}\ \mid\ \texttt{(NUMBERP Y)} = \texttt{T}\ \ \wedge\ \ \texttt{((LESSP Y X)} = \texttt{T}\ \vee\ \texttt{Y} = \texttt{X)}\ \},$$

because (by the THM-analog of axiom (lessp2′) of Example 12 in § 5.2.6) this predicate will contain 0 as a minimal element even for a non-standard natural number X; thus, LESSP is a *proper* super-relation of the reflexive transitive closure of s.

Let us now again follow the Boyer–Moore waterfall (cf. Figure 1) and sketch how the stages of the waterfall are realized in THM in comparison to the PURE LISP THEOREM PROVER. The most relevant change is that previously proved theorems have an effect on the current proof, provided that they have been activated for the purpose they can serve, in which case they are a applied in the reverse order of activation. This means that the performance of the the prover is not doomed to slow down when progressing to less and less elementary results, provided that the user develops the theory stepwise and activates the theorems properly. Moreover, there is also the chance to help the theorem prover directly with a new lemma that becomes necessary during a proof, but is not found by the prover. This is actually the most crucial application of lemmas, because humans have a truly fascinating ability to *understand* a theory *semantically*, and because they can apply this ability to *"see"* why a sub-goal happens to be true. This ability to see the missing lemmas is actually the only aspect where humans still top the machine in the heuristics of typical induction proofs.

5.3.1 Simplification in THM

Regarding the PURE LISP THEOREM PROVER, we have been discussing simplification before in § 5.2.1.

Simplification in THM is covered in Chapters VI–IX of [Boyer and Moore, 1979], and the reader interested in the details is strongly encouraged to read these very well-written descriptions of heuristic procedures for simplification.

To compensate for the extra complication of the untyped approach in THM, which has a much higher number of interesting soft types than the PURE LISP THEOREM PROVER, soft-typing rules are computed for each new function symbol based on types that are disjunctions (actually: bit-vectors) of the following disjoint types: one for `T`, one for `F`, one for each shell, and one for objects not belonging to any of these.[121] These soft-typing rules are pervasively applied in all stages of the theorem prover, which we cannot discuss here in detail. Some of these rules can be expressed in the LISP logic language as a theorem and presented in this form to the human users. Let us see two examples on this.

EXAMPLE 13. *(continuing Example 6 of § 4.1)*
As THM knows `(NUMBERP (FIX X))` and `(NUMBERP (ADD1 X))`, it produces the theorem `(NUMBERP (PLUS X Y))` immediately after the termination proof for the definition of `PLUS` in Example 6. Note that this would neither hold in case of non-termination of `PLUS`, nor if there were a simple `Y` instead of `(FIX Y)` in the definition of `PLUS`. In the latter case, THM would only register that the return-type of `PLUS` is among `NUMBERP` and the types of its second argument `Y`.

EXAMPLE 14. As THM knows that the type of `APPEND` is among `LISTP` and the type of its second argument, it produces the theorem `(LISTP (FLATTEN X))` immediately after the termination proof for the following definition:

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[121] See Chapter VI in [Boyer and Moore, 1979].

```
(FLATTEN X) = (IF (LISTP X)
                  (APPEND (FLATTEN (CAR X)) (FLATTEN (CDR X)))
                  (CONS X NIL))
```
□

The standard representation of an expression of propositional logic has improved from the multifarious LISP representation of the PURE LISP THEOREM PROVER toward today's standard of clausal representation. A *clause* is a disjunctive list of literals. *Literals*, however, deviating from the standard definition of being optionally negated atoms, are just LISP terms here, because every LISP function can be seen as a predicate.

This means that the "water" of the waterfall now consists of clauses, and the conjunction of all clauses in the waterfall represents the proof task.

Based on this clausal representation, we find a full-fledged description of *contextual rewriting* in Chapter IX of [Boyer and Moore, 1979], and its applications in Chapters VII–IX. This description comes some years before the term "contextual rewriting" became popular in automated theorem proving, and the term does not occur yet. It is probably the first description of contextual rewriting in the history of logic, unless one counts the rudimentary contextual rewriting in the "reduction" of the PURE LISP THEOREM PROVER as such.[122]

As indicated before, the essential idea of contextual rewriting is the following: While focusing on one literal of a clause for simplification, we can assume all other literals — the *context* — to be false, simply because the literal in focus is irrelevant otherwise. Especially useful are literals that are negated equations, because they can be used as a ground term-rewrite system. A non-equational literal t can always be taken to be the negated equation $(t \neq \mathtt{F})$. The free universal variables of a clause have to be treated as constants during contextual rewriting.[123]

To bring contextual rewriting to full power, all occurrences of the function symbol `IF` in the literals of a clause are expelled from the literals as follows. If the condition of an `IF`-expression can be simplified to be definitely false `F` or definitely true (i.e. non-`F`, e.g. if `F` is not set in the bit-vector as a potential type), then the `IF`-expression is replaced with its respective case. Otherwise, after the `IF`-expression could not be removed by those rewrite rules for `IF` whose soundness depends on termination,[124] it is moved to the top position (outside-in), by replacing each case with itself in the `IF`'s context, such that the literal $C[\mathtt{(IF}\ t_0\ t_1\ t_2\mathtt{)}]$ is intermediately replaced with $\mathtt{(IF}\ t_0\ C[t_1]\ C[t_2]\mathtt{)}$, and then this literal splits its clause in two: one with the two literals $\mathtt{(NOT}\ t_0\mathtt{)}$ and $C[t_1]$ in place of the old one, and one with t_0 and $C[t_2]$ instead.

[122] Cf. § 5.2.1.

[123] This has the advantage that we could take any well-founded ordering that is total on ground terms and run the terminating ground version of a Knuth–Bendix completion procedure [Knuth and Bendix, 1970] for all literals in a clause representation that have the form $l_i \neq r_i$, and replace the literals of this form with the resulting confluent and terminating rewrite system and normalize the other literals of the clause with it. Note that this transforms a clause into a logically equivalent one. None of the Boyer–Moore theorem provers does this, however.

