German Research Center for Artificial Intelligence

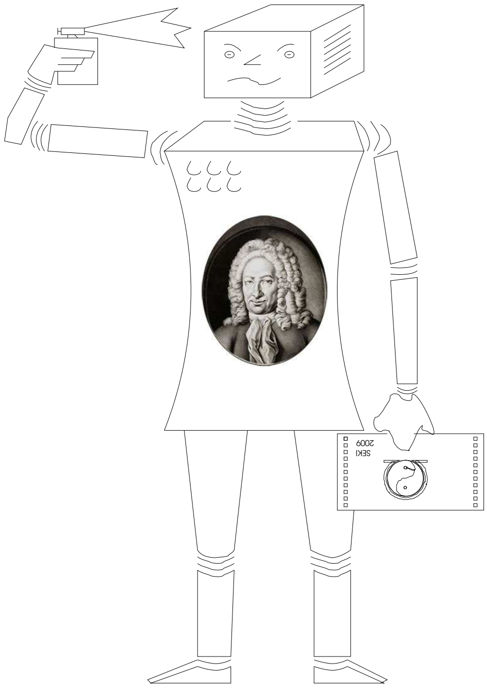
JACOBS UNIVERSITY

UNIVERSITÄT DES SAARLANDES

SEKI Report

http://wirth.bplaced.net/seki.html

ISSN 1437-4447



A Simplified and Improved Free-Variable Framework for HILBERT's epsilon as an Operator of Indefinite Committed Choice

CLAUS-PETER WIRTH

Dept. of Math., ETH Zurich, Rämistr. 101, 8092 Zürich, Switzerland

wirth@logic.at

SEKI Report SR–2011–01

SEKI is published by the following institutions:

German Research Center for Artificial Intelligence (DFKI GmbH), Germany

- Robert Hooke Str. 5, D–28359 Bremen
- Trippstadter Str. 122, D–67663 Kaiserslautern
- Campus D 3 2, D–66123 Saarbrücken

Jacobs University Bremen, School of Engineering & Science, Campus Ring 1, D–28759 Bremen, Germany

Universität des Saarlandes, FR 6.2 Informatik, Campus, D–66123 Saarbrücken, Germany

SEKI Editor:

Claus-Peter Wirth
E-mail: wirth@logic.at
WWW: http://wirth.bplaced.net

Please send surface mail exclusively to:

DFKI Bremen GmbH
Safe and Secure Cognitive Systems
Cartesium
Enrique Schmidt Str. 5
D–28359 Bremen
Germany

This SEKI Report was internally reviewed by:

Murdoch J. Gabbay
School of Mathematical and Computer Sciences
Heriot–Watt University
Edinburgh EH14 4AS, Scotland
murdoch.gabbay@gmail.com

A Simplified and Improved Free-Variable Framework for Hilbert's epsilon as an Operator of Indefinite Committed Choice

Claus-Peter Wirth

Dept. of Math., ETH Zurich, Rämistr. 101, 8092 Zürich, Switzerland

wirth@logic.at

Searchable Online Edition

April 13, 2011

Submitted March 02, 2011
Rev. (Example A.1) and Extd. (Theorem 5.26(7)): Jan. 17, 2012
Thorough revision and addition of § B: May 19, 2015
Rev. and Extd. (Example 4.3 new, Example A.1 extd.): Feb. 01, 2017

Abstract

Free variables occur frequently in mathematics and computer science with *ad hoc* and altering semantics. We present here the most recent version of our free-variable framework for two-valued logics with properly improved functionality, but only two kinds of free variables left (instead of three): implicitly universally and implicitly existentially quantified ones, now simply called "free atoms" and "free variables", respectively. The quantificational expressiveness and the problem-solving facilities of our framework exceed standard first-order logic and even higher-order modal logics, and directly support Fermat's *descente infinie*. With the improved version of our framework, we can now model also Henkin quantification, neither using any binders (such as quantifiers or epsilons) nor raising (Skolemization). Based only on the traditional ε-formula of Hilbert–Bernays, we present our flexible and elegant semantics for Hilbert's ε as a choice operator with the following features: We avoid overspecification (such as right-uniqueness), but admit indefinite choice, committed choice, and classical logics. Moreover, our semantics for the ε supports reductive proof search optimally.

Keywords: Logical Foundations; Theories of Truth and Validity; Formalized Mathematics; Choice; Human-Oriented Interactive Theorem Proving; Automated Theorem Proving; Hilbert's ε-Operator; Henkin Quantification; IF Logic; Fermat's *Descente Infinie*.

Contents

1 Overview

1.1 What is new?

Driven by a weakness in representing Henkin quantification (described in [Wirth, 2012c, § 6.4.1]) and inspired by nominal terms (cf. e.g. [Urban &al., 2004]), in this paper we significantly improve our semantic free-variable framework for two-valued logics:

1. We have replaced the two-layered construction of free δ^+-variables on top of free γ-variables over free δ^--variables of [Wirth, 2004; 2008; 2012c] with a one-layered construction of *free variables* over *free atoms*:

   - Free variables without choice-condition now play the former rôle of the γ-variables.
   - Free variables with choice-condition play the former rôle of the δ^+-variables.
   - Free atoms now play the former rôle of the δ^--variables.

2. As a consequence, the proofs of the lemmas and theorems have shortened by more than a factor of 2. Therefore, we can now present all the proofs in this paper and make it self-contained in this aspect; whereas in [Wirth, 2008; 2012c], we had to point to [Wirth, 2004] for most of the proofs.

3. The difference between free variables and atoms and their names are now more standard and more clear than those of the different free variables before; cf. § 2.1.

4. Compared to [Wirth, 2004], besides shortening the proofs, we have made the meta-level presuppositions more explicit in this paper; cf. § 5.8.

5. Last but not least, we can now treat Henkin quantification in a direct way; cf. § 5.11.

Taking all these points together, the version of our free-variable framework presented in this paper is the version we recommend for further reference, development, and application: it is indeed much easier to handle than its predecessors.

And so we found it appropriate to present most of the material from [Wirth, 2008; 2012c] in this paper in the improved form; we have omitted only the discussions on the tailoring of operators similar to our ε, and on the analysis of natural-language semantics. The material on mathematical induction in the style of Fermat's *descente infinie* in our framework of [Wirth, 2004] is to be reorganized accordingly in a later publication.

