Free Variables and Atoms

Hilbert's ε and Henkin quantification without binding

Claus-Peter Wirth

Free Variables and Atoms

- Occur frequently in math & computer science

- Their function depends on context:

  varying, implicit, ad hoc

- Here:

  disjoint sets of symbols for different functions

- New:

  only two functions of free variables/atoms left

- New:

  Henkin quantification can be modeled directly

Free Atoms (="parameters"="eigenvariables"="δ^-")

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- Advantage of (1) and (3): Mathematical induction (Fermat's *Descente infinie*) becomes possible because the induction hypothesis can refer to the conclusion by means of the free variables.

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- Semantics is uniquely expressed in (2) and (3).

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- Constraints on solutions may be retrieved from a proof by referring to the variables in (1) and (3).

- Semantics is uniquely expressed in (2) and (3).

Free Atoms

- $\mathbb{A}$, Universally quantified (implicitly)

- Arbitrary object in a discourse

- Atomic: black-box, no information on it ever

- Except: Is it an atom?
  Different from another atom?

- Origin of name:
  Set theories with atoms (or urelements)

- Instantiated locally and repeatedly in application
  of lemmas or induction hypotheses

Free Variables

- $\mathbb{V}$, Existentially quantified (implicitly)

- Place-holder in a discourse

- Gather and store information on them

- Replaced with a definition or a description

- Origin of name:
  Fitting's free-variable semantic tableaus

- Rigid: Instantiated globally, once and for all,
  with possible effect on input theorem

What is Hilbert's epsilon?

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- Peano's (inverted) ι:
$$\exists! x^{\mathbb{B}}.\, A \quad \Rightarrow \quad A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}.\, A\}$$

(Reductive) **Inference** (Smullyan's classification)

γ-rule:

$$\frac{\Gamma, \quad \exists x^{\mathbb{B}}.A, \quad \Delta}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\}, \quad \Gamma, \quad \exists x^{\mathbb{B}}.A, \quad \Delta}$$

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δ^--rule:

$$\frac{\Gamma, \quad \forall y^{\mathbb{B}}.A, \quad \Delta}{A\{y^{\mathbb{B}} \mapsto y^{\mathbb{A}}\}, \quad \Gamma, \quad \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

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δ^+-rule:

$$\frac{\Gamma, \quad \forall y^{\mathbb{B}}.A, \quad \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\}, \quad \Gamma, \quad \Delta} \qquad \begin{array}{c}(y^{\mathbb{V}}, \quad \varepsilon y^{\mathbb{B}}. \ \neg A) \\ \mathbb{V}\mathbb{A}(\forall y^{\mathbb{B}}.A) \times \{y^{\mathbb{V}}\}\end{array}$$

1st Example Proof Attempt

δ^--rule:

$$\frac{\Gamma, \ \forall y^{\mathbb{B}}.A, \ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\}, \ \Gamma, \ \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

Proof task: $\qquad\qquad\qquad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

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γ-step: $\qquad\qquad \forall y^{\mathbb{B}}.(x^{\mathbb{V}} = y^{\mathbb{B}}), \ \ \exists x^{\mathbb{V\!A}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

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γ-step: $\qquad\qquad \forall y^{\mathbb{B}}.(x^{\mathbb{V}} = y^{\mathbb{B}}), \ \ \exists x^{\mathbb{VA}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

δ^--step: $\qquad\qquad\qquad (x^{\mathbb{V}} = y^{\mathbb{A}}), \ \ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

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$$\frac{\Gamma, \ \forall y^{\mathbb{B}}.A, \ \Delta}{A\{y^{\mathbb{B}}\!\mapsto\! y^{\mathbb{A}}\}, \ \Gamma, \ \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

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δ^--step: $\qquad\qquad\qquad (x^{\mathbb{V}} = y^{\mathbb{A}}), \ \ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\!\mapsto\! y^{\mathbb{A}}\}$?

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$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \Gamma,\ \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

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γ-step: $\forall y^{\mathbb{B}}.(x^{\mathbb{V}} = y^{\mathbb{B}}),\ \ \exists x^{\mathbb{V}\mathbb{A}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

δ^--step: $(x^{\mathbb{V}} = y^{\mathbb{A}}),\ \ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{A}}\}$?

Record dependency in
negative variable-condition N: $\ \ (x^{\mathbb{V}}, y^{\mathbb{A}}) \in N$

2nd Example Proof Attempt

δ^+-rule:

$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\},\ \Gamma,\ \Delta} \qquad (y^{\mathbb{V}},\ \varepsilon y^{\mathbb{B}}.\ \neg A)$$

$$\mathbb{V\!A}(\forall y^{\mathbb{B}}.A) \times \{y^{\mathbb{V}}\}$$

Proof task: $\qquad\qquad\qquad \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

γ-step: $\qquad\qquad \forall y^{\mathbb{B}}.(x^{\mathbb{V}} = y^{\mathbb{B}}),\ \exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

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Apply $\{x^{\mathbb{V}}\!\mapsto\! y^{\mathbb{V}}\}$?

