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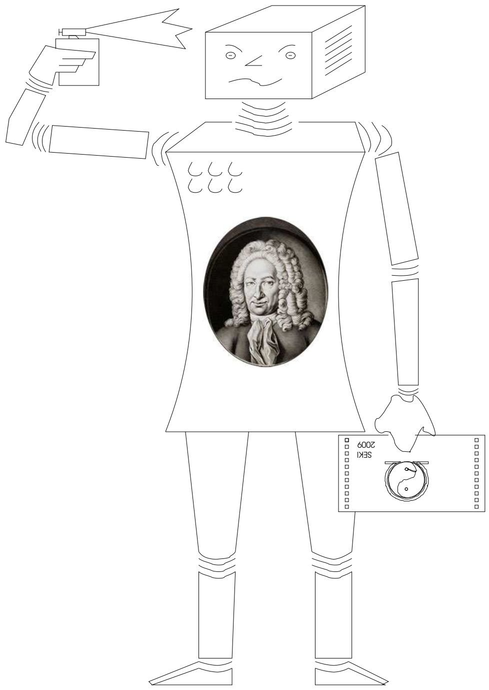

How to Kiss an Angel

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How to Kiss an Angel

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Abstract

We solve the problem of kissing an angel effectively and more efficiently than voodoo man. Some reading of Immanuël Kant, however, is required as a precondition.

Keywords: German Idealism, Dyslexia, Kissing, Angels, Angles, Kant, MSC 03D20, MSC 68Q42, MSC 16S15

1 Introduction

For being concrete, we will work with a simple-typed higher-order framework here.

There is a finite number of *sorts* or *base types*, including the BOOLEan sort bool (denoted with o by other authors). *Types* are inductively defined to be either a base type or or function type $\alpha \to \beta$, for two types α and β. The operator "$\to$" associates to the right.

There is a finite number of *constants*, including true, $\mathsf{false} : \mathsf{bool}$, $\neg : \mathsf{bool} \to \mathsf{bool}$, $\Rightarrow$, $\vee$, $\wedge : \mathsf{bool} \to \mathsf{bool} \to \mathsf{bool}$, and $\forall_\alpha$, $\exists_\alpha : (\alpha \to \mathsf{bool}) \to \mathsf{bool}$, for each type α. Each constant must have a unique type.

There are three disjoint sets of *bound atoms* $\mathbb{B}$, (free) *atoms* $\mathbb{A}$, and (free) *variables* $\mathbb{V}$. Each of their elements has a unique type, and each of the three sets has infinitely many elements of each type. We define:

$$\mathbb{VA} := \mathbb{V} \uplus \mathbb{A}$$

$$\mathbb{VAB} := \mathbb{V} \uplus \mathbb{A} \uplus \mathbb{B}$$

The classification as a free variable or as a free or bound atom will be indicated by adjoining a "$\mathbb{V}$", an "$\mathbb{A}$", or a "$\mathbb{B}$", respectively, as a label to the upper right of the meta-variable for the symbol. If a meta-variable stands for a symbol of the union of some of these sets, we will indicate this by listing all possible sets; e.g. "$x^{\mathbb{VA}}$" is a meta-variable for a symbol that may be either a free variable or a free atom.

Meta-variables with disjoint labels always denote different symbols; e.g. "$x^{\mathbb{V}}$" and "$x^{\mathbb{A}}$" will always denote different symbols; whereas "$x^{\mathbb{VA}}$" may denote the same symbol as "$x^{\mathbb{A}}$". In concrete examples, we will implicitly assume that different meta-variables denote different symbols; whereas in formal discussions, "$x^{\mathbb{A}}$" and "$y^{\mathbb{A}}$" may denote the same symbol.

Definition 1.1 (Terms ($\mathcal{T}$) and [Finite] Atom Substitutions)
Each *term* of the *set of terms* $\mathcal{T}$ will have a unique sort. A term is *closed* if it does not contain unbound occurrences of bound atoms. As *finite atom substitutions* may be parts of our terms, we have to define both notions in a mutual induction.