1.2 Organization

This paper is organized as follows. There are three introductory sections: to our free variables and atoms (§ 2), to their relation to our reductive inference rules (§ 3), and to Hilbert's ε (§ 4). Afterward we explain and formalize our novel approach to the semantics of our free variables and atoms and the ε (§ 5), and summarize and discuss it (§ 6). We conclude in § 7. In an appendix, the reader can find an example on how we can do Henkin quantification and formalize IF-logic quantifiers with our new positive/negative variable-conditions (§ A), as well as a discussion of the literature on extended semantics given to Hilbert's ε-operator in the 2nd half of the 20th century (§ B), in particular on Leisenring's extensionality axiom (E2). The proofs of all lemmas and theorems can be found in § C; and the acknowledgments, notes, and references as well as an index can be found in the appendix as well.

2 Introduction to Free Variables and Atoms

2.1 Outline

Free variables or free atoms occur frequently in practice of mathematics and computer science. The logical function of these free symbols varies locally; it is typically determined *ad hoc* by the context. And the intended semantics is given only *implicitly* and varies from context to context. In this paper, however, we will make the semantics of our free variables and atoms *explicit* by using disjoint sets of symbols for different semantic functions; namely we will use the following sets of symbols:

$\mathbb{V}$ (the set of *free variables*),
$\mathbb{A}$ (the set of *free atoms*),
$\mathbb{B}$ (the set of *bound*[1] *atoms*).

An *atom* typically stands for an arbitrary object in a proof attempt or in a discourse. Nothing else is known on any atom. Atoms are invariant under renaming. And we will never want to know anything about a possible atom but whether it is an atom, and, if yes, whether it is identical to another atom or not. In our context here, for reasons of efficiency, we would also like to know whether an atom is a free or a bound one. The name "atom" for such an object has a tradition in set theories with atoms. (In German, besides "Atom", an atom is also called an "Urelement", but that alternative name puts some emphasis on the origin of creation, in which we are not interested here.)

A *variable*, however, in the sense we will use the word in this paper, is a place-holder in a proof attempt or in a discourse, which gathers and stores information and which may be replaced with a definition or a description during the discourse or proof attempt. The name "free variable" for such a place-holder has a tradition in free-variable semantic tableaus; cf. [Fitting, 1990; 1996].

Both variables and atoms may be instantiated with terms. Only variables, however, may refer to other free variables and atoms, or may depend on them; and only variables have the following properties w.r.t. instantiation:

1. If a variable is instantiated, then this affects *all* of its occurrences in the entire state of the proof attempt (i.e. it is *rigid* in the terminology of semantic tableaus). Thus, if the instantiation is executed eagerly, the variable must be replaced *globally* in all terms of the entire state of the proof attempt with the same term; afterwards the variable can be eliminated from the resulting proof forest completely — without any further effect on the chance to complete it into a successful proof.
2. The instantiation may be relevant for the consequences of a proof because the global replacement may strengthen the input proposition (or query) by providing a witnessing term for an existential property stated in the proposition (or by providing an answer to the query).

By contrast to these properties of variables, atoms cannot refer to any other symbols, nor depend on them in any form. Moreover, free atoms have the following properties w.r.t. instantiation:

1. A free atom may be
   - globally renamed, or else
   - locally and possibly repeatedly instantiated with arbitrary different terms in the application of lemmas or induction hypotheses (provided that the instantiation is admissible in the sense of Theorem 5.26(7)).

   We cannot eliminate a free atom safely, however. Indeed, neither global renaming nor local instantiation can achieve that completely.

2. The question with which terms an atom was actually instantiated can never influence the consequences of a proof (whereas it may be relevant for bookkeeping or for a replay mechanism).

2.2 Notation

The classification as a (free) variable, (free) atom, or bound atom will be indicated by adjoining a "$\mathbb{V}$", an "$\mathbb{A}$", or a "$\mathbb{B}$", respectively, as a label to the upper right of the meta-variable for the symbol. If a meta-variable stands for a symbol of the union of some of these sets, we will indicate this by listing all possible sets; e.g. "$x^{\mathbb{VA}}$" is a meta-variable for a symbol that may be either a free variable or a free atom.

Meta-variables with disjoint labels always denote different symbols; e.g. "$x^{\mathbb{V}}$" and "$x^{\mathbb{A}}$" will always denote different symbols, whereas "$x^{\mathbb{VA}}$" may denote the same symbol as "$x^{\mathbb{A}}$". In formal discussions, also "$x^{\mathbb{A}}$" and "$y^{\mathbb{A}}$" may denote the same symbol. In concrete examples, however, we will implicitly assume that different meta-variables denote different symbols.

2.3 Semantics of Free Variables and Atoms

2.3.1 Semantics of Free Atoms

As already noted in [Russell, 1919, p.155], free symbols of a formula often have an obviously universal intention in mathematical practice, such as the free symbols m, p, and q in the formula

$$(m)^{(p+q)} = (m)^{(p)} * (m)^{(q)}.$$

Moreover, the formula itself is not meant to denote a propositional function, but actually stands for the explicitly universally quantified, closed formula

$$\forall m^{\mathbb{B}}, p^{\mathbb{B}}, q^{\mathbb{B}}. \left(\ (m^{\mathbb{B}})^{(p^{\mathbb{B}}+q^{\mathbb{B}})} = (m^{\mathbb{B}})^{(p^{\mathbb{B}})} * (m^{\mathbb{B}})^{(q^{\mathbb{B}})} \ \right).$$

In this paper, however, we indicate by

$$(m^{\mathbb{A}})^{(p^{\mathbb{A}}+q^{\mathbb{A}})} = (m^{\mathbb{A}})^{(p^{\mathbb{A}})} * (m^{\mathbb{A}})^{(q^{\mathbb{A}})},$$

a proper formula with *free atoms*, which — independent of its context — is equivalent to the explicitly universally quantified formula, but which also admits the reference to the free atoms, which is required for mathematical induction in the style of Fermat's *descente* *infinie*, and which may also be beneficial for solving reference problems in the analysis of natural language. So the third version combines the practical advantages of the first version with the semantic clarity of the second version.

