The Descriptive Operators iota, tau, and epsilon

— On their Origin, Partial Axiomatization, Model-Theoretic Semantics, Practical Applicability

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PART I

- Motivation:
  Problems with Quantifiers and Descriptive Operators
- Introduction:
  The essential axiomatization of Peano's ι and Hilbert's τ and ε

Descriptive Terms instead of Quantifiers

- NL contains no quantifiers, but free symbols and descriptive t.
- Proof theory and proof automation often profit from the removal of quantifiers through partially specified terms, such as ε-terms and partially specified functions.
- The Scoping of quantifiers and binders brings problems:
  - lack of expressiveness (Henkin quantification),
  - enforced overspecification,
  - inefficiency of computation (no incrementality, no in situ handling), e.g. in deep analysis of NL semantics.
- Mathematics often needs (quantified) free symbols, e.g. for Math. Induction (*Descente infinie*) or for computat. of solutions.

Quantifiers are Frege's artificial entities [1879] (not: quantification).

Essential Axiomatizations of ι, τ, ε

- $\exists! x^{\mathbb{B}}.\ A \;\Rightarrow\; A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}.\ A\}$ $\quad(\iota_0)$
- $\exists x^{\mathbb{B}}.\ A \;\Rightarrow\; A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\}$ $\quad(\varepsilon_0)$
  Consequences:
  $\exists x^{\mathbb{B}}.\ A \;\Leftrightarrow\; A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\}$ $\quad(\varepsilon_1)$
  $\neg\exists x^{\mathbb{B}}.\ A \;\Leftrightarrow\; \neg A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\}$ $\quad$(contrap(ε_1))
  Consequence in Classical Logic $(\{A{\mapsto}\neg B\})$:
  $\forall x^{\mathbb{B}}.\ B \;\Leftrightarrow\; B\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg B\}$ $\quad(\varepsilon_2)$
- $\forall x^{\mathbb{B}}.\ B \;\Leftarrow\; B\{x^{\mathbb{B}} \mapsto \tau x^{\mathbb{B}}.\ B\}$ $\quad(\tau_0)$

PART II

- Origins of ι, τ, ε
- How to Specify them
- Choose the ε
- History of Disappointed Expectations

On the History of Peano's ι

- Frege [1893] writes a boldface backslash.
  $\backslash\xi = x^{\mathbb{B}}$ if there is some $x^{\mathbb{B}}$ s.th. $\forall y^{\mathbb{B}}.\,(\xi(y^{\mathbb{B}}) \Leftrightarrow (x^{\mathbb{B}} = y^{\mathbb{B}}))$;
  $\backslash\xi = \xi$ otherwise.
- Peano writes "$\bar{\iota}$" [1896f.] or an inverted "ι" [1899, German].
  His "ι" is "$\iota(x) := \{x\}$", his "$\bar{\iota}$" is the inverse function of his "ι".
  Partial spec. according to (ι_0), but sets instead of predicates.
- Russell & Whitehead [1910ff.] write inverted ι, but with the non-referring semantics of [Russell, 1905]: *On denoting.*
- Hilbert & Bernays [1934] require the completion of a proof of $\exists! x^{\mathbb{B}}.\,A$ before the term $\iota x^{\mathbb{B}}.\,A$ may be formed.
- Quine and many others have ιs with explicit definitions.

Only Peano has always denoting terms + the intended partial spec.

Implicit Partial vs. Explicit Definition

- Peano with his preference on written languages for specification and communication (over calculi) stays within the proper limits:

  Avoid any Overspecification!
  (with all its unintended consequences)
- Frege, Quine, &c. in the tradition of unconditional explicit definition (*"definiendum* $:=$ *definiens"*) (syntactic, always total):

  Eliminability!
  - Eliminability of the ι already requires absurdly powerful logical framework.
  - Eliminability impossible in principle for the ε in general, because of its indefiniteness.

On the History of Hilbert's τ

[Hilbert, 1923]: *Die logischen Grundlagen der Mathematik.* Talk of Sept. 1922.

- τ stands for *transfinite function*
- $\tau : (i \rightarrow o) \rightarrow i, \quad A : i \rightarrow o.$
- Transfinite Axiom: 11. $A(\tau A) \Rightarrow A(a).$
- Acknowledgment for Paul Bernays in footnote:
  "Die Erkenntnis, daß die *eine* Formel 11. zur Herleitung dieser sämtlichen Formeln genügt, verdanke ich P. Bernays."
- Warning 1: A different function is the "τ" in Kneser's private notes to Hilbert's 1921/22 lecture *Grundlagen der Mathematik.*
- Warning 2: An "ε" is written for the τ in Kneser's private notes to Hilbert's 1922/23 lecture *Logische Grundlagen der Math.*
- Warning 3: Nicolas Bourbaki writes "τ", but it is an ε!

