
The Descriptive Operators iota, tau, and epsilon

— On their Origin,
Partial Axiomatization,
Model-Theoretic Semantics,
Practical Applicability

Claus-Peter Wirth

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- PART II: Origin & History
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PART I

- Motivation:

Problems with Quantifiers and Descriptive Operators

- Introduction:

The essential axiomatization of Peano's ι and Hilbert's τ and ε

Descriptive Terms instead of Quantifiers

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 - inefficiency of computation (no incrementality, no in situ handling), e.g. in deep analysis of NL semantics.

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 - inefficiency of computation (no incrementality, no in situ handling), e.g. in deep analysis of NL semantics.
- Mathematics often needs (quantified) free symbols, e.g. for Math. Induction (*Descente infinie*) or for computat. of solutions.

Quantifiers are Frege's artificial entities [1879] (not: quantification).

Essential Axiomatizations of ι , τ , ε

$$\blacksquare \exists! x^{\mathbb{B}}. A \Rightarrow A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}. A\} \quad (\iota_0)$$

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$$\blacksquare \forall x^{\mathbb{B}}. B \Leftarrow B\{x^{\mathbb{B}} \mapsto \tau x^{\mathbb{B}}. B\} \quad (\tau_0)$$

PART II

- Origins of ι , τ , ε
- How to Specify them
- Choose the ε
- History of Disappointed Expectations

On the History of Peano's ι

- Frege [1893] writes a boldface backslash.

$\backslash \xi = x^{\mathbb{B}}$ if there is some $x^{\mathbb{B}}$ s.th. $\forall y^{\mathbb{B}}. (\xi(y^{\mathbb{B}}) \Leftrightarrow (x^{\mathbb{B}} = y^{\mathbb{B}}))$;
 $\backslash \xi = \xi$ otherwise.

- Peano writes “ $\bar{\iota}$ ” [1896f.] or an inverted “ ι ” [1899, German].
His “ ι ” is “ $\iota(x) := \{x\}$ ”, his “ $\bar{\iota}$ ” is the inverse function of his “ ι ”.
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- Hilbert & Bernays [1934] require the completion of a proof of $\exists!x^{\mathbb{B}}. A$ before the term $\iota x^{\mathbb{B}}. A$ may be formed.
- Quine and many others have ι s with explicit definitions.

Only Peano has always denoting terms + the intended partial spec.

Implicit Partial vs. Explicit Definition

- Peano with his preference on written languages for specification and communication (over calculi) stays within the proper limits:

Avoid any Overspecification!
(with all its unintended consequences)

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- Frege, Quine, &c. in the tradition of unconditional explicit definition (“*definiendum := definiens*”) (syntactic, always total):

Eliminability!

- Eliminability of the ι already requires absurdly powerful logical framework.
- Eliminability impossible in principle for the ε in general, because of its indefiniteness.

On the History of Hilbert's τ

[Hilbert, 1923]: *Die logischen Grundlagen der Mathematik.*
Talk of Sept. 1922.

- τ stands for *transfinite function*
- $\tau : (i \rightarrow o) \rightarrow i, \quad A : i \rightarrow o.$
- Transfinite Axiom: $11. \quad A(\tau A) \Rightarrow A(a).$
- Acknowledgment for Paul Bernays in footnote:
“Die Erkenntnis, daß die *eine* Formel 11. zur Herleitung dieser sämtlichen Formeln genügt, verdanke ich P. Bernays.”
- Warning 1: A different function is the “ τ ” in Kneser’s private notes to Hilbert’s 1921/22 lecture *Grundlagen der Mathematik.*
- Warning 2: An “ ε ” is written for the τ in Kneser’s private notes to Hilbert’s 1922/23 lecture *Logische Grundlagen der Math.*
- Warning 3: Nicolas Bourbaki writes “ τ ”, but it is an ε !

On the Early History of Hilbert's ε

- $A(a) \Rightarrow A(\varepsilon A)$ (deduct. equivalent to (ε_0)) is called
 - *transfinite axiom 1* (as binder) [Ackermann, 1925]
 - *axiom of choice* [Hilbert, 1926]
 - *logical ε -axiom* [Hilbert, 1928]
 - *ε -formula* (but as binder) [Hilbert & Bernays, 1939]
- ε is called the
 - *transfinite logical choice function* [Hilbert, 1926]
 - *logical ε -function* [Hilbert, 1928]
 - *ε -symbol* [Hilbert & Bernays, 1939]

[Ackermann, 1925]: *Begründung des „t.n.d.“ mittels der Hilbert'schen Theorie der Widerspruchsfreiheit.* Abstract PhD thesis 1924.