[124] These rewrite rules whose soundness depends on termination are `(IF X Y Y) = Y`; `(IF X X F) = X`; and for Boolean `X`: `(IF X T F) = X` tested for applicability in the given order.

THM already eagerly removes variables in solved form: If the variable `X` does not occur in the term t, but the literal $(\mathtt{X} \neq t)$ occurs in a clause, then we can remove that literal after rewriting all occurrences of `X` in the clause to t. This removal is a logical equivalence transformation, because `X` is universally quantified and so $(\mathtt{X} \neq t)$ must be false because it implies $(t \neq t)$.

It remains to describe the rewriting with function definitions and with lemmas activated for rewriting, where the context of the clause is involved again.

Non-recursive function definitions are always unfolded by THM. Recursive function definitions are treated in a way very similar to that of the PURE LISP THEOREM PROVER. The criteria on the unfolding a function call of a recursively defined function f still depend solely on the terms introduced as arguments in the recursive calls of f in the body of f, which are accessed during the simplification of the body. But now the unfolding is rejected not anymore by destructor terms introduced by the body and not removed by simplification, but actually by the occurrence of terms that (after simplification) do not occur as sub-terms elsewhere in the clause. This means that a new term (`CDR` t) in the simplified argument of a recursive function call, where `CDR` originates in an argument of that recursive call in the body and not in the arguments of the tentative unfolding, does not block unfolding anymore if (`CDR` t) occurs already as a sub-term elsewhere in the clause. There are also further less important criteria to unblock unfolding of recursive function definitions, such an increase of the number arguments that are constructor ground terms.[125]

Rewriting with lemmas that have been proved and then activated for rewriting — so-called *rewrite lemmas* — differs from rewriting with recursive function definitions mainly in one aspect: There is no need to block them because the user has activated them explicitly for rewriting, and because rewrite lemmas have the form of conditional equations instead of unconditional ones. Simplification with lemmas activated for rewriting and the heuristics behind the process are nicely described in [Schmidt-Samoa, 2006c], where a rewrite lemma is not just activated for rewriting, but where the user can also mark the condition literals on how they should be dealt with. In THM there is no lazy rewriting with rewrite lemmas, i.e. no case splits are introduced to be able to apply the lemma. This means that all conditions of the rewrite lemma have to be shown to be fulfilled in the current context. As a partly compensation there is a process of backward chaining, i.e. the conditions can be shown to be fulfilled by the application of further conditional rewrite lemmas. The termination of this backward chaining is achieved by avoiding the generation of conditions into which the previous conditions can be homeomorphically embedded.[126] There are provisions to instantiate extra variables of conditions eagerly, which is necessary because there are no existential variables.[127] Non-termination of evaluation via rewrite lemmas is achieved by simple term orderings, which in particular avoid looping with commutative lemmas.[128]

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[125] See Page 119 of [Boyer and Moore, 1979] for the details and the remaining criteria.
[126] See Page 109ff. of [Boyer and Moore, 1979] for the details.
[127] See Page 111f. of [Boyer and Moore, 1979] for the details.

5.3.2 Destructor Elimination in Thm

We have already seen constructors such as s (in Thm: `ADD1`) and cons (`CONS`) with the destructors p (`SUB1`) and car (`CAR`), and cdr (`CDR`), respectively.

EXAMPLE 15 (From Constructor to Destructor Style and back).
We have presented several function definitions both in constructor and in destructor style. Let us do careful and generalizable equivalence transformations (reverse step justified in parentheses) starting with the constructor-style rule ($\mathsf{ack3}$) of § 3.3:

$$\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) = \mathsf{ack}(x,\mathsf{ack}(\mathsf{s}(x),y)).$$

Introduce (delete) the solved variables x' and y' for the constructor terms $\mathsf{s}(x)$ and $\mathsf{s}(y)$ occurring on the left-hand side, respectively, and add (delete) two further conditions by applying the definition ($\mathsf{p1}'$) (cf. § 3.3) twice.

$$\mathsf{ack}(\mathsf{s}(x),\mathsf{s}(y)) = \mathsf{ack}(x,\mathsf{ack}(\mathsf{s}(x),y)) \;\Leftarrow\; \left(\begin{array}{rl} & x' = \mathsf{s}(x) \;\wedge\; \mathsf{p}(x') = x \\ \wedge & y' = \mathsf{s}(y) \;\wedge\; \mathsf{p}(y') = y \end{array}\right).$$

Normalize conclusion with leftmost equations of the condition from right to left (left to right).

$$\mathsf{ack}(x',y') = \mathsf{ack}(x,\mathsf{ack}(x',y)) \;\Leftarrow\; \left(\begin{array}{rl} & x' = \mathsf{s}(x) \;\wedge\; \mathsf{p}(x') = x \\ \wedge & y' = \mathsf{s}(y) \;\wedge\; \mathsf{p}(y') = y \end{array}\right).$$

Normalize conclusion with rightmost equations of the condition from right to left (left to right).

$$\mathsf{ack}(x',y') = \mathsf{ack}(\mathsf{p}(x'),\mathsf{ack}(x',\mathsf{p}(y'))) \;\Leftarrow\; \left(\begin{array}{rl} & x' = \mathsf{s}(x) \;\wedge\; \mathsf{p}(x') = x \\ \wedge & y' = \mathsf{s}(y) \;\wedge\; \mathsf{p}(y') = y \end{array}\right).$$

Add (Delete) two conditions by applying axiom ($\mathsf{nat2}$) twice.

$$\mathsf{ack}(x',y') = \mathsf{ack}(\mathsf{p}(x'),\mathsf{ack}(x',\mathsf{p}(y'))) \;\Leftarrow\; \left(\begin{array}{rl} & x' = \mathsf{s}(x) \;\wedge\; \mathsf{p}(x') = x \;\wedge\; x' \neq \mathsf{0} \\ \wedge & y' = \mathsf{s}(y) \;\wedge\; \mathsf{p}(y') = y \;\wedge\; y' \neq \mathsf{0} \end{array}\right).$$

