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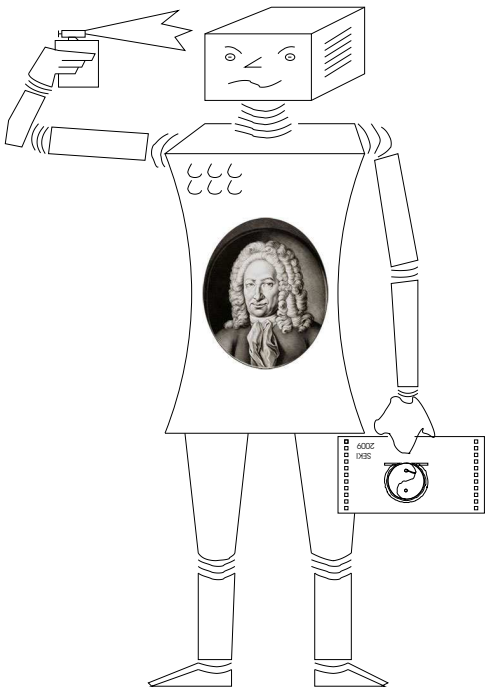

How to Kiss an Angel

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How to Kiss an Angel

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Abstract

We solve the problem of kissing an angel effectively and more efficiently than voodoo man. Some reading of Immanuël Kant, however, is required as a precondition.

Keywords: German Idealism, Dyslexia, Kissing, Angels, Angles, Kant, MSC 03D20, MSC 68Q42, MSC 16S15

1 Introduction

2 Terms and Substitutions

For being concrete, we will work with a simple-typed higher-order framework here.

There is a finite number of *sorts* or *base types*, including the BOOLEan sort bool (denoted with o by other authors). *Types* are inductively defined to be either a base type or a function type $\alpha \to \beta$, for two types α and β. The operator " $\to$ " associates to the right.

There is a finite number of *constants*, including $\mathsf{true}, \mathsf{false} : \mathsf{bool}$, $\neg : \mathsf{bool} \to \mathsf{bool}$, $\Rightarrow, \vee, \wedge : \mathsf{bool} \to \mathsf{bool} \to \mathsf{bool}$, $\forall_\alpha, \exists_\alpha : (\alpha \to \mathsf{bool}) \to \mathsf{bool}$, and $\varepsilon_{\alpha,n} : (\alpha \to \mathsf{bool}) \to \alpha$, for each type α and for each natural number n. Each constant must have a unique type.

There are three disjoint sets of *bound atoms* $\mathbb{B}$, (free) *atoms* $\mathbb{A}$, and (free) *variables* $\mathbb{V}$. Each of their elements has a unique type, and each of the three sets has infinitely many elements of each type. We define:

$$\mathbb{VA} := \mathbb{V} \uplus \mathbb{A}$$

$$\mathbb{VAB} := \mathbb{V} \uplus \mathbb{A} \uplus \mathbb{B}$$

The classification as a free variable or as a free or bound atom will be indicated by adjoining a "$\mathbb{V}$", an "$\mathbb{A}$", or a "$\mathbb{B}$", respectively, as a label to the upper right of the meta-variable for the symbol. If a meta-variable stands for a symbol of the union of some of these sets, we will indicate this by listing all possible sets; e.g. "$x^{\mathbb{VA}}$" is a meta-variable for a symbol that may be either a free variable or a free atom.

Meta-variables with disjoint labels always denote different symbols; e.g. "$x^{\mathbb{V}}$" and "$x^{\mathbb{A}}$" will always denote different symbols; whereas "$x^{\mathbb{VA}}$" may denote the same symbol as "$x^{\mathbb{A}}$". In concrete examples, we will implicitly assume that different meta-variables denote different symbols; whereas in formal discussions, "$x^{\mathbb{A}}$" and "$y^{\mathbb{A}}$" may denote the same symbol.

Definition 2.1 (Terms ($\mathcal{T}$) and [Finite] Atom Substitutions)
Each *term* of the *set of terms* $\mathcal{T}$ will have a unique sort. A term is *closed* if it does not contain unbound occurrences of bound atoms. As *finite atom substitutions* may be parts of our terms, we have to define both notions in a mutual induction.

- Constants, (free) atoms, and bound atoms are terms of their given types.
- If $t_1 : \alpha \to \beta$ and $t_2 : \alpha$ are terms, then the *application* $t_1 t_2 : \beta$ is a term as well. Application associates to the left.
- If $t : \beta$ is a term and $y^{\mathbb{B}} : \alpha$ is a bound atom, then the *abstraction* $\lambda y^{\mathbb{B}}.\, t : \alpha \to \beta$ is a term as well.
- If $z^{\mathbb{V}} : \alpha$ is a variable and θ is a finite atom substitution, then the *pending* $\binom{z^{\mathbb{V}}}{\theta} : \alpha$ is a term. We will use $z^{\mathbb{V}}$ as a shorthand for $\binom{z^{\mathbb{V}}}{\mathbb{A}\rceil \mathrm{id}}$.
- An *atom substitution* θ is a total function from atoms to closed terms respecting types, i.e., for every $x^{\mathbb{A}} : \alpha$, we have $\theta(x^{\mathbb{A}}) : \alpha$. For an atom substitution θ we define its *non-trivial domain* as $\mathcal{NT}(\theta) := \{\, x^{\mathbb{A}} \in \mathbb{A} \mid \theta(x^{\mathbb{A}}) \neq x^{\mathbb{A}} \,\}$. A *finite atom substitution* θ is an atom substitution where $\mathcal{NT}(\theta)$ is finite.

Definition 2.2 (Atom Permutation)
An atom permutation π is an atom substitution that is a permutation in the sense that π is a bijection of $\mathbb{A}$ onto $\mathbb{A}$.