On the Early History of Hilbert's ε

- $A(a) \Rightarrow A(\varepsilon A)$ (deduct. equivalent to (ε_0)) is called
  - *transfinite axiom 1* (as binder) [Ackermann, 1925]
  - *axiom of choice* [Hilbert, 1926]
  - *logical ε-axiom* [Hilbert, 1928]
  - *ε-formula* (but as binder) [Hilbert & Bernays, 1939]
- ε is called the
  - *transfinite logical choice function* [Hilbert, 1926]
  - *logical ε-function* [Hilbert, 1928]
  - *ε-symbol* [Hilbert & Bernays, 1939]

[Ackermann, 1925]: *Begründung des „t.n.d." mittels der Hilbert'schen Theorie der Widerspruchsfreiheit.* Abstract PhD thesis 1924.
[Hilbert, 1926]: *Über das Unendliche.* Talk of June 1925.
[Hilbert, 1928]: *Die Grundlagen der Mathematik.* Talk of July 1927.
[Hilbert & Bernays, 1939]: *Grundlagen der Mathematik, Vol. II.*

The ε is The Choice in Practice

- The ι is no use unless $\exists! x^{\mathbb{B}}.\,A$.
  But given $\exists! x^{\mathbb{B}}.\,A$, (ι_0), (ε_0), we have:
  $$\iota x^{\mathbb{B}}.\,A \;=\; \varepsilon x^{\mathbb{B}}.\,A.$$
  Thus — to obtain weaker proof obligations —
  always use the ε instead of ι (unless eliminability relevant):
  - Less proof work!
  - Easier generalization of proofs!
- τ instead of ε makes sense only in non-classical logics.
  No essential difference in classical logic.

Sharpened 1st ε-Theorem [H. & Bernays, 1939]

Given: a derivation of $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.\ A$
(containing no bound variables besides the ones bound by the prefix $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.$)

from the formulas $P_1, \ldots, P_k$
(containing neither formula variables nor bound variables)

in the predicate calculus
(incl. ε-formula and $=$-substitutability as axiom schemes, plus $=$-reflexivity).

We can construct a (finite) disjunction of the form
$$\bigvee_{i=0}^{s} A\{x_1^{\mathbb{B}}, \ldots, x_r^{\mathbb{B}} \mapsto t_{i,1}, \ldots, t_{i,r}\}$$
and a derivation of it

- in which bound variables do not occur at all
- from $P_1, \ldots, P_k$ and $=$-axioms
  (containing neither formula variables nor bound variables)
- in the quantifier-free predicate calculus (i.e. tautologies plus the inference schema [of modus ponens] and the substitution rule).

Note that r, s range over natural numbers including 0, and that A, $t_{i,j}$, and P_i are ε-free because otherwise they would have to include (additional) bound variables.

ε — A History of Failures? ("twilight of his career")

- [Gödel, 1931]: Gödel's second incompleteness theorem cuts down Hilbert's program.
- New *goal-directed* calculi [Herbrand, 1930] (*Modus Ponens*-free) and [Gentzen, 1935] (Cut-free) do not have an ε.
- Consistency proofs in [Herbrand, 1932] and in [Gentzen, 1936, 1938, 1943] do not use the ε.
- The displaced Ackermann finally [1940] proves the termination of an improved ε-substitution method in arithmetic, only to draw level w.r.t. consistency proofs with [Gentzen, 1938].
- Kreisel's [1958] *unwinding program* (Constructive equiv. math.?) is not properly formulated and followed.
- Leisenring's textbook [1969] renders the ε as too impractical for computer sci. and autom. theorem proving.
- Several projects to translate ε's main reference, Hilbert–Bernays *Grundlagen der Mathematik* [1934/39] [1968/70], failed.