[Hilbert, 1926]: *Über das Unendliche.* Talk of June 1925.

[Hilbert, 1928]: *Die Grundlagen der Mathematik.* Talk of July 1927.

[Hilbert & Bernays, 1939]: *Grundlagen der Mathematik, Vol. II.*

The ε is The Choice in Practice

- The ι is no use unless $\exists!x^{\mathbb{B}}. A$.

But given $\exists!x^{\mathbb{B}}. A$, (ι_0) , (ε_0) , we have:

$$\iota x^{\mathbb{B}}. A = \varepsilon x^{\mathbb{B}}. A.$$

Thus — to obtain weaker proof obligations —
always use the ε instead of ι (unless eliminability relevant):

- Less proof work!
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Thus — to obtain weaker proof obligations —
always use the ε instead of ι (unless eliminability relevant):

- Less proof work!
 - Easier generalization of proofs!
- τ instead of ε makes sense only in non-classical logics.
No essential difference in classical logic.

Sharpened 1st ε -Theorem [H. & Bernays, 1939]

Given: a derivation of $\exists x_1^{\mathbb{B}}. \dots \exists x_r^{\mathbb{B}}. A$

(containing no bound variables besides the ones bound by the prefix $\exists x_1^{\mathbb{B}}. \dots \exists x_r^{\mathbb{B}}.$)

from the formulas P_1, \dots, P_k

(containing neither formula variables nor bound variables)

in the predicate calculus

(incl. ε -formula and $=$ -substitutability as axiom schemes, plus $=$ -reflexivity).

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We can construct a (finite) disjunction of the form

$$\bigvee_{i=0}^s A\{x_1^{\mathbb{B}}, \dots, x_r^{\mathbb{B}} \mapsto t_{i,1}, \dots, t_{i,r}\}$$

and a derivation of it

- in which bound variables do not occur at all
- from P_1, \dots, P_k and $=$ -axioms
(containing neither formula variables nor bound variables)
- in the quantifier-free predicate calculus (i.e. tautologies plus
the inference schema [of modus ponens] and the substitution rule).

Note that r, s range over natural numbers including 0, and that $A, t_{i,j}$, and P_i are ε -free because otherwise they would have to include (additional) bound variables.

ε — A History of Failures? (“twilight of his career”)

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- New *goal-directed* calculi [Herbrand, 1930] (*Modus Ponens*-free) and [Gentzen, 1935] (Cut-free) do not have an ε .
- Consistency proofs in [Herbrand, 1932] and in [Gentzen, 1936, 1938, 1943] do not use the ε .
- The displaced Ackermann finally [1940] proves the termination of an improved ε -substitution method in arithmetic, only to draw level w.r.t. consistency proofs with [Gentzen, 1938].

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- Kreisel’s [1958] *unwinding program* (Constructive equiv. math.?) is not properly formulated and followed.
- Leisenring’s textbook [1969] renders the ε as too impractical for computer sci. and autom. theorem proving.
- Several projects to translate ε ’s main reference, Hilbert–Bernays *Grundlagen der Mathematik* [1934/39] [1968/70], failed.

PART III

- Via Overspecification toward a
Model-Theoretic Semantics for the ε
- Choice: Have really indefinite choice
+ committed choice as an option
- Take Leisenring's Satisfiability,
but not his Notion of Validity

Overspecification: Extensionality (E2)

Ackermann's (II,4), Bourbaki's (S7), Leisenring's (E2):

$$\forall x^{\mathbb{B}}. (A_0 \Leftrightarrow A_1) \quad \Rightarrow \quad \varepsilon x^{\mathbb{B}}. A_0 = \varepsilon x^{\mathbb{B}}. A_1$$

- Good: Syntactical differences between A_1 and A_2 should not matter. Deterrent example: [Asser, 1957, type-3]

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- Bad1: All classical theorems become intuitionistic ones.
- Bad2: Committed choice should be an option, not a must. "A bishop met a bishop."
- Actually committed choice *must* be an option:

$$\begin{aligned} & \exists x^{\mathbb{B}}. (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \\ \varepsilon x^{\mathbb{B}}. (x^{\mathbb{B}} \neq x^{\mathbb{B}}) & \neq \varepsilon x^{\mathbb{B}}. (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \\ & 0 \neq \varepsilon x^{\mathbb{B}}. (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \\ & 0 \neq 1 \end{aligned}$$