Definition 2.3 ([Finite] Variable Substitutions)
A *variable substitution* σ is a total function from variables to closed terms respecting types, i.e., for every $z^{\mathbb{V}} : \alpha$, we have $\sigma(z^{\mathbb{V}}) : \alpha$.
For a variable substitution θ we define its *non-trivial domain* as

$$\mathcal{NT}(\sigma) := \left\{ \; z^{\mathbb{V}} \in \mathbb{V} \;\middle|\; \sigma(z^{\mathbb{V}}) \neq \binom{z^{\mathbb{V}}}{{}_{\mathbb{A}}\restriction \mathrm{id}} \right\}.$$

A *finite variable substitution* θ is a variable substitution where $\mathcal{NT}(\sigma)$ is finite.

Definition 2.4 (Extension to Terms)
We extend any finite atom substitution θ and any variable substitution σ to functions $\theta^{\mathcal{T}}$ and $\sigma^{\mathcal{T}}$ from terms to terms as given in the following box. Note that we write $t\theta$ as a shorthand for $\theta^{\mathcal{T}}(t)$, and also $t\sigma$ as a shorthand for $\sigma^{\mathcal{T}}(t)$, which are recursively defined according to the following table:

t	$t\theta$	$t\sigma$
f (a constant)	f	f
$y^{\mathbb{B}}$ (a bound atom)	$y^{\mathbb{B}}$	$y^{\mathbb{B}}$
$x^{\mathbb{A}}$ (an atom)	$\theta(x^{\mathbb{A}})$	$x^{\mathbb{A}}$
$t_1 t_2$ (an application)	$(t_1\theta)(t_2\theta)$	$(t_1\sigma)(t_2\sigma)$
$\lambda y^{\mathbb{B}}.\ t'$ (an abstraction)	$\lambda y^{\mathbb{B}}.\ (t'\theta)$	$\lambda y^{\mathbb{B}}.\ (t'\sigma)$
$\binom{z^{\mathbb{V}}}{\theta'}$ (a pending)	$\binom{z^{\mathbb{V}}}{\theta' \circ \theta^{\mathcal{T}}}$	$(\sigma(z^{\mathbb{V}}))(\theta' \circ \sigma^{\mathcal{T}})$

Note that $\theta' \circ \theta^{\mathcal{T}}$ may be an infinite atom substitution for a finite atom substitution θ' and an infinite atom substitution θ; so we have to require θ to be finite in Definition 2.4. This finiteness requirement also applies to the atom substitution $\theta' \circ \sigma^{\mathcal{T}}$, where, however, we do not have to require σ to be a finite variable substitution because of $\mathcal{NT}(\theta' \circ \sigma^{\mathcal{T}}) \subseteq \mathcal{NT}(\theta')$. Moreover, note that the term $(\sigma(z^{\mathbb{V}}))(\theta' \circ \sigma^{\mathcal{T}})$ in the last field of the table of Definition 2.4 denotes $(\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}}(\sigma(z^{\mathbb{V}}))$ and not the (in general different) $(\theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}})(\sigma(z^{\mathbb{V}}))$; to see the difference, take $\theta := {}_{\mathbb{A}}\restriction \mathrm{id}$, then $\theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}} = {}_{\mathcal{T}}\restriction \mathrm{id} \circ \sigma^{\mathcal{T}} = \sigma^{\mathcal{T}}$, but $(\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}} = ({}_{\mathbb{A}}\restriction \mathrm{id})^{\mathcal{T}} = {}_{\mathcal{T}}\restriction \mathrm{id}$. For an atom substitution instead of σ, however, this difference does not exist:

Lemma 2.5 *For two finite atom substitutions θ, θ', we have:* $(\theta')^{\mathcal{T}} \circ \theta^{\mathcal{T}} = (\theta' \circ \theta^{\mathcal{T}})^{\mathcal{T}}$.

Proof of Lemma 2.5
It suffices to show $(t\theta')\theta = t(\theta' \circ \theta^{\mathcal{T}})$ by structural induction on the term t.

$\underline{t = f \text{ (a constant)}}$: $(f\theta')\theta = f\theta = f = f(\theta' \circ \theta^{\mathcal{T}})$.

$\underline{t = y^{\mathbb{B}} \text{ (a bound atom)}}$: $(y^{\mathbb{B}}\theta')\theta = y^{\mathbb{B}}\theta = y^{\mathbb{B}} = y^{\mathbb{B}}(\theta' \circ \theta^{\mathcal{T}})$.

$\underline{t = x^{\mathbb{A}} \text{ (an atom)}}$: $(x^{\mathbb{A}}\theta')\theta = (\theta'(x^{\mathbb{A}}))\theta = \theta^{\mathcal{T}}(\theta'(x^{\mathbb{A}})) = (\theta' \circ \theta^{\mathcal{T}})(x^{\mathbb{A}}) = x^{\mathbb{A}}(\theta' \circ \theta^{\mathcal{T}})$.

$\underline{t = t_1 t_2 \text{ (an application)}}$: $((t_1 t_2)\theta')\theta = ((t_1\theta')(t_2\theta'))\theta = ((t_1\theta')\theta)((t_2\theta')\theta) = (t_1(\theta' \circ \theta^{\mathcal{T}}))(t_1(\theta' \circ \theta^{\mathcal{T}})) = (t_1 t_2)(\theta' \circ \theta^{\mathcal{T}})$.

$t = \lambda y^{\mathbb{B}}.\ t'$ (an abstraction): $((\lambda y^{\mathbb{B}}.\ t')\theta')\theta = (\lambda y^{\mathbb{B}}.\ (t'\theta'))\theta = \lambda y^{\mathbb{B}}.\ ((t'\theta')\theta) = \lambda y^{\mathbb{B}}.\ (t'(\theta' \circ \theta^{\mathcal{T}})) = (\lambda y^{\mathbb{B}}.\ t')(\theta' \circ \theta^{\mathcal{T}})$.