2.3.2 Semantics of Free Variables

Changing from universal to existential intention, it is somehow clear that the linear system of the formula

$$\begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}$$

asks us to find the set of solutions for x and y, say $(x, y) \in \{(-38, 29)\}$. The mere existence of such solutions is expressed by the explicitly existentially quantified, closed formula

$$\exists x^{\mathbb{B}}, y^{\mathbb{B}}.\ \left(\ \begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}\begin{pmatrix}x^{\mathbb{B}}\\y^{\mathbb{B}}\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}\ \right).$$

In this paper, however, we indicate by

$$\begin{pmatrix}2 & 3\\5 & 7\end{pmatrix}\begin{pmatrix}x^{\mathbb{V}}\\y^{\mathbb{V}}\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}$$

a proper formula with free variables, which — independent of its context — is equivalent to the explicitly existentially quantified formula, but which admits also the reference to the free variables, which is required for retrieving solutions for $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$ as instantiations for $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$ chosen in a formal proof. So the third version again combines the practical advantages of the first with the semantic clarity of the second.

3 Reductive Inference Rules

We will now present the essential reductive inference rules for our free-variable framework. Regarding form and notation, please note the following items:

- We choose a sequent-calculus representation to enhance the readability of the rules and the explicitness of eliminability of formulas.

  As we restrict ourselves to two-valued logics, we just take the right-hand side of standard sequents. This means that our *sequents* are just disjunctive lists of formulas.

- We assume that all binders have minimal scope; e.g.

  $$\forall x^{\mathbb{B}}, y^{\mathbb{B}}.\ A \wedge B$$

  reads

  $$(\forall x^{\mathbb{B}}.\ \forall y^{\mathbb{B}}.\ A) \wedge B.$$

- Our reductive inference rules will be written "reductively" in the sense that passing the line means reduction. Note that in the good old days when trees grew upward, GERHARD GENTZEN (1909–1945) would have inverted the inference rules such that passing the line means consequence. In our case, passing the line means reduction, and trees grow downward.

- RAYMOND M. SMULLYAN (*1919) has classified reductive inference rules into α-, β-, γ-, and δ-rules, and invented a uniform notation for them; cf. [SMULLYAN, 1968].

In the following rules, let A always be a formula and Γ and Π be sequents.

3.1 α- and β-Rules

α-rules are the non-branching propositional rules, such as

$$\frac{\Gamma \quad \neg\neg A \quad \Pi}{\Gamma \quad A \quad \Pi} \qquad\qquad \frac{\Gamma \quad A \Rightarrow B \quad \Pi}{\Gamma \quad \neg A \quad B \quad \Pi}$$

β-rules are the branching propositional rules, which reduce a sequent to several sequents, such as

$$\begin{array}{llll} \Gamma & \neg(A \Rightarrow B) & & \Pi \\ \hline \Gamma & A & & \Pi \\ \Gamma & & \neg B & \Pi \end{array}$$

3.2 γ-Rules

Suppose we want to prove an existential proposition $\exists y^{\mathbb{B}}.\ A$. Here "$y^{\mathbb{B}}$" is a bound variable according to standard terminology, but as it is an atom according to our classification of § 2.1, we will speak of a "bound atom" instead. Then the γ-rules of old-fashioned inference systems (such as [GENTZEN, 1935] or [SMULLYAN, 1968]) enforce the choice of a witnessing term t as a substitution for the bound atom *immediately* when eliminating the quantifier.

γ-rules: Let t be any term:

$$\frac{\Gamma \quad \exists y^{\mathbb{B}}.\ A \quad \Pi}{A\{y^{\mathbb{B}}\mapsto t\} \quad \Gamma \quad \exists y^{\mathbb{B}}.\ A \quad \Pi} \qquad \frac{\Gamma \quad \neg\forall y^{\mathbb{B}}.\ A \quad \Pi}{\neg A\{y^{\mathbb{B}}\mapsto t\} \quad \Gamma \quad \neg\forall y^{\mathbb{B}}.\ A \quad \Pi}$$

More modern inference systems (such as the ones in [FITTING, 1996]) enable us to delay the crucial choice of the term t until the state of the proof attempt may provide more information to make a successful decision. This delay is achieved by introducing a special kind of variable.

This special kind of variable is called "dummy" in [PRAWITZ, 1960] and [KANGER, 1963], "free variable" in [FITTING, 1990; 1996] and in Footnote 11 of [PRAWITZ, 1960], "meta variable" in the field of planning and constraint solving, and "free γ-variable" in [WIRTH, 2004; 2006; 2008; 2012a; 2012c; 2014] and [WIRTH &AL., 2009; 2014].

In this paper, we call these variables simply "free variables" and write them like "$y^{\mathbb{V}}$". When these additional variables are available, we can reduce $\exists y^{\mathbb{B}}.\ A$ first to $A\{y^{\mathbb{B}} \mapsto y^{\mathbb{V}}\}$ and then sometime later in the proof we may globally replace $y^{\mathbb{V}}$ with an appropriate term.

The addition of these free variables changes the notion of a term, but not the notation of the γ-rules, whereas it will become visible in the δ-rules.

3.3 δ^--Rules

A δ-rule may introduce either a free atom (δ^-*-rule*) or an ε-constrained free variable (δ^+*-rule,* cf. § 3.4).

δ^--rules: Let $x^{\mathbb{A}}$ be a fresh free atom:

$$\frac{\Gamma \quad \forall x^{\mathbb{B}}.\ A \quad \Pi}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \quad \Gamma \quad \Pi} \qquad \mathbb{V}(\Gamma \ \ \forall x^{\mathbb{B}}.\ A \ \ \Pi) \times \{x^{\mathbb{A}}\}$$

$$\frac{\Gamma \quad \neg\exists x^{\mathbb{B}}.\ A \quad \Pi}{\neg A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \quad \Gamma \quad \Pi} \qquad \mathbb{V}(\Gamma \ \ \neg\exists x^{\mathbb{B}}.\ A \ \ \Pi) \times \{x^{\mathbb{A}}\}$$

Note that $\mathbb{V}(\Gamma \ \ \forall x^{\mathbb{B}}.\ A \ \ \Pi)$ stands for the set of all symbols from $\mathbb{V}$ (in this case: the free variables) that occur in the sequent $\Gamma \ \ \forall x^{\mathbb{B}}.\ A \ \ \Pi$.

Let us recall that a free atom typically stands for an arbitrary object in a discourse of which nothing else is known. The free atom $x^{\mathbb{A}}$ introduced by the δ^{-}-rules is sometimes also called "parameter", "eigenvariable", or "free δ-variable". In HILBERT-calculi, however, this free atom is called a "free *variable*", because the non-reductive (i.e. generative) deduction in HILBERT-calculi admits its unrestricted instantiation by the substitution rule, cf. p. 63 of [HILBERT & BERNAYS, 1934] or p. 62 of [HILBERT & BERNAYS, 1968; 2017b]. The equivalents of the δ^{-}-rules in HILBERT–BERNAYS' predicate calculus are Schemata (α) and (β) on p. 103f. of [HILBERT & BERNAYS, 1934] or on p. 102f. of [HILBERT & BERNAYS, 1968; 2017b].