Model-Theoretic Semantics

- Motivation:
Practical Applicability requires a Model-Theoretic Semantics.
- To define Satisfiability, the Existence of
a (generalized) choice function
works fine in the evaluation of ε -terms.
- ε -formula plus (E2) turn predicate calculus sound and
complete for validity w.r.t. all possible choice functions,
cf. [Asser, habil, 1957, type-1], [Leisenring, textbook, 1969]

Main Theses

Practical usefulness as well as formal adequacy of a straightforward model-theoretic specification of the ε require:

- Both satisfiability and validity must refer to the existence of choice functions, not to all of them:

$$\text{Bishop}(\text{Tebarz}) \models \varepsilon x^{\mathbb{B}}. \text{Bishop}(x^{\mathbb{B}}) = \text{Tebarz}$$

- Non-commitment to a choice must be possible where it is not required: $\models \varepsilon_i x^{\mathbb{B}}. \text{Bishop}(x^{\mathbb{B}}) \neq \varepsilon_j x^{\mathbb{B}}. \text{Bishop}(x^{\mathbb{B}})$
(Indexed ε à la Heusinger/Egli)

Better without choice functions: $\models x_0^{\forall} = \text{Tebarz}, \quad x_1^{\forall} \neq x_2^{\forall}$

with choice condition $C(x_i^{\forall}) := \varepsilon x^{\mathbb{B}}. \text{Bishop}(x^{\mathbb{B}})$ for $i \in \{0, 1, 2\}$.

ε -substitution becomes subst. of free variables x_i^{\forall} .

PART IV: Wirth's Free-Variable Framework

- Quantification without Quantifiers
- Hilbert's ε
- Liberalized δ -rules
- Fermat's *Descente infinie*

Free Variables and Atoms

- Occur frequently in math & computer science
- Their function depends on context:
varying, implicit, ad hoc
- Here:
disjoint sets of symbols for different functions
- New:
only two functions of free variables / atoms left:
existentially / universally quantified.
- New:
Henkin quantification can be modeled directly
with positive/*negative* variable-conditions.

Free Atoms (=“parameters”=“eigenvariables”=“ δ^- ”)

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$$(1) \quad \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

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- Semantics is uniquely expressed in (2) and (3).

Free Atoms

- \mathbb{A} , Universally quantified (implicitly)
- Arbitrary object in a discourse
- Atomic: black-box, no information on it ever
- Except: Is it an atom?
Different from another atom?
- Origin of name:
Set theories with atoms (or urelements)
- Instantiated locally and repeatedly in application of lemmas or induction hypotheses

Free Variables

- \forall , Existentially quantified (implicitly)
- Place-holder in a discourse
- Gather and store information
- Replaced with a definition or a description
- Origin of name:
Fitting's free-variable semantic tableaux
- Rigid: Instantiated globally, once and for all,
with possible effect on input theorem

(Reductive) Inference (Smullyan's classification)

γ -rule:

$$\frac{\Gamma, \exists x^{\mathbb{B}}.A, \Delta}{A\{x^{\mathbb{B}} \mapsto x^{\mathbb{V}}\}, \Gamma, \exists x^{\mathbb{B}}.A, \Delta}$$

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$$\delta^{-}\text{-rule: } \frac{\Gamma, \forall y^{\mathbb{B}}.A, \Delta}{A\{y^{\mathbb{B}} \mapsto y^{\mathbb{A}}\}, \Gamma, \Delta} \quad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

(Reductive) Inference (Smullyan's classification)

$$\gamma\text{-rule: } \frac{\Gamma, \exists x^{\mathbb{B}}.A, \Delta}{A\{x^{\mathbb{B}} \mapsto x^{\mathbb{V}}\}, \Gamma, \exists x^{\mathbb{B}}.A, \Delta}$$

$$\delta^{-}\text{-rule: } \frac{\Gamma, \forall y^{\mathbb{B}}.A, \Delta}{A\{y^{\mathbb{B}} \mapsto y^{\mathbb{A}}\}, \Gamma, \Delta} \quad \forall(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

$$\delta^{+}\text{-rule: } \frac{\Gamma, \forall y^{\mathbb{B}}.A, \Delta}{A\{y^{\mathbb{B}} \mapsto y^{\mathbb{V}}\}, \Gamma, \Delta} \quad (y^{\mathbb{V}}, \varepsilon y^{\mathbb{B}}. \neg A) \quad \forall\mathbb{A}(\forall y^{\mathbb{B}}.A) \times \{y^{\mathbb{V}}\}$$