$t = \begin{pmatrix} z^{\mathbb{V}} \\ \theta'' \end{pmatrix}$ (a pending): By induction hypothesis we have $((\theta''(x^{\mathbb{A}}))\theta')\theta = (\theta''(x^{\mathbb{A}}))(\theta' \circ \theta^{\mathcal{T}})$ for all $x^{\mathbb{A}} \in \mathcal{NT}(\theta'')$. And for $x^{\mathbb{A}} \in \mathbb{A} \backslash \mathcal{NT}(\theta'')$, this holds as well because of $\theta''(x^{\mathbb{A}}) = x^{\mathbb{A}}$ and the above proof for atoms. This means that, for all $x^{\mathbb{A}} \in \mathbb{A}$, we have $(\theta'' \circ (\theta')^{\mathcal{T}} \circ \theta^{\mathcal{T}})(x^{\mathbb{A}}) = (\theta'' \circ (\theta' \circ \theta^{\mathcal{T}})^{\mathcal{T}})(x^{\mathbb{A}})$. Thus, we have:

$$\left(\begin{pmatrix} z^{\mathbb{V}} \\ \theta'' \end{pmatrix} \theta' \right) \theta = \begin{pmatrix} z^{\mathbb{V}} \\ \theta'' \circ (\theta')^{\mathcal{T}} \end{pmatrix} \theta = \begin{pmatrix} z^{\mathbb{V}} \\ \theta'' \circ (\theta')^{\mathcal{T}} \circ \theta^{\mathcal{T}} \end{pmatrix} = \begin{pmatrix} z^{\mathbb{V}} \\ \theta'' \circ (\theta' \circ \theta^{\mathcal{T}})^{\mathcal{T}} \end{pmatrix} = \begin{pmatrix} z^{\mathbb{V}} \\ \theta'' \end{pmatrix} (\theta' \circ \theta^{\mathcal{T}}).$$

Q.e.d. (Lemma 2.5)

Lemma 2.6

For a term t, a finite atom substitution θ, and a variable substitution σ, we have:

$$(t\theta)\sigma = (t\sigma)(\theta \circ \sigma^{\mathcal{T}}).$$

Proof of Lemma 2.6

By structural induction on the term t.

$t = f$ (a constant): $(f\theta)\sigma = f\sigma = f = f(\theta \circ \sigma^{\mathcal{T}}) = (f\sigma)(\theta \circ \sigma^{\mathcal{T}})$.

$t = y^{\mathbb{B}}$ (a bound atom): $(y^{\mathbb{B}}\theta)\sigma = y^{\mathbb{B}}\sigma = y^{\mathbb{B}} = y^{\mathbb{B}}(\theta \circ \sigma^{\mathcal{T}}) = (y^{\mathbb{B}}\sigma)(\theta \circ \sigma^{\mathcal{T}})$.

$t = x^{\mathbb{A}}$ (an atom): $(x^{\mathbb{A}}\theta)\sigma = (\theta(x^{\mathbb{A}}))\sigma = (\theta \circ \sigma^{\mathcal{T}})(x^{\mathbb{A}}) = x^{\mathbb{A}}(\theta \circ \sigma^{\mathcal{T}}) = (x^{\mathbb{A}}\sigma)(\theta \circ \sigma^{\mathcal{T}})$.

$t = t_1 t_2$ (an application): $((t_1 t_2)\theta)\sigma = ((t_1\theta)(t_2\theta))\sigma = ((t_1\theta)\sigma)((t_2\theta)\sigma) = ((t_1\sigma)(\theta \circ \sigma^{\mathcal{T}}))((t_2\sigma)(\theta \circ \sigma^{\mathcal{T}})) = ((t_1\sigma)(t_2\sigma))(\theta \circ \sigma^{\mathcal{T}}) = ((t_1 t_2)\sigma)(\theta \circ \sigma^{\mathcal{T}})$.

$t = \lambda y^{\mathbb{B}}.\ t'$ (an abstraction): $((\lambda y^{\mathbb{B}}.\ t')\theta)\sigma = (\lambda y^{\mathbb{B}}.\ (t'\theta))\sigma = \lambda y^{\mathbb{B}}.\ ((t'\theta)\sigma) = \lambda y^{\mathbb{B}}.\ ((t'\sigma)(\theta \circ \sigma^{\mathcal{T}})) = (\lambda y^{\mathbb{B}}.\ (t'\sigma))(\theta \circ \sigma^{\mathcal{T}}) = ((\lambda y^{\mathbb{B}}.\ t')\sigma)(\theta \circ \sigma^{\mathcal{T}})$.

$t = \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}$ (a pending): By induction hypothesis we have $((\theta'(x^{\mathbb{A}}))\theta)\sigma = ((\theta'(x^{\mathbb{A}}))\sigma)(\theta \circ \sigma^{\mathcal{T}})$ for all $x^{\mathbb{A}} \in \mathcal{NT}(\theta')$. And for $x^{\mathbb{A}} \in \mathbb{A} \backslash \mathcal{NT}(\theta')$, this holds as well because of $\theta'(x^{\mathbb{A}}) = x^{\mathbb{A}}$ and the above proof for atoms. This means that, for all $x^{\mathbb{A}} \in \mathbb{A}$, we have $(\theta' \circ \theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}})(x^{\mathbb{A}}) = (\theta' \circ \sigma^{\mathcal{T}} \circ (\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}})(x^{\mathbb{A}})$; i.e. $\theta' \circ \theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}} = \theta' \circ \sigma^{\mathcal{T}} \circ (\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}}$; i.e. $(\theta' \circ \theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}})^{\mathcal{T}} = \left(\theta' \circ \sigma^{\mathcal{T}} \circ (\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}} \right)^{\mathcal{T}} = (\theta' \circ \sigma^{\mathcal{T}})^{\mathcal{T}} \circ (\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}}$ by Lemma 2.5. Thus, we have:

$$\left(\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \theta \right) \sigma = \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \circ \theta^{\mathcal{T}} \end{pmatrix} \sigma = (\sigma(z^{\mathbb{V}}))(\theta' \circ \theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}}) = (\theta' \circ \theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}})^{\mathcal{T}}(\sigma(z^{\mathbb{V}})) = ((\theta' \circ \sigma^{\mathcal{T}})^{\mathcal{T}} \circ (\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}})(\sigma(z^{\mathbb{V}})) = ((\sigma(z^{\mathbb{V}}))(\theta' \circ \sigma^{\mathcal{T}}))(\theta \circ \sigma^{\mathcal{T}}) = \left(\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \sigma \right) (\theta \circ \sigma^{\mathcal{T}}).$$

Q.e.d. (Lemma 2.6)

Lemma 2.7

For a term t, a finite atom permutation π, and a variable substitution σ, we have:

$$(t\pi)\sigma = (t\sigma)\pi.$$

Proof of Lemma 2.7

According to Lemma 2.6 we have $(t\pi)\sigma = (t\sigma)(\pi \circ \sigma^{\mathcal{T}}) = (t\sigma)\pi$. **Q.e.d. (Lemma 2.7)**

Definition 2.8 ($\mathbb{A}(t)$, $\mathbb{V}(t)$)
The (free) *atoms* $\mathbb{A}(t)$ *of a term* t and the (free) *variables* $\mathbb{V}(t)$ *of a term* t are recursively defined according to the following table:

t	$\mathbb{A}(t)$	$\mathbb{V}(t)$
T (a set of terms)	$\bigcup_{t\in T}\mathbb{A}(t)$	$\bigcup_{t\in T}\mathbb{V}(t)$
f (a constant)	$\emptyset$	$\emptyset$
$y^{\mathbb{B}}$ (a bound atom)	$\emptyset$	$\emptyset$
$x^{\mathbb{A}}$ (an atom)	$\{x^{\mathbb{A}}\}$	$\emptyset$
t_1t_2 (an application)	$\bigcup_{i\in\{1,2\}}\mathbb{A}(t_i)$	$\bigcup_{i\in\{1,2\}}\mathbb{V}(t_i)$
$\lambda y^{\mathbb{B}}.\ t'$ (an abstraction)	$\mathbb{A}(t')$	$\mathbb{V}(t')$
$\binom{z^{\mathbb{V}}}{\theta}$ (a pending)	$\emptyset$	$\left\{\ \binom{z^{\mathbb{V}}}{\theta}\ \right\}$

Moreover, we set $\mathbb{VA}(t) := \mathbb{A}(t) \uplus \mathbb{V}(t)$.

Corollary 2.9

1. $\mathbb{A}\big(\bigcup_{i\in I} T_i\big) = \bigcup_{i\in I}\mathbb{A}(T_i)$.
2. $\mathbb{V}\big(\bigcup_{i\in I} T_i\big) = \bigcup_{i\in I}\mathbb{V}(T_i)$.

Lemma 2.10
For a term t *and a finite atom substitution* θ *we have:*

1. $\mathbb{A}(t\theta) = \mathbb{A}(\langle\mathbb{A}(t)\rangle\theta)$.
2. $\mathbb{V}(t\theta) = \langle\mathbb{V}(t)\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\mathbb{A}(t)\rangle\theta)$.

Proof of Lemma 2.10
By structural induction on t.

$t = f$ (a constant): $\mathbb{A}(f\theta) = \mathbb{A}(f) = \emptyset = \mathbb{A}(\emptyset) = \mathbb{A}(\langle\emptyset\rangle\theta) = \mathbb{A}(\langle\mathbb{A}(f)\rangle\theta)$.
$\mathbb{V}(f\theta) = \mathbb{V}(f) = \emptyset = \emptyset \cup \mathbb{V}(\emptyset) = \langle\emptyset\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\emptyset\rangle\theta) = \langle\mathbb{V}(f)\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\mathbb{A}(f)\rangle\theta)$.

$t = y^{\mathbb{B}}$ (a bound atom): $\mathbb{A}(y^{\mathbb{B}}\theta) = \mathbb{A}(y^{\mathbb{B}}) = \emptyset = \mathbb{A}(\emptyset) = \mathbb{A}(\langle\emptyset\rangle\theta) = \mathbb{A}(\langle\mathbb{A}(y^{\mathbb{B}})\rangle\theta)$.
$\mathbb{V}(y^{\mathbb{B}}\theta) = \mathbb{V}(y^{\mathbb{B}}) = \emptyset = \emptyset \cup \mathbb{V}(\emptyset) = \langle\emptyset\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\emptyset\rangle\theta) = \langle\mathbb{V}(y^{\mathbb{B}})\rangle\theta \cup \mathbb{V}(\langle\mathbb{A}(y^{\mathbb{B}})\rangle\theta^{\mathcal{T}})$.

$t = x^{\mathbb{A}}$ (an atom): $\mathbb{A}(x^{\mathbb{A}}\theta) = \mathbb{A}(\theta(x^{\mathbb{A}})) = \mathbb{A}(\langle\{x^{\mathbb{A}}\}\rangle\theta) = \mathbb{A}(\langle\mathbb{A}(x^{\mathbb{A}})\rangle\theta)$.
$\mathbb{V}(x^{\mathbb{A}}\theta) = \mathbb{V}(\theta(x^{\mathbb{A}})) = \emptyset \cup \mathbb{V}(\theta(x^{\mathbb{A}})) = \langle\emptyset\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\{x^{\mathbb{A}}\}\rangle\theta) = \langle\mathbb{V}(x^{\mathbb{A}})\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\mathbb{A}(x^{\mathbb{A}})\rangle\theta)$.