- Constants, (free) atoms, and bound atoms are terms of their given types.

- If $t_1 : \alpha \to \beta$ and $t_2 : \alpha$ are terms, then the *application* $t_1 t_2 : \beta$ is a term as well. Application associates to the left.

- If $t : \beta$ is a term and $y^{\mathbb{B}} : \alpha$ is a bound atom, then the *abstraction* $\lambda y^{\mathbb{B}}.\, t : \alpha \to \beta$ is a term as well.

- If $z^{\mathbb{V}} : \alpha$ is a variable and θ is a finite atom substitution, then the *pending* $\binom{z^{\mathbb{V}}}{\theta} : \alpha$ is a term. We will use $z^{\mathbb{V}}$ as a shorthand for $\binom{z^{\mathbb{V}}}{{}_{\mathbb{A}}\!\upharpoonright\!\mathrm{id}}$.

- An *atom substitution* θ is a total function from atoms to closed terms respecting types, i.e., for every $x^{\mathbb{A}} : \alpha$, we have $\theta(x^{\mathbb{A}}) : \alpha$. For an atom substitution θ we define its *non-trivial domain* as $\mathcal{NT}(\sigma) := \{\, x^{\mathbb{A}} \in \mathbb{A} \mid \theta(x^{\mathbb{A}}) \neq x^{\mathbb{A}} \,\}$. A *finite atom substitution* θ is an atom substitution where $\mathcal{NT}(\theta)$ is finite.

Definition 1.2 ([Finite] Positive/Negative Variable-Condition)
A [*finite*] *positive/negative variable-condition* is a pair (P, N) with

$$P \subseteq \mathbb{VA} \times \mathbb{V}$$

and

$$N \subseteq \mathbb{V} \times \mathbb{A}$$

[and $|P|,\ |N| \in \mathbf{N}$].

Definition 1.3 (Consistency)
A pair (P, N) is *consistent* if

$$P \text{ is well-founded}$$

and

$$P^{+} \circ N \text{ is irreflexive.}$$

Corollary 1.4 *If (P, N) is a finite positive/negative variable-condition, then (P, N) is consistent iff each cycle in the directed graph of $P \uplus N$ contains more than one edge from N. The latter can be effectively tested with time complexity of $|P| + |N|$.*

Note that, in the finite case, the test of Corollary 1.4 seems to be both the most efficient and the most human-oriented way to represent the question of consistency of positive/negative variable-conditions.

Definition 1.5 (Admission Relation)
Let (P, N) be a positive/negative variable-condition.
The *admission relation* $\mathcal{A}_{(P,N)}$ is given as
$\left(x^{\mathbb{A}}, \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right) \in \mathcal{A}_{(P,N)}$ if $\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$ is a pending and $x^{\mathbb{A}} \in \mathbb{A} \cap P^{+}\langle\!\langle\{z^{\mathbb{V}}\}\rangle\!\rangle$.
The *occurrence relation* $\mathcal{O}$ is given as
$\left(x^{\mathbb{A}}, \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right) \in \mathcal{O}$ if $\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$ is a pending and $x^{\mathbb{A}} \in \mathcal{NT}(\theta)$.

The notion of an occurrence of a variable or an atom in a term we need in the following is not the simple one of a participation in the mutually inductive construction of terms and finite atom variables. The reason is that a variable in a pending may stand for any term with atoms from its permission set.

Definition 1.6 ($\mathbb{A}_{(P,N)}(t)$, $\mathbb{V}_{(P,N)}(t)$)
Let A be a binary relation.
The (free) *atoms* $\mathbb{A}_A(t)$ *of a term* t *w.r.t.* A and the (free) *variables* $\mathbb{V}_A(t)$ *of a term* t *w.r.t.* A are recursively defined according to the following table:

t	$\mathbb{A}_A(t)$	$\mathbb{V}_A(t)$
T (a set of terms)	$\bigcup_{t \in T} \mathbb{A}_A(t)$	$\bigcup_{t \in T} \mathbb{V}_A(t)$
f (a constant)	$\emptyset$	$\emptyset$
$x^{\mathbb{A}}$ (an atom)	$\{x^{\mathbb{A}}\}$	$\emptyset$
$y^{\mathbb{B}}$ (a bound atom)	$\emptyset$	$\emptyset$
$t_1 t_2$ (an application)	$\mathbb{A}_A(\{t_1, t_2\})$	$\mathbb{V}_A(\{t_1, t_2\})$
$\lambda y^{\mathbb{B}}.\ t'$ (an abstraction)	$\mathbb{A}_A(t')$	$\mathbb{V}_A(t')$
$\begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix}$ (a pending); setting $\mathrm{X} := A\left\langle \left\{ \begin{pmatrix} z^{\mathbb{V}} \\ \theta \end{pmatrix} \right\} \right\rangle$	$\mathbb{A}_A(\langle \mathrm{X} \rangle \theta)$	$\{z^{\mathbb{V}}\} \cup \mathbb{V}_A(\langle \mathrm{X} \rangle \theta)$

Moreover, we set $\mathbb{VA}_A(t) := \mathbb{A}_A(t) \uplus \mathbb{V}_A(t)$.

Lemma 1.7 $\mathbb{V}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{V}_{\mathcal{O}}(t)$.
$\mathbb{A}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{A}_{\mathcal{O}}(t) \cup P^{+}\langle \mathbb{V}_{\mathcal{O}}(t) \rangle$.

Corollary 1.8 *$\mathbb{VA}_{\mathcal{A}_{(P,N)}}(t)$ is finitary in P in the sense that for every term t there is a finite $P' \subseteq P$ with $\mathbb{VA}_{\mathcal{A}_{(P,N)}}(t) = \mathbb{VA}_{\mathcal{A}_{(P',\emptyset)}}(t)$.*

Corollary 1.9 *Let (P, N) be finite positive/negative variable-condition. Let t be a term. All the sets $\mathbb{A}_{\mathcal{A}_{(P,N)}}(t)$, $\mathbb{A}_{\mathcal{O}}(t)$, $\mathbb{V}_{\mathcal{A}_{(P,N)}}(t)$, $\mathbb{V}_{\mathcal{O}}(t)$ are always finite.*
We have $\mathbb{A}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{A}_{\mathcal{O}}(t) \subseteq \mathbb{A}$, and $\mathbb{V}_{\mathcal{A}_{(P,N)}}(t) \subseteq \mathbb{V}_{\mathcal{O}}(t) \subseteq \mathbb{V}$.

Definition 1.10 ([Finite] Variable Substitution)
A *variable substitution* σ is a total function from variables to closed terms respecting types, i.e., for every $z^{\mathbb{V}} : \alpha$, we have $\sigma(z^{\mathbb{V}}) : \alpha$.
For a variable substitution θ we define its *non-trivial domain* as

$$\mathcal{NT}(\sigma) := \left\{ \ z^{\mathbb{V}} \in \mathbb{V} \ \middle| \ \sigma(z^{\mathbb{V}}) \neq \begin{pmatrix} z^{\mathbb{V}} \\ \mathbb{V}{\upharpoonright}\mathrm{id} \end{pmatrix} \ \right\}.$$

A *finite variable substitution* θ is a variable substitution where $\mathcal{NT}(\sigma)$ is finite.

Definition 1.11 (Extension to Terms)
We extend any finite atom substitution θ and any variable substitution σ to functions $\theta^{\mathcal{T}}$ and $\sigma^{\mathcal{T}}$ from terms to terms as given in the following box. Note that we write $r\theta$ as a shorthand for $\theta^{\mathcal{T}}(r)$, and also $r\sigma$ as a shorthand for $\sigma^{\mathcal{T}}(r)$, which are recursively defined according to the following table:

t	$t\theta$	$t\sigma$
f (a constant)	f	f
$x^{\mathbb{A}}$ (an atom)	$\theta(x^{\mathbb{A}})$	$x^{\mathbb{A}}$
$y^{\mathbb{B}}$ (a bound atom)	$y^{\mathbb{B}}$	$y^{\mathbb{B}}$
$t_1 t_2$ (an application)	$(t_1\theta)(t_2\theta)$	$(t_1\sigma)(t_2\sigma)$
$\lambda y^{\mathbb{B}}.\ t'$ (an abstraction)	$\lambda y^{\mathbb{B}}.\ (t'\theta)$	$\lambda y^{\mathbb{B}}.\ (t'\sigma)$
$\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \end{pmatrix}$ (a pending)	$\begin{pmatrix} z^{\mathbb{V}} \\ \theta' \circ \theta^{\mathcal{T}} \end{pmatrix}$	$(\sigma(z^{\mathbb{V}}))(\theta' \circ \sigma^{\mathcal{T}})$

Note that $\theta' \circ \theta^{\mathcal{T}}$ may be an infinite atom substitution for a finite atom substitution θ' and an infinite atom substitution θ; so we have to require θ to be finite in Definition 1.11. This also applies to the atom substitution $\theta' \circ \sigma^{\mathcal{T}}$, where, however, we do not have to require σ to be a finite variable substitution because of $\mathcal{NT}(\theta' \circ \sigma^{\mathcal{T}}) \subseteq \mathcal{NT}(\theta')$. Moreover, note that the term $(\sigma(z^{\mathbb{V}}))(\theta' \circ \sigma^{\mathcal{T}})$ in the last field of the table of Definition 1.11 denotes $(\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}}(\sigma(X))$ and not the (in general different) $(\theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}})(\sigma(X))$; to see the difference, take $\theta := {}_{\mathbb{A}}\upharpoonright\mathrm{id}$, then $\theta^{\mathcal{T}} \circ \sigma^{\mathcal{T}} = {}_{\mathcal{T}}\upharpoonright\mathrm{id} \circ \sigma^{\mathcal{T}} = \sigma^{\mathcal{T}}$, but $(\theta \circ \sigma^{\mathcal{T}})^{\mathcal{T}} = ({}_{\mathbb{A}}\upharpoonright\mathrm{id})^{\mathcal{T}} = {}_{\mathcal{T}}\upharpoonright\mathrm{id}$.

1.1 Extensions, σ-Updates, and (P, N)-Substitutions

Within a reasoning process, positive/negative variable-conditions may be subject to only one kind of transformation, which we simply call an "extension".

Definition 1.12 ([Weak] Extension)
(P', N') is an $[weak]$ *extension of* (P, N) if
(P', N') is a positive/negative variable-condition, $P \subseteq P'$ [or at least $P \subseteq (P')^{+}$], and $N \subseteq N'$.

As an immediate corollary of Definitions 1.12 and 1.3 we get:

Corollary 1.13
If (P', N') is a consistent positive/negative variable-condition and an [weak] extension of (P, N), then (P, N) is a consistent positive/negative variable-condition as well.

A σ-update is a special form of an extension:

Definition 1.14 (σ-Update, Dependence)
Let (P, N) be a finite positive/negative variable-condition.
Let σ be a finite variable substitution.
The (P, N)-*dependence of* σ is the smallest set D such that $P \subseteq D$ and

$$\{\ (z^{\mathbb{VA}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathcal{NT}(\sigma) \wedge z^{\mathbb{VA}} \in \mathbb{VA}_{\mathcal{A}_{(D,N)}}(\sigma(x^{\mathbb{V}}))\ \} \ \subseteq\ D.$$

The σ-*update of* (P, N) is (D, N).

As a corollary of Corollary 1.9, we get:

Lemma 1.15 *The σ-update (D, N) of a finite positive/negative variable-condition (P, N) and a finite variable substitution σ satisfies*

$$D \ \subseteq\ P\ \cup\ \left(\ \left(\ \bigcup_{x^{\mathbb{V}} \in \mathcal{NT}(\sigma)} \mathbb{VA}_{\mathcal{O}}(\sigma(x^{\mathbb{V}}))\ \right) \times \mathcal{NT}(\sigma)\ \right),$$

so, especially, (D, N) is a finite positive/negative variable-condition.