1st Example Proof Attempt

δ^- -rule:

$$\frac{\Gamma, \forall y^{\mathbb{B}}.A, \Delta}{A\{y^{\mathbb{B}} \mapsto y^{\mathbb{A}}\}, \Gamma, \Delta}$$

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Proof task:

$$\exists x^{\mathbb{B}}. \forall y^{\mathbb{B}}. (x^{\mathbb{B}} = y^{\mathbb{B}})$$

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$$\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$$

γ -step:

$$\forall y^{\mathbb{B}}.(x^{\mathbb{V}} = y^{\mathbb{B}}), \quad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$$

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$$(x^{\mathbb{V}} = y^{\mathbb{A}}), \quad \exists x^{\mathbb{B}}. \forall y^{\mathbb{B}}. (x^{\mathbb{B}} = y^{\mathbb{B}})$$

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Apply $\{x^{\mathbb{V}} \mapsto y^{\mathbb{A}}\}$?

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Record dependency in

negative variable-condition N : $(x^{\mathbb{V}}, y^{\mathbb{A}}) \in N$

2nd Example Proof Attempt

δ^+ -rule:

$$\frac{\Gamma, \forall y^{\mathbb{B}}.A, \Delta}{A\{y^{\mathbb{B}} \mapsto y^{\mathbb{V}}\}, \Gamma, \Delta} \quad (y^{\mathbb{V}}, \varepsilon y^{\mathbb{B}}. \neg A)$$
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We have to prove in advance $\{y^{\mathbb{V}} \mapsto x^{\mathbb{V}}\}$ -instance of:

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$$\text{i.e.} \quad \exists y^{\mathbb{B}}. \neg(x^{\mathbb{V}} = y^{\mathbb{B}}) \Rightarrow \neg(x^{\mathbb{V}} = x^{\mathbb{V}})$$

Variable-Conditions in the Literature

- Wolfgang Bibel's book *Automated Theorem Proving* 1982, (2nd edn. 1987):
 - Two positive relations: awkward & inefficient
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- Lincoln A. Wallen 1990:
 - Single positive relation
 - No liberalized δ -rules

Variable-Conditions in the Literature (contd.)

- Michael Kohlhase's articles
 - With liberalized δ -rules
 - *Higher-Order Tableaux* [TABLEAUX'1995]:
Unsound!
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- Wirth's previous versions with single relation
 - Negative relation [FTP'1998]
 - Positive relation with Variable reuse [J. IGPL 2004]
 - Standard positive relation [J. IGPL 2004], [J. Appl. L. 2008],
SEKI-Report SR–2006–02 [2012]

Positive/Negative Variable-Condition (P, N)

$$P \subseteq (\forall \uplus A) \times \forall$$

$$N \subseteq \forall \times A$$

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$$P \subseteq (\mathbb{V} \uplus \mathbb{A}) \times \mathbb{V}$$

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- *Consistency:*

Each cycle in the directed graph of $P \cup N$ has more than one edge from N .

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Each cycle in the directed graph of $P \cup N$ has more than one edge from N .

- *Admissible substitution σ : $(P \cup D, N)$ is consistent.*

$(x^{\mathbb{A}}, y^{\mathbb{V}}) \in D$ iff

$y^{\mathbb{V}} \in \text{dom}(\sigma)$ and

$x^{\mathbb{A}}$ is a free variable or free atom in $\sigma(y^{\mathbb{V}})$

Whole Proof Search Framework

DATA STRUCTURES:

- A Forest of and/or proof-attempt trees.
Root of each tree carries an [open] proposition.

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“The the solutions of the leaves solve the root.”

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This is more than soundness of problem reduction!

Whole Proof Search Framework (contd.)

OPERATIONS:

- New conjectures get trivial proof-attempt tree.

Whole Proof Search Framework (contd.)

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PS-INVARIANT! But:

$x^{\forall} \neq y^{\forall}$ means that the universe is non-trivial.

It becomes false after application of $\{x^{\forall} \mapsto y^{\forall}\}$

Henkin Quantification

- “Every woman could love someone and every man could love someone, such that these loved ones could love each other.”