PART III

- Via Overspecification toward a
  Model-Theoretic Semantics for the ε
- Choice: Have really indefinite choice + committed choice as an option
- Take Leisenring's Satisfiability, but not his Notion of Validity

Overspecification: Extensionality (E2)

Ackermann's (II,4), Bourbaki's (S7), Leisenring's (E2):

$$\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \quad\Rightarrow\quad \varepsilon x^{\mathbb{B}}.\ A_0 \;=\; \varepsilon x^{\mathbb{B}}.\ A_1$$

- Good: Syntactical differences between A_1 and A_2 should not matter. Deterrent example: [Asser, 1957, type-3]
- Bad1: All classical theorems become intuitionistic ones.
- Bad2: Committed choice should be an option, not a must. "A bishop met a bishop."
- Actually committed choice *must* be an option:

$$\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$
$$\varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$
$$0 \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$
$$0 \neq 1$$

Model-Theoretic Semantics

- Motivation: Practical Applicability requires a Model-Theoretic Semantics.
- To define Satisfiability, the Existence of a (generalized) choice function works fine in the evaluation of ε-terms.
- ε-formula plus (E2) turn predicate calculus sound and complete for validity w.r.t. all possible choice functions, cf. [Asser, habil, 1957, type-1], [Leisenring, textbook, 1969]

Main Theses

Practical usefulness as well as formal adequacy of a straightforward model-theoretic specification of the ε require:

- Both satisfiability and validity must refer to the existence of choice functions, not to all of them:

$$\mathsf{Bishop}(\mathsf{Tebarz}) \models \ \varepsilon x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}}) \;=\; \mathsf{Tebarz}$$

- Non-commitment to a choice must be possible where it is not required: $\models\ \varepsilon_i x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}}) \neq \varepsilon_j x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}})$
  (Indexed ε à la Heusinger/Egli)

Better without choice functions: $\models\ x_0^{\mathrm{v}} = \mathsf{Tebarz}, \quad x_1^{\mathrm{v}} \neq x_2^{\mathrm{v}}$
with choice condition $C(x_i^{\mathrm{v}}) := \varepsilon x^{\mathbb{B}}.\ \mathsf{Bishop}(x^{\mathbb{B}})$ for $i \in \{0,1,2\}$.

ε-substitution becomes subst. of free variables x_i^{v}.

PART IV: Wirth's Free-Variable Framework

- Quantification without Quantifiers
- Hilbert's ε
- Liberalized δ-rules
- Fermat's *Descente infinie*

Free Variables and Atoms

- Occur frequently in math & computer science
- Their function depends on context:
  varying, implicit, ad hoc
- Here:
  disjoint sets of symbols for different functions
- New:
  only two functions of free variables / atoms left:
  existentially / universally quantified.
- New:
  Henkin quantification can be modeled directly
  with positive/*negative* variable-conditions.

Free Atoms (=“parameters”=“eigenvariables”=“δ^-”)

(1) $$(m)^{(p+q)} = (m)^{(p)} * (m)^{(q)}$$

(2) $$\forall m^{\mathbb{B}}, p^{\mathbb{B}}, q^{\mathbb{B}}.\ \left(\ (m^{\mathbb{B}})^{(p^{\mathbb{B}}+q^{\mathbb{B}})} = (m^{\mathbb{B}})^{(p^{\mathbb{B}})} * (m^{\mathbb{B}})^{(q^{\mathbb{B}})}\ \right)$$

(3) $$(m^{\mathbb{A}})^{(p^{\mathbb{A}}+q^{\mathbb{A}})} = (m^{\mathbb{A}})^{(p^{\mathbb{A}})} * (m^{\mathbb{A}})^{(q^{\mathbb{A}})}$$

- Advantage of (1) and (3): Mathematical induction (Fermat’s *Descente infinie*) becomes possible because the induction hypothesis can refer to the conclusion by means of the free variables.
- Semantics is uniquely expressed in (2) and (3).

Free Variables (=“meta”=“dummy”=”γ”=”δ^+”)

(1) $$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

(2) $$\exists x^{\mathbb{B}}, y^{\mathbb{B}}.\ \left(\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{B}} \\ y^{\mathbb{B}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix} \right)$$

(3) $$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{V}} \\ y^{\mathbb{V}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

- Constraints on solutions may be retrieved from a proof by referring to the variables in (1) and (3).
- Semantics is uniquely expressed in (2) and (3).

