Free Variables and Atoms

Hilbert’s ε and Henkin quantification without binding

Claus-Peter Wirth

Free Variables and Atoms

- Occur frequently in math & computer science
- Their function depends on context:
  varying, implicit, ad hoc
- Here:
  disjoint sets of symbols for different functions
- New:
  only two functions of free variables/atoms left
- New:
  Henkin quantification can be modeled directly

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- Advantage of (1) and (3): Mathematical induction (Fermat's *Descente infinie*) becomes possible because the induction hypothesis can refer to the conclusion by means of the free variables.

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- Semantics is uniquely expressed in (2) and (3).

Free Variables (=“meta”=“dummy”=”γ”=”δ^+”)

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- Semantics is uniquely expressed in (2) and (3).

Free Atoms

- $\mathbb{A}$, Universally quantified (implicitly)
- Arbitrary object in a discourse
- Atomic: black-box, no information on it ever
- Except: Is it an atom?
  Different from another atom?
- Origin of name:
  Set theories with atoms (or urelements)
- Instantiated locally and repeatedly in application of lemmas or induction hypotheses

Free Variables

- $\mathbb{V}$, Existentially quantified (implicitly)
- Place-holder in a discourse
- Gather and store information on them
- Replaced with a definition or a description
- Origin of name:
  Fitting's free-variable semantic tableaus
- Rigid: Instantiated globally, once and for all,
  with possible effect on input theorem

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- Peano's (inverted) ι:

$$\exists! x^{\mathbb{B}}.\, A \;\Rightarrow\; A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}.\, A\}$$

(Reductive) Inference (Smullyan's classification)

γ-rule:

$$\frac{\Gamma, \quad \exists x^{\mathbb{B}}.A, \quad \Delta}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\}, \quad \Gamma, \quad \exists x^{\mathbb{B}}.A, \quad \Delta}$$

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$$\frac{\Gamma,\ \forall y^{\mathbb{B}}.A,\ \Delta}{A\{y^{\mathbb{B}}\mapsto y^{\mathbb{A}}\},\ \Gamma,\ \Delta} \qquad \mathbb{V}(\Gamma, \forall y^{\mathbb{B}}.A, \Delta) \times \{y^{\mathbb{A}}\}$$

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$$(y^{\mathbb{V}},\quad \varepsilon y^{\mathbb{B}}.\ \neg A)$$

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1st Example Proof Attempt

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Proof task: $\exists x^{\mathbb{B}}.\forall y^{\mathbb{B}}.(x^{\mathbb{B}} = y^{\mathbb{B}})$

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Apply $\{x^{\mathbb{V}}\mapsto y^{\mathbb{A}}\}$?

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We have to prove in advance $\{y^{\mathbb{V}}\mapsto x^{\mathbb{V}}\}$-instance of:

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i.e. $\qquad \exists y^{\mathbb{B}}.\neg(x^{\mathbb{V}} = y^{\mathbb{B}}) \Rightarrow \neg(x^{\mathbb{V}} = x^{\mathbb{V}})$

Variable-Condition in the Literature

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- Lincoln A. Wallen 1990:
  - Single positive relation
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Variable-Condition in the Literature (contd.)

- Michael Kohlhase's articles
  - With liberalized δ-rule
  - *Higher-Order Tableaux* [TABLEAUX'1995]:
    Unsound!
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- Wirth's previous versions with single relation
  - Negative relation [FTP'1998]
  - Positive relation with Variable reuse [J. IGPL 2004]
  - Standard positive relation [J. IGPL 2004],
    [J. Appl. L. 2008], [SEKI 2012]

Positive/Negative Variable-Condition (P, N)

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\begin{array}{cccccc}
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- *Admissible substitution* σ*:* $(P \cup D, N)$ is consistent.

  $(x^{\mathbb{V}\mathbb{A}}, y^{\mathbb{V}}) \in D$ iff

  $y^{\mathbb{V}} \in \mathrm{dom}(\sigma)$ and

  $x^{\mathbb{V}\mathbb{A}}$ is a free variable or free atom in $\sigma(y^{\mathbb{V}})$

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“The the solutions of the leaves solve the root.”

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The root $(C, (P, N))$-reduces to the leaves.

This is more than soundness of problem reduction!

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  PS-INVARIANT! But:
  $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ means that the universe is non-trivial.
  It becomes false after application of $\{x^{\mathbb{V}} \mapsto y^{\mathbb{V}}\}$

Henkin Quantification

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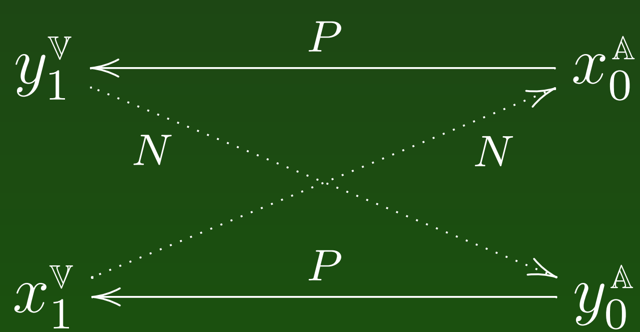
  such that these loved ones could love each other."
- As a Henkin-quantified IF-logic formula:

$$\forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \left(\begin{array}{ll} & \left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array}\right) \\ \Rightarrow & \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.\ \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{array}\right) \end{array}\right)$$