The occurrence of the free atom $x^{\mathbb{A}}$ of the δ^{-}-rules must be disallowed in the terms that may be used to replace those free variables which have already been in use when $x^{\mathbb{A}}$ was introduced by application of the δ^{-}-rule, i.e. the free variables of the upper sequent to which the δ^{-}-rule was applied. The reason for this restriction of instantiation of free variables is that the dependencies (or scoping) of the quantifiers must be somehow reflected in the dependencies of the free variables on the free atoms. In our framework, these dependencies are to be captured in binary relations on the free variables and the free atoms, called *variable-conditions*.

Indeed, it is sometimes unsound to instantiate a free variable $x^{\mathbb{V}}$ with a term containing a free atom $y^{\mathbb{A}}$ that was introduced later than $x^{\mathbb{V}}$:

Example 3.1 (Soundness of δ^{-}-rule)
The formula
$$\exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})$$
is not universally valid. We can start a reductive proof attempt as follows:

γ-step: $\quad \forall x^{\mathbb{B}}.\ (y^{\mathbb{V}} = x^{\mathbb{B}}),\quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})$

δ^{-}-step: $\quad (y^{\mathbb{V}} = x^{\mathbb{A}}),\quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})$

Now, if the free variable $y^{\mathbb{V}}$ could be replaced with the free atom $x^{\mathbb{A}}$, then we would get the tautology $(x^{\mathbb{A}} = x^{\mathbb{A}})$, i.e. we would have proved an invalid formula. To prevent this, as indicated to the lower right of the bar of the first of the δ^{-}-rules, the δ^{-}-step has to record
$$\mathbb{V}(\forall x^{\mathbb{B}}.\ (y^{\mathbb{V}} = x^{\mathbb{B}}),\ \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})) \times \{x^{\mathbb{A}}\} \quad = \quad \{(y^{\mathbb{V}}, x^{\mathbb{A}})\}$$
in a variable-condition, where $(y^{\mathbb{V}}, x^{\mathbb{A}})$ means that $y^{\mathbb{V}}$ is somehow "necessarily older" than $x^{\mathbb{A}}$, so that we may never instantiate the free variable $y^{\mathbb{V}}$ with a term containing the free atom $x^{\mathbb{A}}$. Starting with an empty variable-condition, we extend the variable-condition during proof attempts by δ-steps and by global instantiations of free variables. Roughly speaking, this kind of global instantiation of these *rigid* free variables is *consistent* if the resulting variable-condition (seen as a directed graph) has no cycle after adding, for each free variable $y^{\mathbb{V}}$ instantiated with a term t and for each free variable or atom $x^{\mathbb{VA}}$ occurring in t, the pair $(x^{\mathbb{VA}}, y^{\mathbb{V}})$. This consistency, however, would be violated by the cycle between $y^{\mathbb{V}}$ and $x^{\mathbb{A}}$ if we instantiated $y^{\mathbb{V}}$ with $x^{\mathbb{A}}$.

3.4 δ^+-Rules

There are basically two different versions of the δ-rules: standard δ^--rules (also simply called "δ-rules") and δ^+-rules (also called *"liberalized* δ-rules"). They differ in the kind of symbol they introduce and — crucially — in the way they enlarge the variable-condition, depicted to the lower right of the bar:

δ^+-rules: Let $x^{\mathbb{V}}$ be a fresh free variable:

$$\frac{\Gamma \quad \forall x^{\mathbb{B}}.\ A \quad \Pi}{A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{V}}\} \quad \Gamma \quad \Pi} \qquad \begin{matrix}(\ x^{\mathbb{V}},\ \ \varepsilon x^{\mathbb{B}}.\ \neg A\ ) \\ \mathbb{VA}(\forall x^{\mathbb{B}}.\ A) \times \{x^{\mathbb{V}}\}\end{matrix}$$

$$\frac{\Gamma \quad \neg\exists x^{\mathbb{B}}.\ A \quad \Pi}{\neg A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{V}}\} \quad \Gamma \quad \Pi} \qquad \begin{matrix}(\ x^{\mathbb{V}},\ \ \varepsilon x^{\mathbb{B}}.\ A\ ) \\ \mathbb{VA}(\neg\exists x^{\mathbb{B}}.\ A) \times \{x^{\mathbb{V}}\}\end{matrix}$$

While in the (first) δ^--rule, $\mathbb{V}(\Gamma\ \forall x^{\mathbb{B}}.\ A\ \Pi)$ denotes the set of the free variables occurring in the *entire* upper sequent, in the (first) δ^+-rule, $\mathbb{VA}(\forall x^{\mathbb{B}}.\ A)$ denotes the set of all free variables and all free atoms, but only the ones occurring in particular in the *principal*[2] *formula* $\forall x^{\mathbb{B}}.\ A$.

Therefore, the variable-conditions generated by the δ^+-rules are typically smaller than the ones generated by the δ^--rules. Smaller variable-conditions permit additional proofs. Indeed, the δ^+-rules enable additional proofs on the same level of *γ-multiplicity* (i.e. the maximal number of repeated γ-steps applied to the identical principal formula); cf. e.g. [Wirth, 2004, Example 2.8, p. 21]. For certain classes of theorems, these proofs are exponentially and even non-elementarily shorter than the shortest proofs which apply only δ^--rules; for a short survey cf. [Wirth, 2004, § 2.1.5]. Moreover, the δ^+-rules provide additional proofs that are not only shorter but also more natural and easier to find, both automatically and for human beings; see the discussion on design goals for inference systems in [Wirth, 2004, § 1.2.1], and the formal proof of the limit theorem for + in [Wirth, 2006; 2012b]. All in all, the name "liberalized" for the δ^+-rules is indeed justified: They provide more freedom to the prover.[3]

Moreover, note that the pairs indicated to the upper right of the bar of the δ^+-rules are to augment another global binary relation besides the variable-condition, namely a function called the *choice-condition*. Roughly speaking, the addition of an element $(x^{\mathbb{V}},\ \varepsilon x^{\mathbb{B}}.\ \neg A)$ to the current choice-condition — as required by the first of the δ^+-rules — is to be interpreted as the addition of the equational constraint $x^{\mathbb{V}} = \varepsilon x^{\mathbb{B}}.\ \neg A$. To preserve the soundness of the δ^+-step under subsequent global instantiation of the free variable $x^{\mathbb{V}}$, this constraint must be observed in such instantiations. What this actually means will be explained in § 4.12.