Free Atoms

- $\mathbb{A}$, Universally quantified (implicitly)
- Arbitrary object in a discourse
- Atomic: black-box, no information on it ever
- Except: Is it an atom?
  Different from another atom?
- Origin of name:
  Set theories with atoms (or urelements)
- Instantiated locally and repeatedly in application of lemmas or induction hypotheses

Free Variables

- $\mathbb{V}$, Existentially quantified (implicitly)
- Place-holder in a discourse
- Gather and store information
- Replaced with a definition or a description
- Origin of name:
  Fitting’s free-variable semantic tableaus
- Rigid: Instantiated globally, once and for all,
  with possible effect on input theorem

(Reductive) Inference (Smullyan’s classification)

γ-rule:
$$\frac{\mathit{\Gamma},\quad \exists x^{\mathbb{B}}.A,\quad \mathit{\Delta}}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\},\quad \mathit{\Gamma},\quad \exists x^{\mathbb{B}}.A,\quad \mathit{\Delta}}$$

δ^--rule:
$$\frac{\mathit{\Gamma},\ \forall y^{\mathbb{B}}.A,\ \mathit{\Delta}}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \mathit{\Gamma},\ \mathit{\Delta}} \qquad \mathbb{V}(\mathit{\Gamma}, \forall y^{\mathbb{B}}.A, \mathit{\Delta}) \times \{y^{\mathbb{A}}\}$$

δ^+-rule:
$$\frac{\mathit{\Gamma},\ \forall y^{\mathbb{B}}.A,\ \mathit{\Delta}}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\},\ \mathit{\Gamma},\ \mathit{\Delta}} \qquad \begin{matrix} (y^{\mathbb{V}},\ \ \varepsilon y^{\mathbb{B}}.\ \neg A) \\ \mathbb{V}\mathbb{A}(\forall y^{\mathbb{B}}.A) \times \{y^{\mathbb{V}}\} \end{matrix}$$

1st Example Proof Attempt

δ^--rule:
$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \Gamma,\ \Delta} \qquad \mathbb{V}(\Gamma,\forall y^{\mathbb{B}}.A,\Delta)\times\{y^{\mathbb{A}}\}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

δ^--step: $(x^{\mathbb{V}}=y^{\mathbb{A}}),\ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{A}}\}$?

Record dependency in
negative variable-condition N: $(x^{\mathbb{V}},y^{\mathbb{A}})\in N$

2nd Example Proof Attempt

δ^+-rule:
$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\},\ \Gamma,\ \Delta} \qquad \begin{array}{l}(y^{\mathbb{V}},\ \varepsilon y^{\mathbb{B}}.\ \neg A)\\ \mathbb{VA}(\forall y^{\mathbb{B}}.A)\times\{y^{\mathbb{V}}\}\end{array}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

δ^+-step: $(x^{\mathbb{V}}=y^{\mathbb{V}}),\ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$?

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$?

Record dependency in Apply
positive variable-condition P: $(x^{\mathbb{V}},y^{\mathbb{V}})\in P$
$\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$?

Variable-Conditions in the Literature

- Wolfgang Bibel's book *Automated Theorem Proving* 1982, (2nd edn. 1987):
  - Two positive relations: awkward & inefficient
  - No liberalized δ-rules
- Lincoln A. Wallen 1990:
  - Single positive relation
  - No liberalized δ-rules

Variable-Conditions in the Literature (contd.)

- Michael Kohlhase's articles
  - With liberalized δ-rules
  - *Higher-Order Tableaux* [TABLEAUX'1995]: Unsound!
  - *Higher-Order Theorem Proving* [Bibel & Schmitt, Automated Deduction, Vol. I, 1998]: Unsound!
- Wirth's previous versions with single relation
  - Negative relation [FTP'1998]
  - Positive relation with Variable reuse [J. IGPL 2004]
  - Standard positive relation [J. IGPL 2004], [J. Appl. L. 2008], SEKI-Report SR–2006–02 [2012]

Positive/Negative Variable-Condition (P,N)

$$\begin{array}{lcccl} P & \subseteq & (\mathbb{V}\uplus\mathbb{A}) & \times & \mathbb{V}\\ N & \subseteq & \mathbb{V} & \times & \mathbb{A}\end{array}$$

- *Consistency:*
  Each cycle in the directed graph of $P\cup N$ has more than one edge from N.
- *Admissible substitution* σ*:* $(P\cup D,N)$ is consistent.
  $(x^{\mathbb{VA}},y^{\mathbb{V}})\in D$ iff
  $y^{\mathbb{V}}\in\mathrm{dom}(\sigma)$ and
  $x^{\mathbb{VA}}$ is a free variable or free atom in $\sigma(y^{\mathbb{V}})$

Whole Proof Search Framework

DATA STRUCTURES:

- A Forest of and/or proof-attempt trees.
  Root of each tree carries an [open] proposition.
- A Positive/Negative Variable-Condition (P,N)
- A Choice-Condition C

PS-INVARIANT ("Preservation of solutions"):
"The the solutions of the leaves solve the root."
The root $(C,(P,N))$-reduces to the leaves.
This is more than soundness of problem reduction!