$t = t_1t_2$ (an application): $\mathbb{A}((t_1t_2)\theta) = \mathbb{A}((t_1\theta)(t_2\theta)) = \bigcup_{i\in\{1,2\}}\mathbb{A}(t_i\theta) = \bigcup_{i\in\{1,2\}}\mathbb{A}(\langle\mathbb{A}(t_i)\rangle\theta) = \mathbb{A}\big(\bigcup_{i\in\{1,2\}}\langle\mathbb{A}(t_i)\rangle\theta\big) = \mathbb{A}\big(\langle\bigcup_{i\in\{1,2\}}\mathbb{A}(t_i)\rangle\theta\big) = \mathbb{A}(\langle\mathbb{A}(t_1t_2)\rangle\theta)$.
$\mathbb{V}((t_1t_2)\theta) = \mathbb{V}((t_1\theta)(t_2\theta)) = \bigcup_{i\in\{1,2\}}\mathbb{V}(t_i\theta) = \bigcup_{i\in\{1,2\}}\langle\mathbb{V}(t_i)\rangle\theta^{\mathcal{T}} \cup \bigcup_{i\in\{1,2\}}\mathbb{V}(\langle\mathbb{A}(t_i)\rangle\theta) = \bigcup_{i\in\{1,2\}}\langle\mathbb{V}(t_i)\rangle\theta^{\mathcal{T}} \cup \mathbb{V}\big(\bigcup_{i\in\{1,2\}}\langle\mathbb{A}(t_i)\rangle\theta\big) = \big\langle\bigcup_{i\in\{1,2\}}\mathbb{V}(t_i)\big\rangle\theta^{\mathcal{T}} \cup \mathbb{V}\big(\langle\bigcup_{i\in\{1,2\}}\mathbb{A}(t_i)\rangle\theta\big) = \langle\mathbb{V}(t_1t_2)\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\mathbb{A}(t_1t_2)\rangle\theta)$.

$t = \lambda y^{\mathbb{B}}.\ t'$ (an abstraction): $\mathbb{A}((\lambda y^{\mathbb{B}}.\ t')\theta) = \mathbb{A}(\lambda y^{\mathbb{B}}.\ (t'\theta)) = \mathbb{A}(t'\theta) = \mathbb{A}(\langle\mathbb{A}(t')\rangle\theta) = \mathbb{A}(\langle\mathbb{A}(\lambda y^{\mathbb{B}}.\ t')\rangle\theta)$.
$\mathbb{V}((\lambda y^{\mathbb{B}}.\ t')\theta) = \mathbb{V}(\lambda y^{\mathbb{B}}.\ (t'\theta)) = \mathbb{V}(t'\theta) = \langle\mathbb{V}(t')\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\mathbb{A}(t')\rangle\theta) = \langle\mathbb{V}(\lambda y^{\mathbb{B}}.\ t')\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\mathbb{A}(\lambda y^{\mathbb{B}}.\ t')\rangle\theta)$.

$\underline{t = \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \text{ (a pending):}}$ $\mathbb{A}\Big(\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\theta \Big) = \mathbb{A}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \circ \theta^{\mathcal{T}} \end{pmatrix} = \emptyset = \mathbb{A}(\emptyset) = \mathbb{A}(\langle\emptyset\rangle\theta) = \mathbb{A}\Big(\Big\langle \mathbb{A}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\rangle\theta \Big)$.

$\mathbb{V}\Big(\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\theta \Big) = \mathbb{V}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \circ \theta^{\mathcal{T}} \end{pmatrix} = \Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \circ \theta^{\mathcal{T}} \end{pmatrix} \Big\} = \Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \circ \theta^{\mathcal{T}} \end{pmatrix} \Big\} \cup \emptyset =$
$\Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\theta \Big\} \cup \mathbb{V}(\emptyset) = \Big\langle \Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\} \Big\rangle\theta^{\mathcal{T}} \cup \mathbb{V}(\langle\emptyset\rangle\theta) = \Big\langle \mathbb{V}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\rangle\theta^{\mathcal{T}} \cup \mathbb{V}\Big(\Big\langle \mathbb{A}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\rangle\theta \Big)$.

Q.e.d. (Lemma 2.10)

Lemma 2.11
For a term t and a variable substitution σ we have:

1. $\mathbb{A}(t\sigma) = \mathbb{A}(t) \cup \mathbb{A}(\langle\mathbb{V}(t)\rangle\sigma^{\mathcal{T}})$.
2. $\mathbb{V}(t\sigma) = \mathbb{V}(\langle\mathbb{V}(t)\rangle\sigma^{\mathcal{T}})$.

Proof of Lemma 2.11
By structural induction on t.

$\underline{t = f \text{ (a constant):}}$ $\mathbb{A}(f\sigma) = \mathbb{A}(f) = \mathbb{A}(f) \cup \emptyset = \mathbb{A}(f) \cup \mathbb{A}(\emptyset) = \mathbb{A}(f) \cup \mathbb{A}(\langle\emptyset\rangle\sigma^{\mathcal{T}}) = \mathbb{A}(f) \cup \mathbb{A}(\langle\mathbb{V}(f)\rangle\sigma^{\mathcal{T}})$.
$\mathbb{V}(f\sigma) = \mathbb{V}(f) = \emptyset = \mathbb{V}(\emptyset) = \mathbb{V}(\langle\emptyset\rangle\sigma^{\mathcal{T}}) = \mathbb{V}(\langle\mathbb{V}(f)\rangle\sigma^{\mathcal{T}})$.