All of the three following systems are sound and complete for first-order logic: The one that has (besides the straightforward propositional rules (α-, β-rules) and the γ-rules) only the δ^--rules, the one that has only the δ^+-rules, and the one that has *both* the δ^-- and δ^+-rules.

For a replay of Example 3.1 using the δ^+-rule instead of the δ^--rule, see Example 4.12 in § 4.12.

3.5 SKOLEMization

Note that there is a popular alternative to variable-conditions, namely SKOLEMization, where the δ^{-}- and δ^{+}-rules introduce functions (i.e. the logical order of the replacements for the bound atoms is incremented) which are given the free variables of $\mathbb{V}(\Gamma \quad \forall x^{\mathbb{B}}.\ A \quad \Pi)$ and $\mathbb{V}(\forall x^{\mathbb{B}}.\ A)$ as initial arguments, respectively. Then, the occur-check of unification implements the restrictions on the instantiation of free variables, which are required for soundness. In some inference systems, however, SKOLEMization is unsound (e.g. for higher-order systems such as the one in [KOHLHASE, 1998] or the system in [WIRTH, 2004] for *descente infinie*) or inappropriate (e.g. in the matrix systems of [WALLEN, 1990]).

We prefer inference systems that include variable-conditions to inference systems that offer only SKOLEMization. Indeed, this inclusion provides a more general and often simpler approach, which never results in a necessary reduction in efficiency. Moreover, note that variable-conditions cannot add unnecessary complications here:

- If, in some application, variable-conditions are superfluous, then we can work with empty variable-conditions as if there would be no variable-conditions at all.
- We will need the variable-conditions anyway for our choice-conditions, which again are needed to formalize our approach to HILBERT's ε-operator.

4 Introduction to HILBERT's ε

4.1 Motivation

HILBERT's ε-symbol is an operator or binder that forms terms, just like PEANO's ι-symbol. Roughly speaking, the term $\varepsilon x^{\mathbb{B}}.\,A$, formed from a bound atom (or "bound variable") $x^{\mathbb{B}}$ and a formula A, denotes *just some* object that is *chosen* such that — if possible — A (seen as a predicate on $x^{\mathbb{B}}$) holds for this object.

For ACKERMANN, BERNAYS, and HILBERT, the ε was an intermediate tool in proof theory, to be eliminated in the end. Instead of giving a model-theoretic semantics for the ε, they just specified those axioms which were essential in their proof transformations. These axioms did not provide a complete definition, but left the ε underspecified.

Descriptive terms such as $\varepsilon x^{\mathbb{B}}.\,A$ and $\iota x^{\mathbb{B}}.\,A$ are of universal interest and applicability. Our more elegant and flexible treatment turns out to be useful in many areas where logic is designed or applied as a tool for description and reasoning.

4.2 Requirements Specification

For the usefulness of such descriptive terms we consider the following requirements to be the most important ones.

Requirement I (Indication of Commitment):
The syntax must clearly express where exactly a *commitment* to a choice of a particular object is required, and where, to the contrary, different objects corresponding with the description may be chosen for different occurrences of the same descriptive term.

Requirement II (Reasoning):
It must be possible to replace a descriptive term with a term that corresponds with its description. The correctness of such a replacement must be expressible and should be verifiable in the original calculus.

Requirement III (Semantics):
The semantics should be simple, straightforward, natural, formal, and model-based. Overspecification should be carefully avoided. Furthermore, the semantics should be modular and abstract in the sense that it adds the operator to a variety of logics, independent of the details of a concrete logic.

Our more elegant and flexible, indefinite treatment of the ε-operator is compatible with HILBERT's original one and satisfies these requirements. As it involves novel semantic techniques, it may also serve as the paradigm for the design of similar operators.

4.3 Overview

In § B of the appendix, the reader can find an update of our review form [WIRTH, 2008; 2012c] of the literature on extended semantics given to HILBERT's ε-operator in the 2^{nd} half of the 20^{th} century. In the current § 4, we will now introduce to the ι and the ε (§§ 4.4 and 4.5), to the ε's proof-theoretic origin (§ 4.6), and to our more general semantic objective (§ 4.7) with its emphasis on *indefinite* and *committed choice* (§ 4.8).

4.4 From the ι to the ε

As the ε-operator was developed as an improvement over the still very popular ι-operator, a careful discussion of the ι in this section is required for a deeper understanding of the ε.

4.4.1 The Symbols for the ι-Operator

The probably first descriptive ι-operator occurs in [FREGE, 1893/1903, Vol. I], written as a boldface backslash. As a *boldface* version of the backslash is not easily available in standard typesetting, we will use a simple backslash (\) in § 4.4.4.

A slightly different ι-operator occurs in [PEANO, 1896f.], written as "$\bar{\iota}$", i.e. as an overlined ι. In its German translation [PEANO, 1899b], we also find an alternative symbol with the same denotation, namely an upside-down ι-symbol. Both symbols are meant to indicate the inverse of PEANO's ι-function, which constructs the set of its single argument.

Nowadays, however, "$\{y\}$" is written for PEANO's "ιy", and thus — as a simplifying convention to avoid problems in typesetting and automatic indexing — a simple ι should be used to designate the descriptive ι-operator, without overlining or inversion.

4.4.2 The Essential Idea of the ι-Operator

Let us define the quantifier of *unique existence* by

$$\exists! x^{\mathbb{B}}.\ A \quad := \quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ ((y^{\mathbb{B}}{=}x^{\mathbb{B}}) \Leftrightarrow A),$$

for some fresh $y^{\mathbb{B}}$. All the slightly differing specifications of the ι-operator agree in the following point: If there is the unique $x^{\mathbb{B}}$ such that the formula A (seen as a predicate on $x^{\mathbb{B}}$) holds, then the ι-term $\iota x^{\mathbb{B}}.\ A$ denotes this unique object:

$$\exists! x^{\mathbb{B}}.\ A \quad \Rightarrow \quad A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}.\ A\} \qquad (\iota_0)$$

or in different notation $(\exists! x^{\mathbb{B}}.\ (A(x^{\mathbb{B}}))) \quad \Rightarrow \quad A(\iota x^{\mathbb{B}}.\ (A(x^{\mathbb{B}})))$.