Record dependency in
positive variable-condition P: $(x^{\mathbb{V}}, y^{\mathbb{V}})\in P$

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Apply $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$?

We have to prove in advance $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$-instance of:

(ε_0) $\qquad\qquad\qquad \exists y^{\mathbb{B}}.\neg A \Rightarrow \neg A\{y^{\mathbb{B}}\mapsto y^{\mathbb{V}}\}$

2$^{\text{nd}}$ **Example Proof Attempt**

δ^+-rule:

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i.e. $\qquad\qquad \exists y^{\mathbb{B}}.\neg(x^{\mathbb{V}} = y^{\mathbb{B}}) \Rightarrow \neg(x^{\mathbb{V}} = x^{\mathbb{V}})$

Variable-Condition in the Literature

- Wolfgang Bibel's book *Automated Theorem Proving* 1982, (2nd edn. 1987):

    - Two positive relations: awkward & inefficient

    - No liberalized δ-rule

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- Lincoln A. Wallen 1990:

  - Single positive relation

  - No liberalized δ-rule

Variable-Condition in the Literature (contd.)

- Michael Kohlhase's articles

  - With liberalized δ-rule

  - *Higher-Order Tableaux* [TABLEAUX'1995]:
  Unsound!

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- ■ Wirth's previous versions with single relation
  - ■ Negative relation [FTP'1998]
  - ■ Positive relation with Variable reuse [J. IGPL 2004]
  - ■ Standard positive relation [J. IGPL 2004], [J. Appl. L. 2008], [SEKI 2012]

Positive/Negative Variable-Condition (P, N)

$$P \quad \subseteq \quad (\mathbb{V} \uplus \mathbb{A}) \quad \times \quad \mathbb{V}$$
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- *Consistency:*

Each cycle in the directed graph of $P \cup N$ has more than one edge from N.

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- *Consistency:*

  Each cycle in the directed graph of $P \cup N$ has more than one edge from N.

- *Admissible substitution σ:* $(P \cup D, N)$ is consistent.
  $(x^{\mathbb{A}}, y^{\mathbb{V}}) \in D$ iff

    $y^{\mathbb{V}} \in \mathrm{dom}(\sigma)$ and

    $x^{\mathbb{A}}$ is a free variable or free atom in $\sigma(y^{\mathbb{V}})$

Whole Proof Search Framework

DATA STRUCTURES:

- A Forest of and/or proof-attempt trees.

  Root of each tree carries an [open] proposition.

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"The the solutions of the leaves solve the root."

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The root $(C, (P, N))$-reduces to the leaves.

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PS-INVARIANT ("Preservation of solutions"):

"The the solutions of the leaves solve the root."

The root $(C, (P, N))$-reduces to the leaves.

This is more than soundness of problem reduction!

Whole Proof Search Framework (contd.)

OPERATIONS:

- New conjectures get trivial proof-attempt tree.

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- Global substitution of rigid variables.
  PS-INVARIANT!   But:
  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.

Whole Proof Search Framework (contd.)

OPERATIONS:

- New conjectures get trivial proof-attempt tree.

- Analytic proof steps local to some tree.

- Generative application of open lemmas, either deductively or inductively.

- Global substitution of rigid variables.
  PS-INVARIANT!   But:

  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.
  It becomes false after application of $\{x^{\mathbb{V}} \mapsto y^{\mathbb{V}}\}$