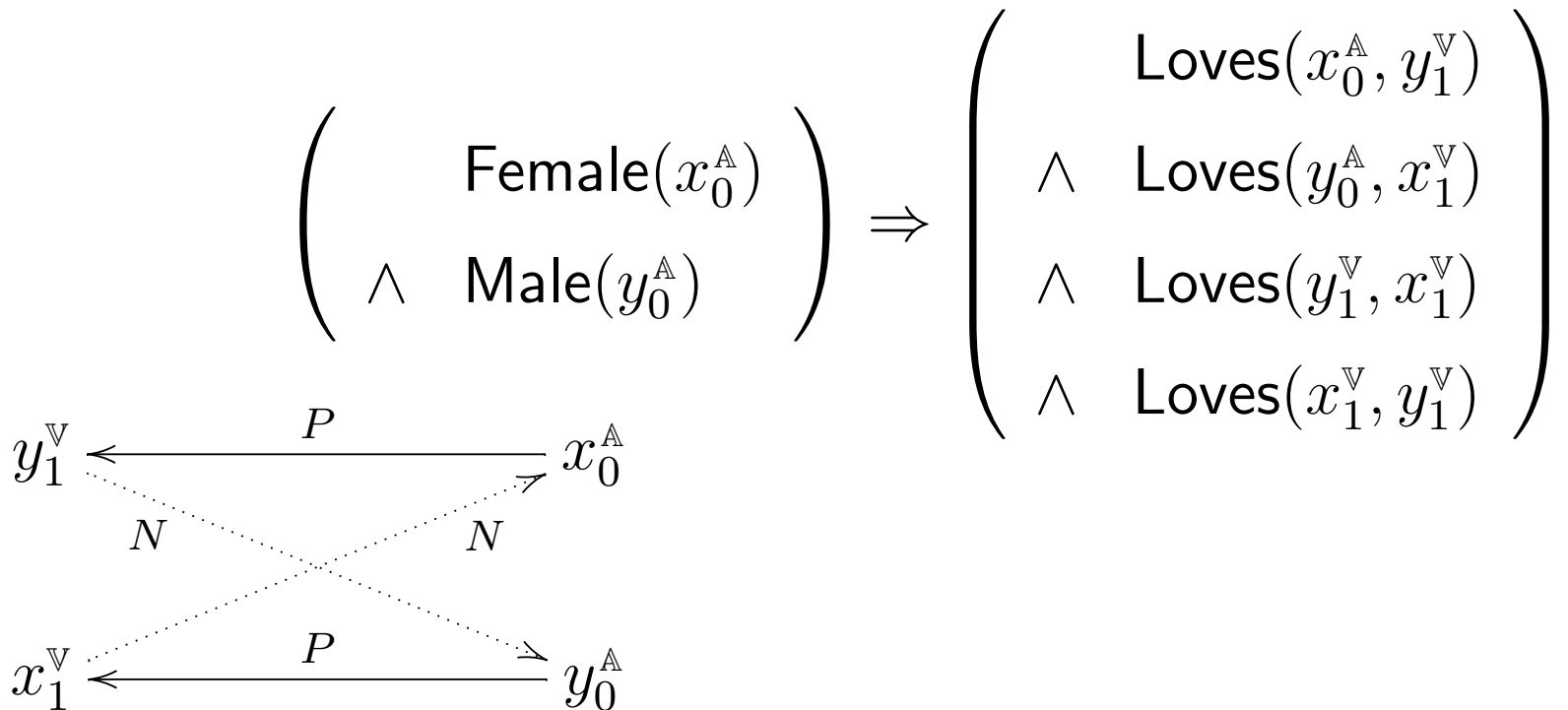
Henkin Quantification

- “Every woman could love someone and every man could love someone, such that these loved ones could love each other.”
- As a Henkin-quantified IF-logic formula:

$$\forall x_0^{\mathbb{B}}. \forall y_0^{\mathbb{B}}. \left(\begin{array}{l} \left(\begin{array}{l} \text{Female}(x_0^{\mathbb{B}}) \\ \wedge \text{Male}(y_0^{\mathbb{B}}) \end{array} \right) \\ \Rightarrow \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}. \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}. \left(\begin{array}{l} \text{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge \text{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge \text{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge \text{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{array} \right) \end{array} \right)$$

Henkin Quantification

- “Every woman could love someone and every man could love someone, such that these loved ones could love each other.”
- Represented in our Framework:



Binders are Bad:

- Quantifiers and the ε mess up formulas
- Quantifiers and the ε make reasoning difficult
- Quantifiers enforce a too primitive form of scoping
- The ε -binder produces terms of unmanageable size

Binding without Binders is Great:

- Free variables and atoms are what we need to manage practical applications
- Positive/Negative Variable-Conditions enable sophisticated scoping
- The term-sharing of free variables admits ε -binding that is manageable w.r.t. term size
- Our semantics for the ε is existential (!) and admits indefinite committed choice
- Free atoms admit mathematical induction in the liberal style of Fermat's *Descente Infinie*

Conclusion

Want your reasoning applications to be successful in practice?