Delete (Introduce) the leftmost equations of the condition by applying lemma ($\mathsf{s1}'$) (cf. § 3.3) twice, and delete (introduce) the solved variables x and y for the destructor terms $\mathsf{p}(x')$ and $\mathsf{p}(y')$ occurring in the left-hand side of the equation in the conclusion, respectively.

$$\mathsf{ack}(x',y') = \mathsf{ack}(\mathsf{p}(x'),\mathsf{ack}(x',\mathsf{p}(y'))) \;\Leftarrow\; x' \neq \mathsf{0} \wedge y' \neq \mathsf{0}.$$

Up to renaming of the variables, this is the destructor-style rule ($\mathsf{ack3}'$) of Example 11 (cf. § 5.2.6). ☐

Our data types are defined inductively over constructors.[129] Therefore constructors play the main rôle in our semantics, and practice shows that step cases of induction proofs work out much better with constructors than with the respective destructors, which are secondary (i.e. defined) operators in our semantics and have a more complicated case analysis in application.

[128] See Page 104f. of [Boyer and Moore, 1979] for the details.

[129] Here the term “inductive” means the following: We start with the empty set and take the smallest fixpoint under application of the constructors, which contains only finite structures, such as natural numbers and lists. Co-inductively over the destructors we would obtain different data types, because we start with the universal class and obtain the greatest fixed point under inverse application of the destructors, which typically contains infinite structures. For instance, for the unrestricted destructors car, cdr of the list of natural numbers $\mathsf{list}(\mathsf{nat})$ of § 3.4, we co-inductively obtain the data type of infinite streams of natural numbers.

There are two further positive effects of destructor elimination:

1. It tends to standardize the representation of a clauses in the sense that the numbers of occurrences of identical sub-terms tend to be increased.
2. Destructor elimination also brings the sub-term property in line with the sub-structure property; e.g., `Y` is both a sub-structure of `(CONS X Y)` and a sub-term of it, whereas `(CDR Z)` is a sub-structure of `Z` in case of `(LISTP Z)`, but not a sub-term of `Z`.

Both effects improve the chances that the clause passes the follow-up stages of cross-fertilization and generalization with good success.[130]

For these reasons, the PURE LISP THEOREM PROVER did induction using step cases with constructors, such as $P(\mathsf{s}(x)) \Leftarrow P(x)$. THM, however, does induction using step cases with destructors, such as

$$\big(\ P(x)\ \Leftarrow\ P(\mathsf{p}(x))\ \big)\ \Leftarrow\ x \neq \mathsf{0}.$$

So destructor elimination was not so urgent in the PURE LISP THEOREM PROVER, simply because there were less destructors around. Indeed, the stage "destructor elimination" does not exist in the PURE LISP THEOREM PROVER.

THM does not do induction with constructors because there are generalized destructors that do not have a straightforward constructor, and because the induction rule of explicit induction has to fix in advance whether the step cases are destructor or constructor style. So with destructor style in all step cases and in all function definitions, explicit induction and recursion in THM chooses the style that is always applicable.

EXAMPLE 16 (A Generalized Destructor Without Constructor).
A generalized destructor that does not have a straightforward constructor is the function delfirst defined in § 3.4. To verify the correctness of a deletion-sort algorithm based on delfirst, a useful step case for an induction proof[131] is of the form

$$\big(\ P(l)\ \Leftarrow\ P(\mathsf{delfirst}(\mathsf{max}(l), l))\ \big)\ \Leftarrow\ l \neq \mathsf{nil}.$$

A constructor version of this induction scheme would need something like an insertion function with an additional free variable for a natural number indicating the position of insertion, resulting in a step case that is too complicated for the proof to be successful. □

Proper destructor functions take only one argument. The generalized destructor delfirst we have seen in Example 16 has actually two arguments; the second one is the *proper destructor argument* and the first is a *parameter*. After the elimination of set of destructors, the terms at the parameter positions of the destructors are typically still present, where terms at the proper destructor argument are removed.

EXAMPLE 17 (Division with Remainder as a pair of Generalized Destructors).
In case of $y \neq \mathsf{0}$, we can construct each natural number in the form of $(q * y) + r$ with $\mathsf{lessp}(r, y) = \mathsf{true}$. The related generalized destructors are the the quotient

[130] See Page 114ff. of [Boyer and Moore, 1979] for a nice example for the advantage of destructor elimination for cross-fertilization.
[131] See Page 143f. of [Boyer and Moore, 1979].

$\mathsf{div}(x,y)$ of x by y, and its remainder $\mathsf{rem}(x,y)$. Note that in both functions, the first argument is the proper destructor argument and the second the parameter, which must not be $\mathsf{0}$. The rôle that the definition $(\mathsf{p1}')$ and the lemma $(\mathsf{s1}')$ of § 3.3 have in Example 15 (and which the definitions $(\mathsf{car1}')$, $(\mathsf{cdr1}')$ and the lemma $(\mathsf{cons1}')$ of § 3.4 have in the equivalence transformations between constructor and destructor style for lists) is here taken by the following definitions of div and rem and the following lemma on the constructor $\lambda q, r.\ ((q*y)+r)$:

$$
\begin{array}{lllll}
(\mathsf{div1}') & \mathsf{div}(x,y) & = q & \Leftarrow & y \neq \mathsf{0} \ \wedge\ (q*y)+r = x \ \wedge\ \mathsf{lessp}(r,y) = \mathsf{true} \\
(\mathsf{rem1}') & \mathsf{rem}(x,y) & = r & \Leftarrow & y \neq \mathsf{0} \ \wedge\ (q*y)+r = x \ \wedge\ \mathsf{lessp}(r,y) = \mathsf{true} \\
(+9') & (q*y)+r & = x & \Leftarrow & y \neq \mathsf{0} \ \wedge\ q = \mathsf{div}(x,y) \ \wedge\ r = \mathsf{rem}(x,y)
\end{array}
$$

