Free Variables and Atoms

Hilbert's ε and Henkin quantification without binding

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Free Variables and Atoms

- Occur frequently in math & computer science
- Their function depends on context: varying, implicit, ad hoc
- Here: disjoint sets of symbols for different functions
- New: only two functions of free variables/atoms left
- New: Henkin quantification can be modeled directly

Free Atoms (="parameters"="eigenvariables"="δ^-")

(1) $$(m)^{(p+q)} = (m)^{(p)} * (m)^{(q)}$$

(2) $$\forall m^{\mathbb{B}}, p^{\mathbb{B}}, q^{\mathbb{B}}.\ \left((m^{\mathbb{B}})^{(p^{\mathbb{B}}+q^{\mathbb{B}})} = (m^{\mathbb{B}})^{(p^{\mathbb{B}})} * (m^{\mathbb{B}})^{(q^{\mathbb{B}})} \right)$$

(3) $$(m^{\mathbb{A}})^{(p^{\mathbb{A}}+q^{\mathbb{A}})} = (m^{\mathbb{A}})^{(p^{\mathbb{A}})} * (m^{\mathbb{A}})^{(q^{\mathbb{A}})}$$

- Advantage of (1) and (3): Mathematical induction (Fermat's *Descente infinie*) becomes possible because the induction hypothesis can refer to the conclusion by means of the free variables.
- Semantics is uniquely expressed in (2) and (3).

Free Variables (="meta"="dummy"="γ"="δ^+")

(1) $$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

(2) $$\exists x^{\mathbb{B}}, y^{\mathbb{B}}.\ \left(\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{B}} \\ y^{\mathbb{B}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix} \right)$$

(3) $$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{V}} \\ y^{\mathbb{V}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

- Constraints on solutions may be retrieved from a proof by referring to the variables in (1) and (3).
- Semantics is uniquely expressed in (2) and (3).

Free Atoms

- $\mathbb{A}$, Universally quantified (implicitly)
- Arbitrary object in a discourse
- Atomic: black-box, no information on it ever
- Except: Is it an atom? Different from another atom?
- Origin of name: Set theories with atoms (or urelements)
- Instantiated locally and repeatedly in application of lemmas or induction hypotheses

Free Variables

- $\mathbb{V}$, Existentially quantified (implicitly)
- Place-holder in a discourse
- Gather and store information on them
- Replaced with a definition or a description
- Origin of name: Fitting's free-variable semantic tableaus
- Rigid: Instantiated globally, once and for all, with possible effect on input theorem

What is Hilbert's epsilon?

- A binder similar to Peano's (inverted) ι-operator, binding a variable in a formula, returning a *term*
- (Under-) specified by Hilbert by the ε-*formula*:
  $$A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \;\;\Rightarrow\;\; A\{x^{\mathbb{B}}\mapsto \varepsilon x^{\mathbb{B}}.\,A\}$$
- Alternative specification by (ε_0):
  $$\exists x^{\mathbb{B}}.\,A \;\Rightarrow\; A\{x^{\mathbb{B}}\mapsto \varepsilon x^{\mathbb{B}}.\,A\}$$
- Peano's (inverted) ι:
  $$\exists! x^{\mathbb{B}}.\,A \;\Rightarrow\; A\{x^{\mathbb{B}}\mapsto \iota x^{\mathbb{B}}.\,A\}$$

(Reductive) Inference (Smullyan's classification)

γ-rule:
$$\frac{\varGamma,\;\; \exists x^{\mathbb{B}}.A,\;\; \varDelta}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\},\;\; \varGamma,\;\; \exists x^{\mathbb{B}}.A,\;\; \varDelta}$$

δ^--rule:
$$\frac{\varGamma,\;\forall y^{\mathbb{B}}.A,\;\varDelta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\;\varGamma,\;\varDelta} \qquad \mathbb{V}(\varGamma,\forall y^{\mathbb{B}}.A,\varDelta)\times\{y^{\mathbb{A}}\}$$

δ^+-rule:
$$\frac{\varGamma,\;\forall y^{\mathbb{B}}.A,\;\varDelta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\},\;\varGamma,\;\varDelta} \qquad \begin{array}{c}(y^{\mathbb{V}},\;\; \varepsilon y^{\mathbb{B}}.\,\neg A)\\ \mathbb{VA}(\forall y^{\mathbb{B}}.A)\times\{y^{\mathbb{V}}\}\end{array}$$