$\underline{t = y^{\mathbb{B}} \text{ (a bound atom):}}$ $\mathbb{A}(y^{\mathbb{B}}\sigma) = \mathbb{A}(y^{\mathbb{B}}) = \mathbb{A}(y^{\mathbb{B}}) \cup \emptyset = \mathbb{A}(y^{\mathbb{B}}) \cup \mathbb{A}(\emptyset) = \mathbb{A}(y^{\mathbb{B}}) \cup \mathbb{A}(\langle\emptyset\rangle\sigma^{\mathcal{T}}) = \mathbb{A}(y^{\mathbb{B}}) \cup \mathbb{A}(\langle\mathbb{V}(y^{\mathbb{B}})\rangle\sigma^{\mathcal{T}})$.
$\mathbb{V}(y^{\mathbb{B}}\sigma) = \mathbb{V}(y^{\mathbb{B}}) = \emptyset = \mathbb{V}(\emptyset) = \mathbb{V}(\langle\emptyset\rangle\sigma^{\mathcal{T}}) = \mathbb{V}(\langle\mathbb{V}(y^{\mathbb{B}})\rangle\sigma^{\mathcal{T}})$.

$\underline{t = x^{\mathbb{A}} \text{ (an atom):}}$ $\mathbb{A}(x^{\mathbb{A}}\sigma) = \mathbb{A}(x^{\mathbb{A}}) = \mathbb{A}(x^{\mathbb{A}}) \cup \emptyset = \mathbb{A}(x^{\mathbb{A}}) \cup \mathbb{A}(\emptyset) = \mathbb{A}(x^{\mathbb{A}}) \cup \mathbb{A}(\langle\emptyset\rangle\sigma^{\mathcal{T}}) = \mathbb{A}(x^{\mathbb{A}}) \cup \mathbb{A}(\langle\mathbb{V}(x^{\mathbb{A}})\rangle\sigma^{\mathcal{T}})$.
$\mathbb{V}(x^{\mathbb{A}}\sigma) = \mathbb{V}(x^{\mathbb{A}}) = \emptyset = \mathbb{V}(\emptyset) = \mathbb{V}(\langle\emptyset\rangle\sigma^{\mathcal{T}}) = \mathbb{V}(\langle\mathbb{V}(x^{\mathbb{A}})\rangle\sigma^{\mathcal{T}})$.

$\underline{t = t_1t_2 \text{ (an application):}}$ $\mathbb{A}((t_1t_2)\sigma) = \mathbb{A}((t_1\sigma)(t_2\sigma)) = \bigcup_{i\in\{1,2\}} \mathbb{A}(t_i\sigma) = \bigcup_{i\in\{1,2\}} \mathbb{A}(t_i) \cup \bigcup_{i\in\{1,2\}} \mathbb{A}(\langle\mathbb{V}(t_i)\rangle\sigma^{\mathcal{T}}) =$
$\mathbb{A}(t_1t_2) \cup \mathbb{A}\big(\bigcup_{i\in\{1,2\}} \langle\mathbb{V}(t_i)\rangle\sigma^{\mathcal{T}} \big) = \mathbb{A}(t_1t_2) \cup \mathbb{A}\big(\langle\bigcup_{i\in\{1,2\}} \mathbb{V}(t_i)\rangle\sigma^{\mathcal{T}} \big) =$
$\mathbb{A}(t_1t_2) \cup \mathbb{A}(\langle\mathbb{V}(t_1t_2)\rangle\sigma^{\mathcal{T}})$.
$\mathbb{V}((t_1t_2)\sigma) = \mathbb{V}((t_1\sigma)(t_2\sigma)) = \bigcup_{i\in\{1,2\}} \mathbb{V}(t_i\sigma) = \bigcup_{i\in\{1,2\}} \mathbb{V}(\langle\mathbb{V}(t_i)\rangle\sigma^{\mathcal{T}}) =$
$\mathbb{V}\big(\bigcup_{i\in\{1,2\}} \langle\mathbb{V}(t_i)\rangle\sigma^{\mathcal{T}} \big) = \mathbb{V}\big(\langle\bigcup_{i\in\{1,2\}} \mathbb{V}(t_i)\rangle\sigma^{\mathcal{T}} \big) = \mathbb{V}(\langle\mathbb{V}(t_1t_2)\rangle\sigma^{\mathcal{T}})$.

$\underline{t = \lambda y^{\mathbb{B}}.\ t' \text{ (an abstraction):}}$ $\mathbb{A}((\lambda y^{\mathbb{B}}.\ t')\sigma) = \mathbb{A}(\lambda y^{\mathbb{B}}.\ (t'\sigma)) = \mathbb{A}(t'\sigma) = \mathbb{A}(t') \cup \mathbb{A}(\langle\mathbb{V}(t')\rangle\sigma^{\mathcal{T}}) = \mathbb{A}(\lambda y^{\mathbb{B}}.\ t') \cup \mathbb{A}(\langle\mathbb{V}(\lambda y^{\mathbb{B}}.\ t')\rangle\sigma^{\mathcal{T}})$.
$\mathbb{V}((\lambda y^{\mathbb{B}}.\ t')\sigma) = \mathbb{V}(\lambda y^{\mathbb{B}}.\ (t'\sigma)) = \mathbb{V}(t'\sigma) = \mathbb{V}(\langle\mathbb{V}(t')\rangle\sigma^{\mathcal{T}}) = \mathbb{V}(\langle\mathbb{V}(\lambda y^{\mathbb{B}}.\ t')\rangle\sigma^{\mathcal{T}})$.