If we have a clause with the literal $y=0$, in which the destructor terms $\mathsf{div}(x,y)$ or $\mathsf{rem}(x,y)$ occur, we can — just as in the of Example 15 (reverse direction) — introduce the new literals $\mathsf{div}(x,y) \neq q$ and $\mathsf{rem}(x,y) \neq r$ for fresh q, r, and apply lemma $(+9')$ to introduce also the literal $x \neq (q*y)+r$. Then we can normalize with the first two literals, and afterwards with the third. Then all occurrences of $\mathsf{div}(x,y)$, $\mathsf{rem}(x,y)$, and x are gone.[132] □

To enable the form of elimination of generalized destructors described in Example 17, THM admits the user to activate lemmas of the form $(\mathsf{s1}')$, $(\mathsf{cons1}')$, or $(+9')$ as *elimination lemmas* to perform destructor elimination. In clause representation, this form is in general the following: The first literal of the clause is of the form $(t^{\mathrm{c}} = x)$, where x is a variable which does not occur in the constructor term t^{c}. Moreover, t^{c} contains some distinct variables $y_0, \ldots, y_n$, which occur only on the left-hand sides of the first literal and of the last $n{+}1$ literals of the clause, which are of the form $(y_0 \neq t^{\mathrm{d}}_0)$, $\ldots$, $(y_n \neq t^{\mathrm{d}}_n)$, for distinct (generalized) destructor terms $t^{\mathrm{d}}_0, \ldots, t^{\mathrm{d}}_n$.[133]

The idea of application for destructor elimination in a given clause is, of course, the following: If the non-mentioned literals can be matched to literals of the given clause, and if t^{d}_0, $\ldots$, t^{d}_n occur in the given clause as sub-terms, rewrite all their occurrences with $(y_0 \neq t^{\mathrm{d}}_0)$, $\ldots$, $(y_n \neq t^{\mathrm{d}}_n)$ from right to left and then use the first literal of the elimination lemma from right to left for further normalization.[134]

[132] For a nice, but non-trivial example on why proofs tend to work out much easier after this transformation, see Page 135ff. of [Boyer and Moore, 1979].

[133] THM adds one more restriction, namely that the generalized destructor terms have to consist of a function symbol applied to a list containing exactly the variables of the clause, beside $y_0, \ldots, y_n$.

Moreover, note that THM actually does not use our flattened form of the elimination lemmas, but the one that results from replacing each y_i in the clause with t^{d}_i, and then removing the literal $(y_i \neq t^{\mathrm{d}}_i)$. Thus, THM would accept as elimination lemmas only the non-flattened versions of our elimination lemmas, such as $(\mathsf{s1})$ instead of $(\mathsf{s1}')$ (cf. § 3.3) and such as $(\mathsf{cons1})$ instead of $(\mathsf{cons1}')$ (cf. § 3.4).

[134] If we add the last literals of the elimination lemma to the given clause, use them for contextual rewriting, and remove them only if this can be achieved safely via application of the definitions of the destructors (as we could do in all our examples), then the elimination of destructors is an equivalence transformation. Destructor elimination in THM, however, may (over-) generalize the conjecture, because these last literals are not present in the non-flattened elimination lemma of THM and its variables y_i are actually introduced in THM by generalization. Thus, instead of trying to delete the last literals of our deletion lemmas safely, THM never adds them.

In the destructor elimination of THM, the most simple destructor (actually: the one defined first) is removed first, possibly in several steps. Then the clauses are re-introduced to the waterfall, but — to avoid looping — the terms with variables introduced by destructor elimination are blocked for simplification until the next induction.[135]

5.3.3 (Cross-) Fertilization in THM

This stage has already been described in § 5.2.3 because there is no noticeable difference between the PURE LISP THEOREM PROVER and THM here, beside some heuristic fine tuning.[136]

5.3.4 Generalization in THM

THM adds only one new rule to the universally applicable heuristic rules for generalization on a term t mentioned already in § 3.8. This new rule makes sense in particular after the preceding stage of destructor elimination: Never generalize on a term t consisting in the application of a destructor function symbol to a list of distinct variables!

The main improvement of generalization in THM over the PURE LISP THEOREM PROVER, however, is the following: Suppose again that the term t is to be replaced at all its occurrences in the clause $T[t]$ with the fresh variable z. Now the the predicate p that is to express essential properties of the term t symbolically, now added by two new means, different from the synthesis algorithm of the PURE LISP THEOREM PROVER:

1. Assuming all literals of the clause $T[t]$ to be false, the bit-vector describing the soft type of t is computed and if only one bit is set, then, for the respective type predicate, say the bit expressing `NUMBERP`, then a new literal is added to the clause, such as `(NOT (NUMBERP` t`))`.
2. The user can activate certain theorems as *generalization lemmas*; such as `(SORTEDP (SORT X))` for a sorting function `SORT`; and if `(SORT X)` matches t, the respective instance of `(NOT (SORTEDP (SORT X)))` is added to $T[t]$.[137] In general, for the addition of such a literal `(NOT` t''`)`, a proper sub-term of a generalization lemma t' must match t.[138]

[135] See Page 139 of [Boyer and Moore, 1979]. In general, for more sophisticated details of destructor elimination in THM, we have to refer the reader to Chapter X of [Boyer and Moore, 1979].

[136] See Page 149 of [Boyer and Moore, 1979].

[137] Cf. Note 103.

[138] Moreover, the literal is actually added to the generalized clause only if the top function symbol of t does not occur in the literal anymore after replacing t with x. This means that, for a generalization lemma `(EQUAL (FLATTEN (GOPHER X)) (FLATTEN X))`, the literal `(NOT (EQUAL (FLATTEN (GOPHER` t'''`)) (FLATTEN` t'''`)))` is added to $T[t]$ in case of t being of the form `(GOPHER` t'''`)`, but not in case of t being of the form `(FLATTEN` t'''`)`, where the first occurrence of `FLATTEN` is not removed by the generalization. See Page 156f. of [Boyer and Moore, 1979] for the details.