Example 4.1 (ι-operator)
For an informal introduction to the ι-operator, consider Father to be a predicate for which $\mathsf{Father}(\mathsf{Heinrich\,III}, \mathsf{Heinrich\,IV})$ holds, i.e. "Heinrich III is father of Heinrich IV".
Now, "*the* father of Heinrich IV" is designated by $\iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV})$, and because this is nobody but Heinrich III, i.e. $\mathsf{Heinrich\,III} = \iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV})$, we know that $\mathsf{Father}(\iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV}), \mathsf{Heinrich\,IV})$. Similarly,

$$\mathsf{Father}(\iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Adam}), \mathsf{Adam}), \qquad (4.1.1)$$

and thus $\exists y^{\mathbb{B}}.\ \mathsf{Father}(y^{\mathbb{B}}, \mathsf{Adam})$, but, oops! Adam and Eve do not have any fathers.
If you do not agree, you probably appreciate the following problem that occurs when somebody has God as an additional father.

$$\mathsf{Father}(\mathsf{Holy\,Ghost}, \mathsf{Jesus}) \quad \wedge \quad \mathsf{Father}(\mathsf{Joseph}, \mathsf{Jesus}). \qquad (4.1.2)$$

Then the Holy Ghost is *the* father of Jesus and Joseph is *the* father of Jesus:

$$\mathsf{Holy\,Ghost} = \iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus}) \ \wedge \ \mathsf{Joseph} = \iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus}) \qquad (4.1.3)$$

This implies something *the* Pope may not accept, namely $\mathsf{Holy\,Ghost} = \mathsf{Joseph}$,
and he anathematized Heinrich IV in the year 1076:

$$\mathsf{Anathematized}(\iota x^{\mathbb{B}}.\ \mathsf{Pope}(x^{\mathbb{B}}), \mathsf{Heinrich\,IV}, 1076). \qquad (4.1.4)$$

4.4.3 Elementary Semantics Without Straightforward Overspecification

Semantics without a straightforward form of overspecification can be given to the ι-terms in the following three elementary ways:

Russell's non-referring ι-operator, [Russell, 1905]:

In *Principia Mathematica* [1910–1913] by Bertrand Russell (1872–1970) and Alfred North Whitehead (1861–1947), an ι-term is given a meaning only in form of quantifications over contexts $C[\cdots]$ of the occurrences of the ι-term:

$$C[\iota x^{\mathbb{B}}.\ A] \text{ is defined as a short form for } \exists y^{\mathbb{B}}.\ \big(\forall x^{\mathbb{B}}.\ \big((y^{\mathbb{B}}{=}x^{\mathbb{B}}) \Leftrightarrow A\big) \wedge C[y^{\mathbb{B}}]\big).$$

This definition is peculiar because the *definiens* is not of the expected form $C[t]$ (for some term t), and because an ι-term on its own — i.e. without a context $C[\cdots]$ — cannot *directly refer* to an object that it may be intended to denote.

This was first presented as a linguistic theory of descriptions in [Russell, 1905] — but without using any symbol for the ι.

Russell's *On Denoting* [1905] became so popular that the term "non-referring" had to be introduced to make aware of the fact that Russell's ι-terms are not denoting (in spite of the title), and that Russell's theory of descriptions ignores the fundamental reference aspect of descriptive terms, cf. Strawson's *On Referring* [1950].

Hilbert–Bernays' presuppositional ι-operator [Hilbert & Bernays, 1934]:

To overcome the complex difficulties of Russell's non-referring semantics, in § 8 of the first volume of the two-volume monograph *Foundations of Mathematics* (*Grundlagen der Mathematik,* 1st edn. 1934, 2nd edn. 1968) by David Hilbert (1862–1943) and Paul Bernays (1888–1977), a completed proof of $\exists! x^{\mathbb{B}}.\ A$ is required to precede each formation of a term $\iota x^{\mathbb{B}}.\ A$, which otherwise is not considered a well-formed term at all.

This way of defining the ι is nowadays called *"presuppositional".* This word occurs in relation to Hilbert–Bernays' ι in [Slater, 2007a] and [Slater, 2009, §§ 1, 6, and 8f.], but it does not occur in [Strawson, 1950], and we do not know where it occurs first with this meaning.

Peano's partially specified ι-operator [Peano, 1896f.]:

Since Hilbert–Bernays' presuppositional treatment makes the ι quite impractical and the formal syntax of logic undecidable in general, in § 1 of the second volume of Hilbert–Bernays' *Foundations of Mathematics* (1st edn. 1939, 2nd edn. 1970), Hilbert's ε, however, is already given a more flexible treatment: The simple idea is to leave the ε-terms uninterpreted. This will be described below. In this paper, we will present this more flexible treatment also for the ι.

After all, this treatment is the original one of Peano's ι, found already in the article *Studii di Logica Matematica* [1896f.] by Guiseppe Peano (1858–1932).[4]

It cannot surprise that it was Peano — interested in written languages for specification and communication, but hardly in calculi — who came up with the only practical specification of ι-terms (unlike Russell and Hilbert–Bernays).

Moreover, by the partiality of his specification, PEANO avoided also the other pitfall, namely overspecification, and all its unintended consequences (unlike FREGE and QUINE, cf. § 4.4.4). As the symbol "ι" was invented by PEANO as well (cf. § 4.4.1), we have good reason to speak of "PEANO's ι", at least as much as we have reason to speak of "HILBERT's ε".

It must not be overlooked that PEANO's ι — in spite of its partiality — always denotes: It is not a partial operator, it is just partially specified.

At least in non-modal classical logics, it is a well justified standard that *each term denotes.* More precisely — in each model or structure $\mathcal{S}$ under consideration — each occurrence of a proper term must denote an object in the universe of $\mathcal{S}$. Following that standard, to be able to write down $\iota x^{\mathbb{B}}.\ A$ without further consideration, we have to treat $\iota x^{\mathbb{B}}.\ A$ as an uninterpreted term about which we only know axiom (ι_0) from § 4.4.2.

With (ι_0) as the only axiom for the ι, the term $\iota x^{\mathbb{B}}.\ A$ has to satisfy A (seen as a predicate on $x^{\mathbb{B}}$) only if there exists a unique object such that A holds for it. The price, however, we have to pay for the avoidance of non-referringness, presuppositionality, and overspecification is that — roughly speaking — the term $\iota x^{\mathbb{B}}.\ A$ is of no use unless the unique existence $\exists! x^{\mathbb{B}}.\ A$ can be derived.