- Ask for a tailored version of a free-variable framework!
- Do not introduce free variables and atoms ad hoc for operational purposes,
but give them a clear semantics
- Get *both* free variables *and* free atoms:
- The liberalized δ -rule (δ^+ -rule) is a practical improvement only if the non-liberalize δ -rule (δ^- -rule) remains available (Henkin, Fermat)
- Use an ε with existential semantics

Related Publications

- *Descente Infinie + Deduction*

Logic J. of the IGPL 12:1–96, 2004, Oxford Univ. Press

- *Hilbert's epsilon as an Operator of Indefinite Committed Choice (IDC)*

J. Applied Logic 6:287–317, 2008, Elsevier

- *A Simplified and Improved Free-Variable Framework of Hilbert's ε as an Operator of IDC*

SEKI-Report SR–2011–01, Revised May 2015.

<http://arxiv.org/abs/1104.2444>

- *Hilbert–Bernays: Grundlagen der Mathematik.*

1st English translation. With comments and German facsimile.

<http://wirth.bplaced.net/p/hilbertbernays>

Semantic treatment of Variable-Conditions

$$e(\pi)(\delta)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})(S_{\pi}\langle\{x^{\mathbb{V}}\}\rangle\delta).$$

$$\pi : \mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}, \quad \delta : \mathbb{A} \rightsquigarrow \mathcal{S}, \quad x \in \mathbb{V}$$

$$e : (\mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \rightarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightarrow \mathbb{V} \rightsquigarrow \mathcal{S}$$

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$$S_{\pi} := \{ (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \text{dom}(\pi) \wedge y^{\mathbb{A}} \in \text{dom}(\bigcup(\text{dom}(\pi(x^{\mathbb{V}})))) \}$$

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π is \mathcal{S} -compatible with $(C, (P, N))$ if ... and

$(P \cup S_{\pi}, N)$ is consistent and

π respects C in \mathcal{S}

Reduction, Def.

$G_0 (C, (P, N))$ -reduces to G_1 in \mathcal{S} if

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Let G_0 and G_1 be sets of sequents.

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G_0 $(C, (P, N))$ -reduces to G_1 in \mathcal{S} if

for each π that is \mathcal{S} -compatible with $(C, (P, N))$:

if G_1 is (π, \mathcal{S}) -valid,
then G_0 is (π, \mathcal{S}) -valid.

Reduction PS-Invariant under Substitution

If $G_0 (C, (P, N))$ -reduces to G_1 in \mathcal{S} ,
then $G_0\sigma (C', (P', N'))$ -reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$
in \mathcal{S} .

Reduction PS-Invariant under Substitution

For an (P, N) -substitution σ on \mathbb{V} ,
for the extended σ -update $(C', (P', N'))$
of $(C, (P, N))$:

If $G_0 (C, (P, N))$ -reduces to G_1 in \mathcal{S} ,
then $G_0\sigma (C', (P', N'))$ -reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$
in \mathcal{S} .

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}} \cdot \varepsilon y^{\mathbb{B}} \cdot (v_0^{\mathbb{B}} = y^{\mathbb{B}} + 1) :$

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$Q_C(y^{\mathbb{V}})$

$$= \forall v_0^{\mathbb{B}}. \left(\begin{array}{l} \exists y^{\mathbb{B}}. (v_0^{\mathbb{B}} = y^{\mathbb{B}} + 1) \\ \Rightarrow (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}}) + 1) \end{array} \right)$$

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$(Q_C(y^{\mathbb{V}}))\{y^{\mathbb{V}} \mapsto \mathbf{p}\}$

$$= \forall v_0^{\mathbb{B}}. \left(\begin{array}{l} \exists y^{\mathbb{B}}. (v_0^{\mathbb{B}} = y^{\mathbb{B}} + 1) \\ \Rightarrow (v_0^{\mathbb{B}} = \mathbf{p}(v_0^{\mathbb{B}}) + 1) \end{array} \right)$$