5.3.5 Elimination of Irrelevance in Thm

Before we come to the next stage "induction", we should remove irrelevant literals from clauses. After generalization, this is again a stage that may turn a valid clause into an invalid one. The main reason for taking this risk here is that the subsequent heuristic procedures for induction assume all literals to be relevant, and so they get confused by the presence of irrelevant literals and suggest the wrong step cases, which results in a failure of the induction proof. Moreover, if all literals seem to be irrelevant, then we know that we are going to prove a probably invalid clause, for which we should not do a costly induction, but had better ask the user to check whether he has supplied the theorem prover with an invalid lemma, possibly missing a side condition.[139]

Let us call two literals *connected* if they both have an occurrence of the same variable. Consider the partition of a clause into its equivalence classes w.r.t. the transitive closure of connectedness. If we have more than one equivalence class in a clause, this is an alarm signal for irrelevance: if the original clause is valid, then a sub-clause consisting only of the literals of one of these equivalence classes must be valid as well. This is a consequence of the logical equivalence of $\forall x.\ (A \vee B)$ with $A \vee \forall x.\ B$, provided that x does not occur in A. Then we should remove one of the irrelevant equivalence classes after the other from the original clause. To this end, Thm has two heuristic tests for irrelevance.

1. An equivalence class of literals is irrelevant if it does not contain any properly recursive function symbol.

   Based on the assumption that the previous stages of the waterfall are sufficiently powerful, the justification for this heuristic test is the following: If the clause of the equivalence class were valid, then the previous stages of the waterfall should already have established the validity of this equivalence class.

2. An equivalence class of literals is irrelevant if it consists of only one literal and if this literal is the application of a properly recursive function to a list of distinct variables.

   Based on the assumption that the soft typing rules are sufficiently powerful and that the user has not defined a tautological, but tricky predicate,[140] the justification for this heuristic test is the following: The bit-vector of this literal must contain the singleton type of `F`; otherwise the validity of the literal and the clause would have been recognized by the stage "simplification". This means that `F` is most probably a possible value for some combination of arguments.

---

[139]See Page 160f. of [Boyer and Moore, 1979] for a typical example of this.

[140]This assumption is critical because it often occurs that updated program code contains recursive predicates that are actually trivially true, but very tricky. See § 3.2 of [Wirth, 2004] for such an example.

5.3.6 Induction in THM *as compared to the* PURE LISP THEOREM PROVER

As we have seen in § 5.2.6, the *recursion analysis* in the PURE LISP THEOREM PROVER is only rudimentary. Indeed, the whole information on the body of the recursive function definitions comes out of the poor feedback of the "evaluation" procedure of the simplification stage of the PURE LISP THEOREM PROVER. Roughly speaking, this information consists only in the two facts

1. that a destructor symbol occurring as an argument of the recursive function call in the body is not removed by the "evaluation" procedure in the context of the current goal and in the local environment, and

2. that it is not possible to derive that this recursive function call is unreachable in this context and environment.

In THM, however, the first part of recursion analysis is done at *definition time*, i.e. at the time the function is defined, and applied at *proof time*, i.e. at the time the induction rule produces the base and step cases. Surprisingly, there is no reachability analysis for the recursive calls in this second part of the recursion analysis in THM. While the information in Item 1 is thoroughly improved as compared to the PURE LISP THEOREM PROVER, the information in Item 2 is partly weaker because all recursive function calls are assumed to be reachable during recursion analysis. The overwhelming success of THM means that the heuristic decision to abandon reachability analysis in THM was appropriate.[141]

5.3.7 Induction Templates generated by Definition-Time Recursion Analysis

The first part of recursion analysis in THM consists in a termination analysis of every recursive function at the time of its definition. The system does not only look for one termination proof that is sufficient for the admissibility of the function definition, but actually looks through all termination proofs in a finite search space and gathers from them all information required for justifying the termination of the recursive function definition, as well as for justifying the soundness and for improving the feasibility of the step cases to be generated by the induction rule.

To this end, THM constructs valid induction templates very similar to our description in § 4.4.[142] Let us approach the idea of a valid induction template with some typical examples, which are actually the templates for the constructor-style examples of § 4.4, but now for the destructor-style definitions of lessp and ack, because THM admits only destructor-style definitions.

[141] Note that in most cases the step formula of the reachable cases works somehow in THM, as long as no better step case was canceled due to unreachable step cases, which, of course, are trivial to prove, simply because their condition is false. Moreover, note that, contrary to *descente infinie* which can make do with the first part of recursion analysis, the heuristics of explicit induction have to guess the induction steps eagerly, which is always a fault-prone procedure, to be corrected by additional induction proofs, as we have seen in Example 4 of § 3.7.1.

EXAMPLE 18 (Two Induction Templates with different Measured Positions).
For the ordering predicate lessp as defined by $(\mathsf{lessp1'}\text{–}3')$ in Example 12 of § 5.2.6, we get two induction templates with the sets of measured positions $\{1\}$ and $\{2\}$, respectively, both for the well-founded ordering $\lambda x, y.\ (\mathsf{lessp}(x, y) = \mathsf{true})$. The first template has the weight term (1) and the relational description

$$\{\ (\ \mathsf{lessp}(x, y),\ \ \{\{\mathsf{lessp}(\mathsf{p}(x), \mathsf{p}(y))\}\},\ \ x \neq 0\ )\ \}.$$

The second one has the weight term (2) and the relational description

$$\{\ (\ \mathsf{lessp}(x, y),\ \ \{\{\mathsf{lessp}(\mathsf{p}(x), \mathsf{p}(y))\}\},\ \ y \neq 0\ )\ \}.$$

□

EXAMPLE 19 (One Induction Template with Two Measured Positions).
For the Ackermann function ack as defined by $(\mathsf{ack1'}\text{–}3')$ in Example 11 of § 5.2.6, we get only one appropriate induction template. The set of its measured positions is $\{1, 2\}$, because of the weight function $\mathsf{cons}((1), \mathsf{cons}((2), \mathsf{nil}))$ (in Thm actually: `(CONS` x y`)`) in the well-founded lexicographic ordering

$$\lambda l, k.\ (\mathsf{lexlimless}(l, k, \mathsf{s}(\mathsf{s}(\mathsf{s}(0)))) = \mathsf{true}).$$