Finally, let us come back to Example 4.1 of § 4.4.2. The problems presented there do not actually appear if (ι_0) is the only axiom for the ι, because (4.1.1) and (4.1.3) are not valid. Indeed, the description of (4.1.1) lacks existence and the descriptions of (4.1.3) and (4.1.4) lack uniqueness.

4.4.4 Overspecified ι-Operators

From FREGE to QUINE, we find a multitude of ι-operators with definitions that overspecify the ι in different ways for the sake of *complete definedness* and *syntactic eliminability.*

As we already stated in Requirement III (Semantics) of § 4.2, overspecification should be carefully avoided. Indeed, any overspecification leads to puzzling, arbitrary consequences, which may cause harm to the successful application of descriptive operators in practice.

Frege's arbitrarily overspecified ι-operator [FREGE, 1893/1903]:

> The first occurrence of a descriptive ι-operator in the literature seems to be in 1893, namely in § 11 of the first volume of the two-volume monograph *Grundgesetze der Arithmetik — Begriffsschriftlich abgeleitet* [1893/1903] by GOTTLOB FREGE (1848–1925):
>
> For A seen as a function from objects to truth values, $\backslash A$ (in our notation $\iota x^{\mathbb{B}}.\ A$) is defined to be the object Δ if A is extensionally equal to the function that checks for equality to Δ, i.e. if $A = \lambda x^{\mathbb{B}}.\ (\Delta = x^{\mathbb{B}})$.
>
> In the case that there is no such Δ, FREGE overspecified his ι-operator pretty arbitrarily by defining $\backslash A$ to be A, which is not even an object, but a function.
>
> (Note that FREGE actually wrote an ε (having nothing to do with the ε-operator) instead of our $x^{\mathbb{B}}$, and a *spiritus lenis* over it instead of a modern λ-operator before and a dot after it. Moreover, he wrote a ξ for the A.)

Quine's overspecified ι-operator [QUINE, 1981]:

In set theories without urelements, such as in [QUINE, 1981], the ι-operator can be defined by something like

$$\iota x^{\mathbb{B}}.\ A \quad := \quad \left\{\ z^{\mathbb{B}}\ \middle|\ \exists y^{\mathbb{B}}.\ \left(\ \forall x^{\mathbb{B}}.\ ((y^{\mathbb{B}}{=}x^{\mathbb{B}}) \Leftrightarrow A) \quad \wedge \quad z^{\mathbb{B}} \in y^{\mathbb{B}}\ \right)\ \right\},$$

for fresh $y^{\mathbb{B}}$ and $z^{\mathbb{B}}$.

This is again an overspecification resulting in $\iota x^{\mathbb{B}}.\, A \;=\; \emptyset$ if there is no such $y^{\mathbb{B}}$ (which otherwise is always unique).

4.4.5 A Completely Defined, but Not Overspecified ι-Operator

The complete definitions of the ι in § 4.4.4 take place in *possibly inconsistent* logical frameworks, namely FREGE's Begriffsschrift and QUINE's set theory.

That neither overspecification nor possible inconsistency are necessary for complete definitions of the ι is witnessed by the following complete, but non-elementary definition of the ι, which is also referring and non-presuppositional.

The ε-calculus' ι-operator [HILBERT & BERNAYS, 1939]:

In the ε-calculus, which is a conservative extension of first-order predicate calculus, first elaborated in the second volume of HILBERT–BERNAYS' *Foundations of Mathematics* [1939], we can define the ι simply by

$$\iota x^{\mathbb{B}}.\ A \quad := \quad \varepsilon y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ ((y^{\mathbb{B}} = x^{\mathbb{B}}) \Leftrightarrow A)$$

(for a fresh $y^{\mathbb{B}}$), i.e. as a unique $x^{\mathbb{B}}$ such that A holds (provided there is such an $x^{\mathbb{B}}$).

This definition is non-elementary, however, because it introduces ε-terms, which cannot be eliminated in first-order logic in general.

Note that this definition is — to the best of our knowledge — the most useful and elegant way to introduce the ι, although it is somehow *ex eventu,* because the development of the ε was started two dozen years after the first publications on FREGE's and PEANO's ι-operators.

4.5 The ε as an Improvement over the ι

Compared to the ι, the ε is more useful because — instead of (ι_0) — it comes with the stronger axiom

$$\exists x^{\mathbb{B}}.\ A \ \Rightarrow\ A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \qquad (\varepsilon_0)$$

More specifically, as the formula $\exists x^{\mathbb{B}}.\ A$ (which has to be true to guarantee an interpretation of the ε-term $\varepsilon x^{\mathbb{B}}.\ A$ that is meaningful in the sense that it satisfies its formula A) is weaker than the corresponding formula $\exists! x^{\mathbb{B}}.\ A$ (for the respective ι-term), the area of useful application is wider for the ε- than for the ι-operator. Indeed, we have already seen in § 4.4.5 that the ι can be defined in terms of the ε, but not vice versa.

Moreover, in case of $\exists! x^{\mathbb{B}}.\ A$, the ε-operator picks the same element as the ι-operator:

$$\exists! x^{\mathbb{B}}.\ A \ \Rightarrow\ \left(\ \varepsilon x^{\mathbb{B}}.\ A \ =\ \iota x^{\mathbb{B}}.\ A\ \right).$$

Thus, unless eliminability is relevant, we should replace all useful occurrences of the ι with the ε: As a consequence, among other advantages, the arising proof obligations become weaker and both human and automated generation and generalization of proofs become more efficient.

4.6 On the ε's Proof-Theoretic Origin

4.6.1 The ε-Formula and the Historical Sources of the ε

The main historical source on the ε is the second volume of the *Foundations of Mathematics* [Hilbert & Bernays, 1934; 1939; 1968; 1970], the fundamental work which summarizes the foundational and proof-theoretic contributions of David Hilbert and his mathematical-logic group.

The preferred specification for Hilbert's ε in proof-theoretic investigations is not the axiom (ε_0), but actually the following formula:

$$A\{x^{\mathbb{B}} \mapsto x^{\mathbb{A}}\} \ \Rightarrow\ A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \qquad (\varepsilon\textit{-formula})$$

The ε-formula is equivalent to (ε_0), but it gets along without any quantifier.