1st Example Proof Attempt

δ^--rule:
$$\frac{\varGamma,\;\forall y^{\mathbb{B}}.A,\;\varDelta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\;\varGamma,\;\varDelta} \qquad \mathbb{V}(\varGamma,\forall y^{\mathbb{B}}.A,\varDelta)\times\{y^{\mathbb{A}}\}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\;\; \exists x^{\mathbb{VA}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

δ^--step: $(x^{\mathbb{V}}=y^{\mathbb{A}}),\;\; \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{A}}\}$?

Record dependency in
negative variable-condition N: $(x^{\mathbb{V}},y^{\mathbb{A}})\in N$

2nd Example Proof Attempt

δ^+-rule:
$$\frac{\varGamma,\;\forall y^{\mathbb{B}}.A,\;\varDelta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\},\;\varGamma,\;\varDelta} \qquad \begin{array}{c}(y^{\mathbb{V}},\;\; \varepsilon y^{\mathbb{B}}.\,\neg A)\\ \mathbb{VA}(\forall y^{\mathbb{B}}.A)\times\{y^{\mathbb{V}}\}\end{array}$$

Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}}=y^{\mathbb{B}}),\;\; \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

δ^+-step: $(x^{\mathbb{V}}=y^{\mathbb{V}}),\;\; \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}}=y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$?

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$?

Record dependency in Apply
positive variable-condition P: $(x^{\mathbb{V}},y^{\mathbb{V}})\in P$
$\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$?

Variable-Condition in the Literature

- Wolfgang Bibel's book *Automated Theorem Proving* 1982, (2nd edn. 1987):
  - Two positive relations: awkward & inefficient
  - No liberalized δ-rule
- Lincoln A. Wallen 1990:
  - Single positive relation
  - No liberalized δ-rule

Variable-Condition in the Literature (contd.)

- Michael Kohlhase's articles
  - With liberalized δ-rule
  - *Higher-Order Tableaux* [TABLEAUX'1995]: Unsound!
  - *Higher-Order Theorem Proving* [Bibel & Schmitt, Automated Deduction, Vol. I, 1998]: Unsound!
- Wirth's previous versions with single relation
  - Negative relation [FTP'1998]
  - Positive relation with Variable reuse [J. IGPL 2004]
  - Standard positive relation [J. IGPL 2004], [J. Appl. L. 2008], [SEKI 2012]

Positive/Negative Variable-Condition (P, N)

$$\begin{array}{lclcl} P & \subseteq & (\mathbb{V} \uplus \mathbb{A}) & \times & \mathbb{V} \\ N & \subseteq & \mathbb{V} & \times & \mathbb{A} \end{array}$$

- *Consistency:*
  Each cycle in the directed graph of $P \cup N$ has more than one edge from N.
- *Admissible substitution* σ*:* $(P \cup D, N)$ is consistent.
  $(x^{\mathbb{VA}}, y^{\mathbb{V}}) \in D$ iff
  $y^{\mathbb{V}} \in \mathrm{dom}(\sigma)$ and
  $x^{\mathbb{VA}}$ is a free variable or free atom in $\sigma(y^{\mathbb{V}})$

Whole Proof Search Framework

DATA STRUCTURES:

- A Forest of and/or proof-attempt trees.
  Root of each tree carries an [open] proposition.
- A Positive/Negative Variable-Condition (P, N)
- A Choice-Condition C

PS-INVARIANT (“Preservation of solutions”):
“The the solutions of the leaves solve the root.”
The root $(C, (P, N))$-reduces to the leaves.
This is more than soundness of problem reduction!

Whole Proof Search Framework (contd.)