The relational description has two elements: For the equation $(\mathsf{ack2'})$ we get

$$(\ \mathsf{ack}(x, y),\ \ \{\mathsf{ack}(\mathsf{p}(x), \mathsf{s}(0))\},\ \ x \neq 0\ ),$$

and for the equation $(\mathsf{ack3'})$ we get

$$(\ \mathsf{ack}(x, y),\ \ \{\mathsf{ack}(x, \mathsf{p}(y)),\ \ \mathsf{ack}(\mathsf{p}(x), \mathsf{ack}(x, \mathsf{p}(y)))\},\ \ x \neq 0 \wedge y \neq 0\ ).$$

□

To find valid induction templates automatically by exhaustive search, Thm admits the user to activate certain theorems as *“induction lemmas”*. An induction lemma consists of the application of a well-founded relation to two terms with the same top function symbol w, playing the rôle of the weight term; plus a condition without extra variables, which is used to generate the case conditions of the induction template. Moreover, the arguments of the application of w occurring as the second argument must be distinct variables in Thm, mirroring the left-hand side of its function definitions in destructor style.

Certain induction lemmas are generically activated when a shell is declared. Such an induction lemma generated for the shell `CONS`, which is roughly

```
(LESSP (COUNT (SUB1 X)) (COUNT X))   ⇐   (NOT (ZEROP X)).
```

suffices for generating the two templates of Example 18. Note that `COUNT`, playing the rôle of w here, is a special function in Thm, which is generically extended by every shell declaration in an object-oriented style for the elements of the new shell. On the natural numbers here, `COUNT` is the identity. On other shells, `COUNT` is defined similar to our function count from § 3.4.[143]

---

[142] Those parts of the condition of the equation that contain the new function symbol f_1 must be ignored in the case conditions of the induction template because the definition of the function f is admitted in Thm only *after* it has passed the termination proof.

That Thm ignores the governing conditions that contain the new function symbol f_1 is described in the 2nd paragraph on Page 165 of [Boyer and Moore, 1979]. Moreover, an example for this is the definition of `OCCUR` on Page 166 of [Boyer and Moore, 1979].

After one successful termination proof, however, the function can be admitted in Thm, and the system QuodLibet does not even require termination for admissibility. In any case, these conditions could then actually be admitted in the templates. So the actual reason why Thm ignores these conditions in the templates is that it generates the templates with the help of previously proved *induction lemmas*, which, of course, cannot contain the new function yet.

5.3.8 Proof-Time Recursion Analysis in THM

The induction rule uses the information from the induction templates as follows: For each recursive function occurring in the input formula, all *applicable* induction templates are retrieved and turned into *induction schemes* as described in § 4.5. Those induction schemes that are *subsumed* by others are deleted; the remaining schemes are *merged* into new ones with a higher score, and finally, after the *failed* schemes are deleted, the scheme with the highest score will be used by the induction rule to generate the base and step cases.

EXAMPLE 20 (Applicable Induction Templates).
Let us consider the conjecture (ack4) from § 3.3. From the three induction templates of Examples 18 and 19, only the second one of Example 18 is not applicable because the second position of lessp (which is the only measured position of that template) is changeable, but filled by the term $\mathsf{ack}(x, y)$. □

EXAMPLE 21 (Induction Schemes).
The two induction templates of Example 20 applicable to (ack4) are augmented to become in their respective induction schemes as follows:

The first template for lessp of Example 18 is augment by the position set $\{1\}$, containing the top position of the left-hand side of (ack4) of § 3.3 and the pair $\big(\,\{\mathsf{lessp}(\mathsf{p}(x), \mathsf{p}(y))\},\;\; x \neq \mathsf{0}\,\big)$ of the template becomes

$$\big(\,\{(\mathsf{lessp}(\mathsf{p}(y), \mathsf{p}(\mathsf{ack}(x, y))), \mu_1)\},\;\; y \neq \mathsf{0}\,\big),$$

where $\mu_1 = \{x{\mapsto}x,\;\; y{\mapsto}\mathsf{p}(y)\}$, and the score is $\frac{1}{2}$. This can be seen as follows: The substitution called σ in the above discussion is $\{x{\mapsto}y,\;\; y{\mapsto}\mathsf{ack}(x, y)\}$. So the constraint for the first (measured) position is $y\mu_1 = \mathsf{p}(y)$, and the constraint for the second (unmeasured) position is $\mathsf{ack}(x, y)\mu_1 = \mathsf{p}(\mathsf{ack}(x, y))$, which cannot be achieved.

The template for ack of Example 19 is augmented by the position set $\{1.2\}$. The pair $\big(\;\{\mathsf{ack}(\mathsf{p}(x), \mathsf{s}(\mathsf{0}))\},\;\; x \neq \mathsf{0}\;\big)$ (generated by the equation (ack2′)) becomes

$$\big(\;\{(\mathsf{ack}(\mathsf{p}(x), \mathsf{s}(\mathsf{0})), \mu'_{1,1})\},\;\; x \neq \mathsf{0}\;\big).$$

where $\mu'_{1,1} = \{x{\mapsto}\mathsf{p}(x),\;\; y{\mapsto}\mathsf{s}(\mathsf{0})\}$. This can be seen as follows: The substitution called σ in the above discussion is $\{x{\mapsto}x,\;\; y{\mapsto}y\}$. So the constraints for the (measured) positions are $x\mu'_{1,1} = \mathsf{p}(x)$ and $y\mu'_{1,1} = \mathsf{s}(\mathsf{0})$.

Moreover, the pair $\big(\;\{\mathsf{ack}(x, \mathsf{p}(y)),\;\; \mathsf{ack}(\mathsf{p}(x), \mathsf{ack}(x, \mathsf{p}(y)))\},\;\; x \neq \mathsf{0} \wedge y \neq \mathsf{0}\;\big)$ (generated by the equation (ack3′)) becomes

$$\big(\;\{(\mathsf{ack}(x, \mathsf{p}(y)), \mu'_{2,1}),\;\; (\mathsf{ack}(\mathsf{p}(x), \mathsf{ack}(x, \mathsf{p}(y))), \mu'_{2,2})\},\;\; x \neq \mathsf{0} \wedge y \neq \mathsf{0}\;\big),$$

where $\mu'_{2,1} = \{x{\mapsto}x,\;\; y{\mapsto}\mathsf{p}(y)\}$, and $\mu'_{2,2} = \{x{\mapsto}\mathsf{p}(x),\;\; y{\mapsto}\mathsf{ack}(x, \mathsf{p}(y))\}$.