Whole Proof Search Framework (contd.)

OPERATIONS:

- New conjectures get trivial proof-attempt tree.
- Analytic proof steps local to a tree.
- Generative application of open lemmas, either deductively or inductively.
- Global substitution of rigid variables.
  PS-INVARIANT! But:
  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.
  It becomes false after application of $\{x^{\mathbb{V}} \mapsto y^{\mathbb{V}}\}$

Henkin Quantification

- "Every woman could love someone and every man could love someone, such that these loved ones could love each other."
- As a Henkin-quantified IF-logic formula:
- Represented in our Framework:

$$\forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}. \left(\begin{array}{l} \left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array} \right) \\ \Rightarrow\ \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}. \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{array} \right) \end{array} \right)$$

Binders are Bad:

- Quantifiers and the ε mess up formulas
- Quantifiers and the ε make reasoning difficult
- Quantifiers enforce a too primitive form of scoping
- The ε-binder produces terms of unmanageable size

Binding without Binders is Great:

- Free variables and atoms are what we need to manage practical applications
- Positive/Negative Variable-Conditions enable sophisticated scoping
- The term-sharing of free variables admits ε-binding that is manageable w.r.t. term size
- Our semantics for the ε is existential (!) and admits indefinite committed choice
- Free atoms admit mathematical induction in the liberal style of Fermat's *Descente Infinie*

Conclusion

Want your reasoning applications to be successful in practice?

- Ask for a tailored version of a free-variable framework!
- Do not introduce free variables and atoms ad hoc for operational purposes, but give them a clear semantics
- Get *both* free variables *and* free atoms:
- The liberalized δ-rule (δ^{+}-rule) is a practical improvement only if the non-liberalize δ-rule (δ^{-}-rule) remains available (Henkin, Fermat)
- Use an ε with existential semantics

Related Publications

- *Descente Infinie + Deduction*
  Logic J. of the IGPL 12:1–96, 2004, Oxford Univ. Press
- *Hilbert's epsilon as an Operator of Indefinite Committed Choice (IDC)*
  J. Applied Logic 6:287–317, 2008, Elsevier
- *A Simplified and Improved Free-Variable Framework of Hilbert's ε as an Operator of IDC*
  SEKI-Report SR–2011–01, Revised May 2015.
  `http://arxiv.org/abs/1104.2444`
- *Hilbert–Bernays: Grundlagen der Mathematik.*
  1st English translation. With comments and German facsimile.
  `http://wirth.bplaced.net/p/hilbertbernays`

Semantic treatment of Variable-Conditions

$$\mathsf{e}(\pi)(\delta)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\delta).$$

$$\pi : \mathbb{V} \leadsto (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}, \qquad \delta : \mathbb{A} \leadsto \mathcal{S}, \quad x \in \mathbb{V}$$

$$\mathsf{e} \;:\; (\mathbb{V} \leadsto (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}) \;\to\; (\mathbb{A} \leadsto \mathcal{S}) \;\to\; \mathbb{V} \;\leadsto\; \mathcal{S}$$

$$S_\pi := \{\, (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\pi) \wedge y^{\mathbb{A}} \in \mathrm{dom}(\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}})))) \,\}$$

π is $\mathcal{S}$*-compatible with* $(C, (P, N))$ if ... and
$(P \cup S_\pi, N)$ is consistent and
π respects C in $\mathcal{S}$

Reduction, Def.

Let G_0 and G_1 be sets of sequents.
Let $\mathcal{S}$ be a Σ-structure.
Let C be an (P, N)-choice-condition.

G_0 $(C, (P, N))$*-reduces to* G_1 *in* $\mathcal{S}$ if
for each π that is $\mathcal{S}$-compatible with $(C, (P, N))$:

if G_1 is $(\pi, \mathcal{S})$-valid,
then G_0 is $(\pi, \mathcal{S})$-valid.

Reduction PS-Invariant under Substitution

For an (P, N)-substitution σ on $\mathbb{V}$,
for the extended σ-update $(C', (P', N'))$
of $(C, (P, N))$:

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$
in $\mathcal{S}$.

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1)$:

$Q_C(y^{\mathbb{V}})$

$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$

$(Q_C(y^{\mathbb{V}}))\{y^{\mathbb{V}} \mapsto \mathsf{p}\}$

$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$