Proof of Lemma 1.15 By Corollary 1.8, the closure operator of Definition 1.14 is finitary for a *finite* variable substitution σ. Moreover, it is monotonic. Thus, we get the set D as the ω-closure of P. By Corollary 1.9, it now suffices to show that all finite applications of the closure operator satisfy the displayed equation of Lemma 1.15. So let us consider a pending $\binom{z^{\mathbb{V}}}{\theta}$. Set $\mathrm{X} := \mathcal{A}_{(D,N)}\left\langle\left\{\binom{z^{\mathbb{V}}}{\theta}\right\}\right\rangle$. Then $\mathrm{X} = \mathbb{A} \cap D^{+}\langle\{z^{\mathbb{V}}\}\rangle$. Then $\langle\mathrm{X}\rangle\theta = \langle\mathrm{X} \cap \mathcal{NT}(\theta)\rangle\theta \cup \langle\mathrm{X} \setminus \mathcal{NT}(\theta)\rangle\theta = \langle\mathrm{X} \cap \mathcal{NT}(\theta)\rangle\theta \cup (\mathrm{X} \setminus \mathcal{NT}(\theta)) \subseteq \langle\mathcal{NT}(\theta)\rangle\theta \cup (\mathrm{X} \setminus \mathcal{NT}(\theta))$. Thus, $\mathbb{V}_{\mathcal{A}_{(D,N)}}(\langle\mathrm{X}\rangle\theta) \subseteq \mathbb{V}_{\mathcal{A}_{(D,N)}}(\langle\mathcal{NT}(\theta)\rangle\theta)$ and $\mathbb{A}_{\mathcal{A}_{(D,N)}}(\langle\mathrm{X}\rangle\theta) \subseteq \mathbb{A}_{\mathcal{A}_{(D,N)}}(\langle\mathcal{NT}(\theta)\rangle\theta) \cup (\mathrm{X} \setminus \mathcal{NT}(\theta))$. **Q.e.d. (Lemma 1.15)**

Definition 1.16 ((P, N)-Substitution)
Let (P, N) be a positive/negative variable-condition.
σ is a (P, N)-*substitution* if
σ is a variable substitution and the σ-update of (P, N) is consistent.

Syntactically, $(x^{\mathbb{V}}, a^{\mathbb{A}}) \in N$ is to express that a (P, N)-substitution σ must not replace $x^{\mathbb{V}}$ with a term in which $a^{\mathbb{A}}$ could ever occur; i.e. that $a^{\mathbb{A}}$ is fresh for $x^{\mathbb{V}}$: $a^{\mathbb{A}} \# x^{\mathbb{V}}$. This is indeed guaranteed if any σ-update (P', N') of (P, N) is again required to be consistent, and so on. We can see this as follows: For $z^{\mathbb{V}} \in \mathbb{V}(\sigma(x^{\mathbb{V}}))$, we get

$$z^{\mathbb{V}} \quad P' \quad x^{\mathbb{V}} \quad N' \quad a^{\mathbb{A}}.$$

If we now try to apply a second substitution σ' with $a^{\mathbb{A}} \in \mathbb{A}(\sigma'(z^{\mathbb{V}}))$ (so that $a^{\mathbb{A}}$ occurs in $(x^{\mathbb{V}}\sigma)\sigma'$, contrary to what we initially expressed as our freshness intention), then σ' is not a (P', N')-substitution because, for the σ'-update (P'', N'') of (P', N'), we have

$$a^{\mathbb{A}} \quad P'' \quad z^{\mathbb{V}} \quad P'' \quad x^{\mathbb{V}} \quad N'' \quad a^{\mathbb{A}};$$

so $(P'')^{+} \circ N''$ is not irreflexive. All in all, the positive/negative variable-condition

- (P', N') blocks any instantiation of $(x^{\mathbb{V}}\sigma)$ resulting in a term containing $a^{\mathbb{A}}$, just as
- (P, N) blocked $x^{\mathbb{V}}$ before the application of σ.