The score is $\frac{4}{4} = 1$. □

An induction scheme subsumed by another induction scheme adds its set of positions in the input formula and its score to the subsumer's, respectively, and is then deleted.

[143] For more details on the recursion analysis a definition time in THM, see Page 180ff. of [Boyer and Moore, 1979].

The most important case of subsumption are schemes that are identical except for their position in the input formula.

The more general case is that the subsumer provides the essential structure of the subsumee. More precisely, this more general case of subsumption requires the following: The case conditions of the subsumee can be injectively mapped to case conditions of the subsumer, such that each case condition C is mapped to a condition C' that differs only in the addition of conjunctive elements, and such that the substitutions $\{\ \mu_j \mid j \in J\ \}$ belonging to C can be injectively mapped to the substitutions $\{\ \mu'_j \mid j \in J \uplus J'\ \}$ belonging to C', such that for each $x \in \mathrm{dom}(\mu_j)$ we have $x \in \mathrm{dom}(\mu'_j)$, $x\mu_j$ is a subterm of $x\mu'_j$, and $x\mu_j = x$ implies $x\mu'_j = x$.

For instance, in Example 21, the induction scheme for lessp is subsumed by the induction scheme for ack because the pair of the scheme for lessp is subsumed be the second pair for ack, more precisely by $\mu'_{2,1}$. So the scheme for lessp is deleted and the scheme for ack is updated to have the position set $\{1,\ 1.2\}$ and the score $\frac{3}{2}$.

It is remarkable that the well-founded relation that is expressed by the subsumer is smaller than that of the subsumee, such that a proof by *descente infinie* with the subsumer immediately provides a proof with the subsumee, but not the other way round in general. This means that the subsumer does not represent a more powerful induction ordering, but actually achieves an improvement w.r.t. the eager instantiations of the induction hypothesis (both for a direct proof and for generalization), and with a case condition that admits for a better generalization without further case analysis.

This section is still not okay, moreover, induction variables and "failed" induction schemes are missing

5.4 NQTHM

Short section on quantification. Just describe what "quantification" is and that we cannot treat it here.

[Boyer and Moore, 1988c]

5.5 ACL2 *and the practical challenges*

Short section. We should demonstrate the economic applications if they can be made public. We should also discuss that induction is not the critical step in this applications, as Matt had put it.

5.6 INKA *and other discontinued explicit induction projects*

Very short section similar to the one in [Bundy, 1999]

6 ALTERNATIVE APPROACHES TO THE AUTOMATION OF INDUCTION

6.1 Proof Planning

Suggestions on how to overcome an envisioned dead end in automated theorem proving were summarized at the end of the 1980s under the keyword *proof planning*. Beside its human-science aspects,[144] the main idea of proof planning[145] is to add a smaller and more human-oriented *higher-level search space* to the theorem-proving system on top of the *low level search space* of the logic calculus. We do not cover this subject here, and refer the reader to the article by Alan Bundy and Jörg Siekmann in this volume.

6.2 Rippling

Rippling is a technique for augmenting rewrite rules with information that helps to find a way to rewrite one expression (*goal*) into another (*target*), more precisely to reduce the difference between the goal and the target by rewriting the goal. We cannot cover this very well-documented subject here, but refer the reader to the monograph [Bundy *et al.*, 2005].[146] Let us explain here, however, why rippling can be most helpful in the automation of simple inductive proofs.

Roughly speaking, the remarkable success in proving *simple* theorems by induction automatically, can be explained as follows: If we look upon the task of proving a theorem as reducing it to a tautology, then we have more heuristic guidance when we know that we probably have to do it by mathematical induction: Tautologies can have arbitrary subformulas, but the induction hypothesis we are going to apply can restrict the search space tremendously.

In a cartoon of Alan Bundy's, the original theorem is pictured as a zigzagged mountainscape and the reduced theorem after the unfolding of recursive operators according to recursion analysis as a lake with ripples (goal). To apply the induction hypothesis (target), instead of the uninformed search for an arbitrary tautology, we have to *get rid of the ripples* to be able to apply an instance of the theorem as induction hypothesis, mirrored by the calmed surface of the lake.

6.3 Implicit Induction

Proof planning and rippling were applied to the automation of induction within the paradigm of explicit induction. The alternative approaches to mechanize mathematical induction *not* subsumed by explicit induction, however, are united under the name "implicit induction". Triggered by the success of Boyer and Moore [1979], work on these alternative approaches started already in the year 1980 in purely

[144] Cf. [Bundy, 1989].
[145] Cf. [Bundy, 1988] [Dennis *et al.*, 2005].
[146] Historically important are also the following publications on rippling: [Hutter, 1990], [Bundy *et al.*, 1991], [Basin and Walsh, 1996].

equational theories.[147] A sequence of papers on technical improvements[148] was topped by [Bachmair, 1988], which gave rise to a hope to develop the method into practical usefulness, although it was still restricted to purely equational theories. Inspired by this paper, in the end of the 1980s and the first half of the 1990s several researchers tried to understand more clearly what implicit induction means from a theoretical point of view and whether it could be useful in practice.[149]

While it is generally accepted that [Bachmair, 1988] is about implicit induction and [Boyer and Moore, 1979] is about explicit induction, there are the following three different viewpoints on what the essential aspect of implicit induction actually is.

Proof by Consistency:[150] Systems for proof by consistency run some Knuth–Bendix[151] or superposition[152] completion procedure that produces a huge number of irrelevant inferences under which the ones relevant for establishing the induction steps can hardly be made explicit. A proof attempt is successful when the prover has drawn all necessary inferences and stops without having detected an inconsistency.