OPERATIONS:

- New conjectures get trivial proof-attempt tree.
- Analytic proof steps local to some tree.
- Generative application of open lemmas,
  either deductively or inductively.
- Global substitution of rigid variables.
  PS-INVARIANT! But:
  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.
  It becomes false after application of $\{x^{\mathbb{V}} \mapsto y^{\mathbb{V}}\}$

Henkin Quantification

- “Every woman could love someone and
  every man could love someone,
  such that these loved ones could love each other.”
- As a Henkin-quantified IF-logic formula:
- Represented in our Framework:

$$\forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \left(\begin{array}{l} \left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array}\right) \\ \Rightarrow\ \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.\ \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{array}\right) \end{array}\right)$$

Binders are Bad:

- Quantifiers and the ε mess up formulas
- Quantifiers and the ε make reasoning difficult
- Quantifiers enforce a too primitive form of scoping
- The ε-binder produces terms of unmanageable size

Binding without Binders is Great:

- Free variables and atoms are what we need to manage practical applications
- Positive/Negative Variable-Conditions enable sophisticated scoping
- The term-sharing of free variables admits ε-binding that is manageable w.r.t. term size
- Our semantics for the ε is existential (!) and admits indefinite committed choice
- Free atoms admit mathematical induction in the liberal style of Fermat’s *Descente Infinie*

Conclusion

Want your reasoning applications to successful in practice?

- Do not buy a logic off-the-shelf!
- Ask for a tailored version of a free-variable framework!
- Do not introduce free variables and atoms ad hoc for operational purposes, but give them a clear semantics
- Get *both* free variables *and* free atoms:
- The liberalized δ-rule (δ^+-rule) is a practical improvement only if the non-liberalize δ-rule (δ^--rule) remains available (Henkin, Fermat)

Related Publications

- *Descente Infinie + Deduction*
  Logic J. of the IGPL 12:1–96, 2004, Oxford Univ. Press
- *Hilbert's epsilon as an Operator of Indefinite Committed Choice.*
  - Shorter version:
    J. Applied Logic 6:287–317, 2008, Elsevier,
    `http://dx.doi.org/10.1016/j.jal.2007.07.009`
  - Long version, revised 2012
    SEKI-Report SR–2006–02 (ISSN 1437–4447), 72 pp.,
    `http://arxiv.org/abs/0902.3749`

Related Publications (contd.)

- $\lim +$, δ^+, *and Non-Permutability of* β*-steps*
  J. Symbolic Computation, 47, Issue 9, pp.1109–1135, 2012
- *Jacques Herbrand: Life, Logic, and Automated Deduction.*
  Handbook of the History of Logic, Vol. 5, North-Holland, 2009.
- *Hilbert & Bernays: Foundations of Mathematics.*
  First English edition.
  With English comments and German facsimile.
  `http://wirth.bplaced.net/p/hilbertbernays`

Semantic treatment of Variable-Conditions

$$\epsilon(\pi)(\delta)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\delta).$$

$$\pi : \mathbb{V} \leadsto (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}, \qquad \delta : \mathbb{A} \leadsto \mathcal{S}, \quad x \in \mathbb{V}$$

$$\epsilon \quad : \quad (\mathbb{V} \leadsto (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}) \quad \to \quad (\mathbb{A} \leadsto \mathcal{S}) \quad \to \quad \mathbb{V} \quad \leadsto \quad \mathcal{S}$$

$$S_\pi \ := \ \{\ (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\pi) \wedge y^{\mathbb{A}} \in \mathrm{dom}(\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}}))))\ \}$$

π is *$\mathcal{S}$-compatible with* $(C, (P, N))$ if $\ldots$ and
$(P \cup S_\pi, N)$ is consistent and
π respects C in $\mathcal{S}$

Reduction, Def.

Let G_0 and G_1 be sets of sequents.
Let $\mathcal{S}$ be a Σ-structure.
Let C be an (P, N)-choice-condition.

G_0 $(C, (P, N))$*-reduces to* G_1 *in* $\mathcal{S}$ if
for each π that is $\mathcal{S}$-compatible with $(C, (P, N))$:

if G_1 is $(\pi, \mathcal{S})$-valid,
then G_0 is $(\pi, \mathcal{S})$-valid.

Reduction PS-Invariant under Substitution

For an (P, N)-substitution σ on $\mathbb{V}$,
for the extended σ-update $(C', (P', N'))$
of $(C, (P, N))$:

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$
in $\mathcal{S}$.

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1)$:

$Q_C(y^{\mathbb{V}})$

$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$

$(Q_C(y^{\mathbb{V}}))\{y^{\mathbb{V}} {\mapsto} \mathsf{p}\}$

$$= \quad \forall v_0^{\mathbb{B}}. \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$