$\underline{t = \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \text{ (a pending):}}$ $\mathbb{A}\Big(\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\sigma \Big) = \mathbb{A}\Big(\Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\sigma \Big\} \Big) = \emptyset \cup \mathbb{A}\Big(\Big\langle \Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\} \Big\rangle\sigma^{\mathcal{T}} \Big) = \mathbb{A}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \cup \mathbb{A}\Big(\Big\langle \mathbb{V}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\rangle\sigma^{\mathcal{T}} \Big)$.
$\mathbb{V}\Big(\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\sigma \Big) = \mathbb{V}\Big(\Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}\sigma \Big\} \Big) = \mathbb{V}\Big(\Big\langle \Big\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\} \Big\rangle\sigma^{\mathcal{T}} \Big) = \mathbb{V}\Big(\Big\langle \mathbb{V}\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix} \Big\rangle\sigma^{\mathcal{T}} \Big)$.

Q.e.d. (Lemma 2.11)

Definition 2.12 ([Finite] Positive/Negative Variable-Condition)
A [*finite*] *positive/negative variable-condition* is a pair (P, N) with

$$P \subseteq \mathbb{VA} \times \mathbb{V}$$

and

$$N \subseteq \mathbb{V} \times \mathbb{A}$$

[and $|P|, \ |N| \in \mathbf{N}$].

Definition 2.13 (Consistency)
A pair (P, N) is *consistent* if

$$P \text{ is well-founded}$$

and

$$P^+ \circ N \text{ is irreflexive.}$$

Corollary 2.14 *If (P, N) is a finite positive/negative variable-condition, then (P, N) is consistent iff each cycle in the directed graph of $P \uplus N$ contains more than one edge from N. The latter can be effectively tested with time complexity of $|P| + |N|$.*

Note that, in the finite case, the test of Corollary 2.14 seems to be both the most efficient and the most human-oriented way to represent the question of consistency of positive/negative variable-conditions.

Definition 2.15 (Admission Relation)
Let (P, N) be a positive/negative variable-condition.
The *admission relation* $\mathcal{A}_{(P,N)}$ is given as
$\left(x^{\mathbb{A}}, \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right) \in \mathcal{A}_{(P,N)}$ if $\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$ is a pending and $x^{\mathbb{A}} \in \mathbb{A} \cap P^+\langle\{z^{\mathbb{V}}\}\rangle$.
The *occurrence relation* $\mathcal{O}$ is given as
$\left(x^{\mathbb{A}}, \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right) \in \mathcal{O}$ if $\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$ is a pending and $x^{\mathbb{A}} \in \mathcal{NT}(\theta)$.

The notion of an occurrence of a variable or an atom in a term we need in the following is not the simple one of a participation in the mutually inductive construction of terms and finite atom variables. The reason is that a variable in a pending may stand for any term with atoms from its permission set.

Definition 2.16 ($\mathbb{A}_{(P,N)}(t)$, $\mathbb{V}_{(P,N)}(t)$)
Let A be a binary relation.
The (free) *atoms* $\mathbb{A}_A(t)$ *of a term* t *w.r.t.* A and the (free) *variables* $\mathbb{V}_A(t)$ *of a term* t *w.r.t.* A are recursively defined according to the following table:

t	$\mathbb{A}_A(t)$	$\mathbb{V}_A(t)$
T (a set of terms)	$\bigcup_{t\in T} \mathbb{A}_A(t)$	$\bigcup_{t\in T} \mathbb{V}_A(t)$
f (a constant)	$\emptyset$	$\emptyset$
$x^{\mathbb{A}}$ (an atom)	$\{x^{\mathbb{A}}\}$	$\emptyset$
$y^{\mathbb{B}}$ (a bound atom)	$\emptyset$	$\emptyset$
$t_1 t_2$ (an application)	$\mathbb{A}_A(\{t_1, t_2\})$	$\mathbb{V}_A(\{t_1, t_2\})$
$\lambda y^{\mathbb{B}}.\ t'$ (an abstraction)	$\mathbb{A}_A(t')$	$\mathbb{V}_A(t')$
$\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$ (a pending); setting $\mathrm{X} := A\left\langle \left\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right\} \right\rangle$	$\mathbb{A}_A(\langle \mathrm{X} \rangle \theta)$	$\{z^{\mathbb{V}}\} \cup \mathbb{V}_A(\langle \mathrm{X} \rangle \theta)$

Moreover, we set $\mathbb{VA}_A(t) := \mathbb{A}_A(t) \uplus \mathbb{V}_A(t)$.

Lemma 2.17 $\mathbb{V}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{V}_{\mathcal{O}}(t)$.
$\mathbb{A}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{A}_{\mathcal{O}}(t) \cup P^+\langle \mathbb{V}_{\mathcal{O}}(t) \rangle$.

Corollary 2.18 $\mathbb{VA}_{\mathcal{A}_{(P,N)}}(t)$ *is finitary in* P *in the sense that for every term* t *there is a finite* $P' \subseteq P$ *with* $\mathbb{VA}_{\mathcal{A}_{(P,N)}}(t) = \mathbb{VA}_{\mathcal{A}_{(P',\emptyset)}}(t)$.

Corollary 2.19 *Let* (P, N) *be finite positive/negative variable-condition. Let* t *be a term. All the sets* $\mathbb{A}_{\mathcal{A}_{(P,N)}}(t)$, $\mathbb{A}_{\mathcal{O}}(t)$, $\mathbb{V}_{\mathcal{A}_{(P,N)}}(t)$, $\mathbb{V}_{\mathcal{O}}(t)$ *are always finite.*
We have $\mathbb{A}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{A}_{\mathcal{O}}(t) \subseteq \mathbb{A}$, *and* $\mathbb{V}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{V}_{\mathcal{O}}(t) \subseteq \mathbb{V}$.