Henkin Quantification

- "Every woman could love someone and
  every man could love someone,
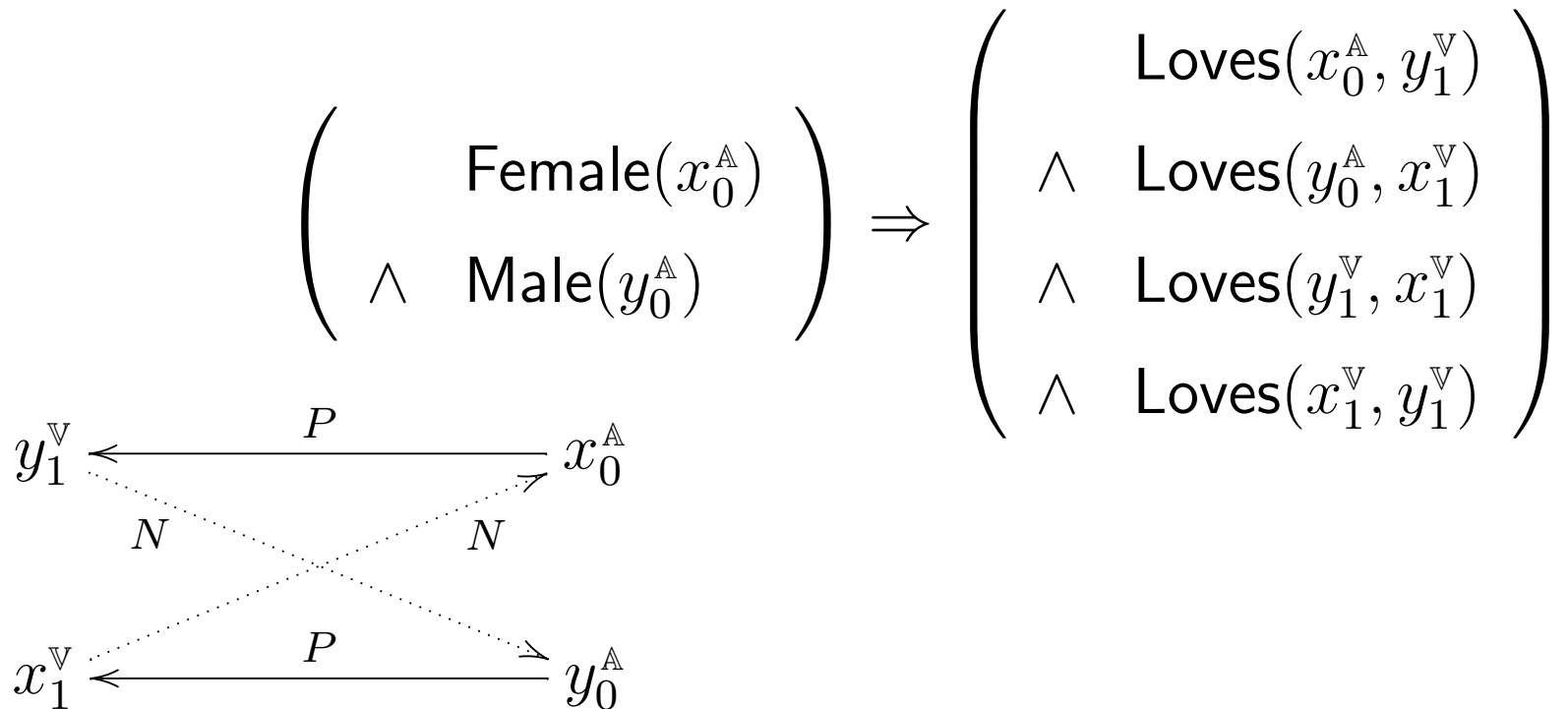 such that these loved ones could love each other."

Henkin Quantification

- "Every woman could love someone and
every man could love someone,
such that these loved ones could love each other."

- As a Henkin-quantified IF-logic formula:

$$\forall x_0^{\mathbb{B}}. \ \forall y_0^{\mathbb{B}}. \left(\begin{pmatrix} \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge \quad \mathsf{Male}(y_0^{\mathbb{B}}) \end{pmatrix} \Rightarrow \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}. \ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}. \begin{pmatrix} \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge \quad \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge \quad \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge \quad \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{pmatrix} \right)$$

Henkin Quantification

- "Every woman could love someone and
  every man could love someone,
  such that these loved ones could love each other."

- Represented in our Framework:

$$\begin{pmatrix} \mathsf{Female}(x_0^{\mathbb{A}}) \\ \wedge \quad \mathsf{Male}(y_0^{\mathbb{A}}) \end{pmatrix} \Rightarrow \begin{pmatrix} \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}) \\ \wedge \quad \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \\ \wedge \quad \mathsf{Loves}(y_1^{\mathbb{V}}, x_1^{\mathbb{V}}) \\ \wedge \quad \mathsf{Loves}(x_1^{\mathbb{V}}, y_1^{\mathbb{V}}) \end{pmatrix}$$

Binders are Bad:

- Quantifiers and the ε mess up formulas

- Quantifiers and the ε make reasoning difficult

- Quantifiers enforce a too primitive form of scoping

- The ε-binder produces terms of unmanageable size

Binding without Binders is Great:

- Free variables and atoms are what we need to manage practical applications

- Positive/Negative Variable-Conditions enable sophisticated scoping

- The term-sharing of free variables admits ε-binding that is manageable w.r.t. term size

- Our semantics for the ε is existential (!) and admits indefinite committed choice

- Free atoms admit mathematical induction in the liberal style of Fermat's *Descente Infinie*

Conclusion

Want your reasoning applications to successful in practice?

- Do not buy a logic off-the-shelf!

- Ask for a tailored version of a free-variable framework!