Proof by consistency has shown to be not competitive with explicit induction in practice, mainly due to too many superfluous inferences, typically infinite runs, and too restrictive admissibility conditions. Roughly speaking, the conceptual flaw in proof by consistency is that, instead of finding a sufficient set of reasonable inferences, the research follows the paradigm of ruling out as many irrelevant inferences as possible.

Implicit Induction Ordering: In the early implicit-induction systems, induction proceeds over a syntactical term ordering, which typically cannot be made explicit in the sense that there would be some predicate in the logical syntax that denotes this ordering in the intended models of the specification. The semantical orderings of explicit induction, however, cannot depend precisely on the syntactical term structure of a weight w, but only on the value of w under an evaluation in the intended models.

The price one has to pay for the possibility to have induction orderings that can also depend on the precise syntax is surprisingly high for powerful inference systems.[153]

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[147] Cf. [Goguen, 1980], [Huet and Hullot, 1980], [Lankford, 1980], [Musser, 1980].
[148] Cf. [Göbel, 1985], [Jouannaud and Kounalis, 1986], [Fribourg, 1986], [Küchlin, 1989].
[149] Cf. e.g. [Zhang *et al.*, 1988], [Kapur and Zhang, 1989], [Bevers and Lewi, 1990], [Reddy, 1990], [Gramlich and Lindner, 1991], [Ganzinger and Stuber, 1992], [Bouhoula and Rusinowitch, 1995], [Padawitz, 1996].

[150] The name "proof by consistency" was coined in the title of [Kapur and Musser, 1987], which is the later published forerunner of its outstanding improved version [Kapur and Musser, 1986].
[151] Cf. [Gramlich and Lindner, 1991].
[152] Cf. [Ganzinger and Stuber, 1992].

[153] Cf. [Wirth, 1997].

The early implicit-induction systems needed such sophisticated term orderings,[154] because they started from the induction conclusion and every inference step reduced the formulas w.r.t. the induction ordering again and again, but an application of an induction hypothesis was admissible to greater formulas only. This deterioration of the ordering information with every inference step was overcome by the introduction of explicit weight terms,[155] after which the need for syntactical term orderings as induction orderings does not exist anymore.

Descente Infinie ("Lazy Induction"): Contrary to explicit induction, where induction is introduced into an otherwise merely deductive inference system only by the explicit application of induction axioms in the induction rule, the cyclic arguments and their termination in implicit induction need not be confined to single inference steps.[156] The induction rule of explicit induction combines several induction hypotheses in a single inference step. To the contrary, in implicit induction, the inference system "knows" what an induction hypothesis is, i.e. it includes inference rules that provide or apply induction hypotheses, given that certain ordering conditions resulting from these applications can be met by an induction ordering. Because this aspect of implicit induction can facilitate the human-oriented induction method described in § 3, the name *descente infinie* was coined for it in [Wirth, 2004]. Researchers introduced to this aspect by [Protzen, 1994] (entitled "Lazy Generation of Induction Hypotheses") sometimes speak of "lazy induction" instead of *descente infinie*.

The interest in proof by consistency and implicit induction orderings today is either merely theoretical or merely historical, especially because these approaches cannot be combined with the paradigm of explicit induction. For more information on these viewpoints on implicit induction see the handbook article [Comon, 2001] and its partial correction [Wirth, 2005].

In § 6.4 we will show, however, how *Descente infinie* ("lazy induction") goes well together with explicit induction and why we can hope that both the restrictions implied by induction axioms can be overcome and the usefulness of the excellent heuristic knowledge developed in explicit induction can be improved.[157]

6.4 QuodLibet

(along [Wirth, 2009] and [Schmidt-Samoa, 2006c])

[154] Cf. e.g. [Bachmair, 1988], [Steinbach, 1995].

[155] Cf. [Wirth and Becker, 1995].

[156] For this reason, the funny name "inductionless induction" was originally coined for implicit induction in the titles of [Lankford, 1980; 1981] as a short form for "induction without induction rule". See also the title of [Goguen, 1980] for a similar phrase.

[157] Cf. [Wirth, 2013].

7 BEYOND INDUCTION

(short)

7.1 Beyond Noetherian induction (Full axiom of choice instead of principle of dependent choices)

7.2 What the incredible success of NQTHM *meant for the fields of ATP and AI*

7.3 Lessons Learned beyond Induction

Boyer and Moore never gave a name to their provers and so they became most popular under the name *the Boyer–Moore theorem prover*. So here is an advice to the young researchers who want to become popular: Build a good system, but do not give it a name, so that people have to attach your name to it.

From the development of THM via the intermediate stage of the PURE LISP THEOREM PROVER, which somehow contains most essential stages, procedures, and heuristics of NQTHM in a rudimentary and partly inferior form, we can learn that it may be beneficial at an early stage of a true creation to proceed as follows:

1. Learn the essential ideas in a rudimentary form from the simpler, most frequent standard scenarios.
2. Implement and test these ideas.
3. Refine the resulting procedures and structures in later work.

This section on Lessons needs revision and possibly extension.

8 CONCLUSION

(very short)

In [Boyer and Moore, 2012], in their noble humbleness, Moore said the following, and Boyer agreed laughingly:

> "One of the reasons our theorem prover is successful is that we trick the user into telling us the proof. And the best example of that, that I know, is: If you want to prove that there exists a prime factorization — that is to say a list of primes whose product is any given number — then the way you state it is: You define a function that takes a natural number and delivers a list of primes, and then you prove that it does that. And, of course, the definition of that function is everybody else's proof. The absence of quantifiers and the focus on constructive, you know, recursive definitions forces people to do the work. And so then, when the theorem prover proves it, they say 'Oh what wonderful theorem prover!' without even realizing they sweated bullets to express the theorem in that impoverished logic."

This section on Lessons needs revision and extension.

ACKNOWLEGDGEMENTS

We would like to thank Anne O. Boyer, Robert S. Boyer, Warren A. Hunt, Matt Kaufmann, Marianeh Rezaei.

We could split the bibliography into subsections, but this seems to be inappropriate here, because alternative approaches to explicit induction have few references, and because the sections on INKA, NQTHM, ✓ERIFUN, ACL2, λCLAM would heavily overlap)

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