Henkin Quantification

- "Every woman could love someone and
  every man could love someone,
  such that these loved ones could love each other."
- Represented in our Framework:

$$\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{A}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{A}}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{V}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{V}}, y_1^{\mathbb{V}}) \end{array}\right)$$



Binders are Bad:

- Quantifiers and the ε mess up formulas
- Quantifiers and the ε make reasoning difficult
- Quantifiers enforce a too primitive form of scoping
- The ε-binder produces terms of unmanageable size

Binding without Binders is Great:

- Free variables and atoms are what we need to manage practical applications
- Positive/Negative Variable-Conditions enable sophisticated scoping
- The term-sharing of free variables admits ε-binding that is manageable w.r.t. term size
- Our semantics for the ε is existential (!) and admits indefinite committed choice
- Free atoms admit mathematical induction in the liberal style of Fermat's *Descente Infinie*

Conclusion

Want your reasoning applications to successful in practice?

- Do not buy a logic off-the-shelf!
- Ask for a tailored version of a free-variable framework!
- Do not introduce free variables and atoms ad hoc for operational purposes,
  but give them a clear semantics
- Get *both* free variables *and* free atoms:
- The liberalized δ-rule (δ^{+}-rule) is a practical improvement only if the non-liberalize δ-rule (δ^{-}-rule) remains available (Henkin, Fermat)

Related Publications

- *Descente Infinie + Deduction*

  Logic J. of the IGPL 12:1–96, 2004, Oxford Univ. Press

- *Hilbert's epsilon*
  as an Operator of Indefinite Committed Choice.
  - Shorter version:
    J. Applied Logic 6:287–317, 2008, Elsevier,
    `http://dx.doi.org/10.1016/j.jal.2007.07.009`
  - Long version, revised 2012
    SEKI-Report SR–2006–02 (ISSN 1437–4447), 72 pp.,
    `http://arxiv.org/abs/0902.3749`

Related Publications (contd.)

- *$\lim +$, δ^+, and Non-Permutability of β-steps*

  J. Symbolic Computation, 47, Issue 9, pp.1109–1135, 2012

- *Jacques Herbrand:*
  Life, Logic, and Automated Deduction.

  Handbook of the History of Logic, Vol. 5, North-Holland, 2009.

- *Hilbert & Bernays: Foundations of Mathematics*.

  First English edition.

  With English comments and German facsimile.

  `http://wirth.bplaced.net/p/hilbertbernays`

Semantic treatment of Variable-Conditions

$$\epsilon(\pi)(\delta)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})({}_{S_{\pi}\langle\{x^{\mathbb{V}}\}\rangle}{\upharpoonleft}\delta).$$

$$\pi : \mathbb{V} \leadsto (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}, \qquad \delta : \mathbb{A} \leadsto \mathcal{S}, \qquad x \in \mathbb{V}$$

$$\epsilon \quad : \quad (\mathbb{V} \leadsto (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}) \quad \longrightarrow \quad (\mathbb{A} \leadsto \mathcal{S}) \quad \longrightarrow \quad \mathbb{V} \quad \leadsto \quad \mathcal{S}$$

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$$S_\pi \;:=\; \{\; (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\pi) \wedge y^{\mathbb{A}} \in \mathrm{dom}(\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}})))) \;\}$$

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π is $\mathcal{S}$-*compatible with* $(C, (P, N))$ if … and

$(P \cup S_\pi, N)$ is consistent and

π respects C in $\mathcal{S}$

Reduction, Def.

G_0 $(C, (P, N))$*-reduces to* G_1 *in* $\mathcal{S}$ if

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Let G_0 and G_1 be sets of sequents.

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Reduction, Def.

Let G_0 and G_1 be sets of sequents.
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G_0 $(C, (P, N))$*-reduces to* G_1 *in* $\mathcal{S}$ if
for each π that is $\mathcal{S}$-compatible with $(C, (P, N))$:

if G_1 is $(\pi, \mathcal{S})$-valid,
then G_0 is $(\pi, \mathcal{S})$-valid.

Reduction PS-Invariant under Substitution

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$, then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$ in $\mathcal{S}$.

Reduction PS-Invariant under Substitution

For an (P, N)-substitution σ on $\mathbb{V}$,
for the extended σ-update $(C', (P', N'))$
of $(C, (P, N))$:

If G_0 $(C, (P, N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C', (P', N'))$-reduces to $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$
in $\mathcal{S}$.

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}+1)$:

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$$Q_C(y^{\mathbb{V}})$$

$$= \quad \forall v_0^{\mathbb{B}}.\ \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = y^{\mathbb{V}}(v_0^{\mathbb{B}}){+}1) \end{array} \right)$$

Example

In case of $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \varepsilon y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1)$:

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$$(Q_C(y^{\mathbb{V}}))\{y^{\mathbb{V}}{\mapsto}\mathsf{p}\}$$

$$= \quad \forall v_0^{\mathbb{B}}.\ \left(\begin{array}{ll} & \exists y^{\mathbb{B}}.\ (v_0^{\mathbb{B}} = y^{\mathbb{B}}{+}1) \\ \Rightarrow & (v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}}){+}1) \end{array}\right)$$