2.1 Extensions, σ-Updates, and (P,N)-Substitutions

Within a reasoning process, positive/negative variable-conditions may be subject to only one kind of transformation, which we simply call an "extension".

Definition 2.20 ([Weak] Extension)
(P',N') is an [*weak*] *extension of* (P,N) if
(P',N') is a positive/negative variable-condition, $P \subseteq P'$ [or at least $P \subseteq (P')^+$], and $N \subseteq N'$.

As an immediate corollary of Definitions 2.20 and 2.13 we get:

Corollary 2.21
If (P',N') is a consistent positive/negative variable-condition and an [weak] extension of (P,N), then (P,N) is a consistent positive/negative variable-condition as well.

A σ-update is a special form of an extension:

Definition 2.22 (σ-Update, Dependence)
Let (P,N) be a finite positive/negative variable-condition.
Let σ be a finite variable substitution.
The (P,N)-*dependence of* σ is the smallest set D such that $P \subseteq D$ and

$$\{\ (z^{\mathbb{VA}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathcal{NT}(\sigma) \wedge z^{\mathbb{VA}} \in \mathbb{VA}_{\mathcal{A}_{(D,N)}}(\sigma(x^{\mathbb{V}}))\ \} \subseteq D.$$

The σ-*update of* (P,N) is (D,N).

As a corollary of Corollary 2.19, we get:

Lemma 2.23 *The σ-update (D,N) of a finite positive/negative variable-condition (P,N) and a finite variable substitution σ satisfies*

$$D \subseteq P \cup \left(\left(\bigcup_{x^{\mathbb{V}} \in \mathcal{NT}(\sigma)} \mathbb{VA}_{\mathcal{O}}(\sigma(x^{\mathbb{V}})) \right) \times \mathcal{NT}(\sigma) \right),$$

so, especially, (D,N) is a finite positive/negative variable-condition.

Proof of Lemma 2.23 By Corollary 2.18, the closure operator of Definition 2.22 is finitary for a *finite* variable substitution σ. Moreover, it is monotonic. Thus, we get the set D as the ω-closure of P. By Corollary 2.19, it now suffices to show that all finite applications of the closure operator satisfy the displayed equation of Lemma 2.23. So let us consider a pending $\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$. Set $\mathrm{X} := \mathcal{A}_{(D,N)}\left\langle \left\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right\} \right\rangle$. Then $\mathrm{X} = \mathbb{A} \cap D^+\langle\{z^{\mathbb{V}}\}\rangle$. Then $\langle \mathrm{X} \rangle\theta = \langle \mathrm{X} \cap \mathcal{NT}(\theta) \rangle\theta \cup \langle \mathrm{X} \setminus \mathcal{NT}(\theta) \rangle\theta = \langle \mathrm{X} \cap \mathcal{NT}(\theta) \rangle\theta \cup (\mathrm{X} \setminus \mathcal{NT}(\theta)) \subseteq \langle \mathcal{NT}(\theta) \rangle\theta \cup (\mathrm{X} \setminus \mathcal{NT}(\theta))$. Thus, $\mathbb{V}_{\mathcal{A}_{(D,N)}}(\langle \mathrm{X} \rangle\theta) \subseteq \mathbb{V}_{\mathcal{A}_{(D,N)}}(\langle \mathcal{NT}(\theta) \rangle\theta)$ and $\mathbb{A}_{\mathcal{A}_{(D,N)}}(\langle \mathrm{X} \rangle\theta) \subseteq \mathbb{A}_{\mathcal{A}_{(D,N)}}(\langle \mathcal{NT}(\theta) \rangle\theta) \cup (\mathrm{X} \setminus \mathcal{NT}(\theta))$. **Q.e.d. (Lemma 2.23)**

Definition 2.24 ((P,N)-Substitution)
Let (P,N) be a positive/negative variable-condition.
σ is a (P,N)-*substitution* if
σ is a variable substitution and the σ-update of (P,N) is consistent.

Syntactically, $(x^{\mathbb{V}}, a^{\mathbb{A}}) \in N$ is to express that a (P,N)-substitution σ must not replace $x^{\mathbb{V}}$ with a term in which $a^{\mathbb{A}}$ could ever occur; i.e. that $a^{\mathbb{A}}$ is fresh for $x^{\mathbb{V}}$: $a^{\mathbb{A}} \# x^{\mathbb{V}}$. This is indeed guaranteed if any σ-update (P',N') of (P,N) is again required to be consistent, and so on. We can see this as follows: For $z^{\mathbb{V}} \in \mathbb{V}(\sigma(x^{\mathbb{V}}))$, we get

$$z^{\mathbb{V}} \;\; P' \;\; x^{\mathbb{V}} \;\; N' \;\; a^{\mathbb{A}}.$$

If we now try to apply a second substitution σ' with $a^{\mathbb{A}} \in \mathbb{A}(\sigma'(z^{\mathbb{V}}))$ (so that $a^{\mathbb{A}}$ occurs in $(x^{\mathbb{V}}\sigma)\sigma'$, contrary to what we initially expressed as our freshness intention), then σ' is not a (P',N')-substitution because, for the σ'-update (P'',N'') of (P',N'), we have

$$a^{\mathbb{A}} \;\; P'' \;\; z^{\mathbb{V}} \;\; P'' \;\; x^{\mathbb{V}} \;\; N'' \;\; a^{\mathbb{A}};$$

so $(P'')^+ \circ N''$ is not irreflexive. All in all, the positive/negative variable-condition

- (P',N') blocks any instantiation of $(x^{\mathbb{V}}\sigma)$ resulting in a term containing $a^{\mathbb{A}}$, just as
- (P,N) blocked $x^{\mathbb{V}}$ before the application of σ.