- Do not introduce free variables and atoms ad hoc for operational purposes,
  but give them a clear semantics

- Get *both* free variables *and* free atoms:

- The liberalized δ-rule (δ^+-rule) is a practical improvement only if the non-liberalize δ-rule (δ^--rule) remains available (Henkin, Fermat)

Related Publications

- *Descente Infinie + Deduction*

  Logic J. of the IGPL 12:1–96, 2004, Oxford Univ. Press

- *Hilbert's epsilon*
  as an Operator of Indefinite Committed Choice.

  - Shorter version:

    J. Applied Logic 6:287–317, 2008, Elsevier,

    `http://dx.doi.org/10.1016/j.jal.2007.07.009`

  - Long version, revised 2012

    SEKI-Report SR–2006–02 (ISSN 1437–4447), 72 pp.,

    `http://arxiv.org/abs/0902.3749`

Related Publications (contd.)

- $\lim +$, δ^+, *and Non-Permutability of β-steps*

  J. Symbolic Computation, 47, Issue 9, pp.1109–1135, 2012

- *Jacques Herbrand:*
  Life, Logic, and Automated Deduction.

  Handbook of the History of Logic, Vol. 5, North-Holland, 2009.

- *Hilbert & Bernays: Foundations of Mathematics.*

  First English edition.

  With English comments and German facsimile.

  `http://wirth.bplaced.net/p/hilbertbernays`

Semantic treatment of Variable-Conditions

$$\epsilon(\pi)(\delta)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})\big(_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}\!\!\upharpoonright\!\delta\big).$$

$$\pi : \mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}, \qquad \delta : \mathbb{A} \rightsquigarrow \mathcal{S}, \quad x \in \mathbb{V}$$

$$\epsilon \; : \; (\mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \; \longrightarrow \; (\mathbb{A} \rightsquigarrow \mathcal{S}) \; \longrightarrow \; \mathbb{V} \; \rightsquigarrow \; \mathcal{S}$$

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$$\epsilon \;:\; (\mathbb{V} \rightsquigarrow (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \;\rightarrow\; (\mathbb{A} \rightsquigarrow \mathcal{S}) \;\rightarrow\; \mathbb{V} \;\rightsquigarrow\; \mathcal{S}$$

$$S_\pi \;:=\; \{\, (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\pi) \wedge y^{\mathbb{A}} \in \mathrm{dom}(\textstyle\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}})))) \,\}$$

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π is $\mathcal{S}$-*compatible with* $(C, (P, N))$ if … and

$(P \cup S_\pi, N)$ is consistent and

π respects C in $\mathcal{S}$

Reduction, Def.

G_0 $(C, (P, N))$-*reduces to* G_1 *in* $\mathcal{S}$ if

Reduction, Def.

Let G_0 and G_1 be sets of sequents.

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Reduction, Def.

Let G_0 and G_1 be sets of sequents.

Let $\mathcal{S}$ be a Σ-structure.

Let C be an (P, N)-choice-condition.

G_0 $(C, (P, N))$-*reduces to* G_1 *in* $\mathcal{S}$ if

for each π that is $\mathcal{S}$-compatible with $(C, (P, N))$:

$$\text{if } G_1 \text{ is } (\pi, \mathcal{S})\text{-valid,}$$
$$\text{then } G_0 \text{ is } (\pi, \mathcal{S})\text{-valid.}$$

Reduction PS-Invariant under Substitution

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$
in $\mathcal{S}$.

Reduction PS-Invariant under Substitution

For an (P, N)-substitution σ on $\mathbb{V}$,

for the extended σ-update $(C', (P', N'))$

of $(C, (P, N))$:

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,

then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$

in $\mathcal{S}$.

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ \left(v_0^{\mathbb{B}} = y^{\mathbb{B}}+1\right):$

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$$Q_C(y^{\mathbb{V}})$$

$$= \quad \forall v_0^{\mathbb{B}}.\ \begin{pmatrix} \exists y^{\mathbb{B}}.\ \left(v_0^{\mathbb{B}} = y^{\mathbb{B}}+1\right) \\ \Rightarrow \ \left(v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}})+1\right) \end{pmatrix}$$

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}+1)$:

$$Q_C(y^{\mathbb{V}})$$

$$= \quad \forall v_0^{\mathbb{B}}.\ \left(\begin{array}{l} \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}+1) \\[2ex] \Rightarrow\ (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}})+1) \end{array} \right)$$

$$(Q_C(y^{\mathbb{V}}))\{y^{\mathbb{V}} \mapsto \mathsf{p}\}$$

$$= \quad \forall v_0^{\mathbb{B}}.\ \left(\begin{array}{l} \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}+1) \\[2ex] \Rightarrow\ (v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}})+1) \end{array} \right)$$