The name "ε-formula" originates in [Hilbert & Bernays, 1939, p. 13], where the ε-operator is simply called "Hilbert's ε-symbol".

For historical correctness, note that the notation in the original is closer to

$$A(x^{\mathbb{A}}) \Rightarrow A(\varepsilon x^{\mathbb{B}}.\ A(x^{\mathbb{B}})),$$

where the A is a concrete singulary predicate atom (called "formula variable" in the original) and comes with several extra rules for its instantiation, cf. [Hilbert & Bernays, 1939, p. 13f.].

The exact notation actually is

$$A(a) \Rightarrow A(\varepsilon_x\, A(x)),$$

and the deductive equivalence is straightforward to the exact notation of (ε_0), i.e. to

$$(Ex)\, A(x) \Rightarrow A(\varepsilon_x\, A(x)),$$

cf. [Hilbert & Bernays, 1939, pp. 13–15].

In our notation, however, (ε_0) and the ε-formula are axiom *schemata* where the A is a meta-variable for a formula (which, contrary to the predicate atom, may contain occurrences of $x^{\mathbb{A}}$). Nevertheless, their deductive equivalence is given for versions of (ε_0) and the ε-formula where the A is replaced with $A\{x^{\mathbb{A}}{\mapsto}y^{\mathbb{A}}\}$ for some fresh (free) atom $y^{\mathbb{A}}$, from which both (ε_0) and the ε-formula can be obtained by instantiation.

The ε-formula already occurs, however under different names, in the pioneering papers on the ε, i.e. in [Ackermann, 1925] as "transfinite axiom 1", in [Hilbert, 1926] as "axiom of choice" (in the operator form $A(a) \Rightarrow A(\varepsilon A)$, where the ε is called "transfinite logical choice function"), and in [Hilbert, 1928] as "logical ε-axiom" (again in operator form, where the ε is called "logical ε-function").

4.6.2 The Original Explanation of the ε

As the basic methodology of Hilbert's program is to treat all symbols as meaningless, no semantics is required besides the one given by the single axiom (ε_0). To further the understanding, however, we read on p.12 of [Hilbert & Bernays, 1939; 1970]:

> $\varepsilon x^{\mathbb{B}}.\ A$ ... "ist ein Ding des Individuenbereichs, und zwar ist dieses Ding gemäß der inhaltlichen Übersetzung der Formel (ε_0) *ein solches, auf das jenes Prädikat A zutrifft, vorausgesetzt, daß es überhaupt auf ein Ding des Individuenbereichs zutrifft.*"
>
> $\varepsilon x^{\mathbb{B}}.\ A$ ... "is a thing of the domain of individuals for which — according to the contentual translation of the formula (ε_0) — *the predicate A holds, provided that A holds for any thing of the domain of individuals at all.*"

Example 4.2 (ε instead of ι) *(continuing Example 4.1 of § 4.4.2)*
Just as for the ι, for the ε we have $\mathsf{Heinrich\ III} = \varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\ IV})$ and

$$\mathsf{Father}(\varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\ IV}), \mathsf{Heinrich\ IV}).$$

But, from the contrapositive of (ε_0) and $\neg\mathsf{Father}(\varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Adam}), \mathsf{Adam})$, we now conclude that $\neg\exists y^{\mathbb{B}}.\ \mathsf{Father}(y^{\mathbb{B}}, \mathsf{Adam})$.

4.6.3 Defining the Quantifiers via the ε

Hilbert and Bernays did not need any semantics or precise intention for the ε-symbol because it was introduced merely as a formal syntactic device to facilitate proof-theoretic investigations, motivated by the possibility to get rid of the existential and universal quantifiers via two direct consequences of axiom (ε_0):

$$\exists x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \qquad (\varepsilon_1)$$

$$\forall x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg A\} \qquad (\varepsilon_2)$$

These equivalences can be seen as definitions of the quantifiers because innermost rewriting with (ε_1), (ε_2) yields a normal form after as many steps as there are quantifiers in the input formula. Moreover, also arbitrary rewriting is confluent and terminating, cf. [WIRTH, 2016].

It should be noted, however, that rewriting with (ε_1), (ε_2) must not be taken for granted *under* modal operators, at least not under the assumption that ε-terms are to remain *rigid,* i.e. independent in their interpretation from their modal contexts. For this assumption there are very good reasons, nicely explained e.g. in [SLATER, 2007a; 2009].

Example 4.3 Consider the first-order modal logic formula $\Box\exists x^{\mathbb{B}}.\,A$. Moreover, to simplify matters, let us assume that we have constant domains, i.e. that all modal contexts have the same domain of individuals.

Under this condition and for a formula of this structure, it is suggested in [SLATER, 2007a, p.153] to apply (ε_1) to the considered formula, resulting in $\Box A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\,A\}$, from which we can doubtlessly conclude $\exists x^{\mathbb{B}}.\,\Box A$, e.g. by Formula (a) in § 4.6.4.

Let us interpret the $\Box$ as "believes" and A as "$x^{\mathbb{B}}$ is the number of rice corns in my car", and let our constant domain be the one of the standard model of the natural numbers. Note that I do not believe of any concrete and definite number that it numbers the rice corns in my car just because I believe that their number is finite.

This interpretation shows that our rewriting with (ε_1) *under* the operator $\Box$ is incorrect for modal logic in general, at least for rigid ε-terms.

On the other hand, rewriting with (ε_1), (ε_2) *above* modal operators is uncritical: $\exists x^{\mathbb{B}}.\,\Box A$ is indeed equivalent to $\Box A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\,\Box A\}$.

4.6.4 The ε-Theorems

When we remove all quantifiers in a derivation of the HILBERT-style predicate calculus of the *Foundations of Mathematics* along (ε_1) and (ε_2), the following transformations occur:

Tautologies are turned into tautologies.

The axioms

$$A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \quad\Rightarrow\quad \exists x^{\mathbb{B}}.\,A \qquad (\textit{Formula (a)})$$

and

$$\forall x^{\mathbb{B}}.\,A \quad\Rightarrow\quad A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \qquad (\textit{Formula (b)})$$

(cf. p. 100f. of [HILBERT & BERNAYS, 1934] or on p. 99f. of [HILBERT & BERNAYS, 1968; 2017b]), are turned into the ε-formula (cf. § 4.6.1) and, roughly speaking, its contrapositive, respectively. Indeed, for the case of Formula (b), we can replace first all A with $\neg A$, and after applying (ε_2), replace $\neg\neg A$ with A, and thus obtain the contrapositive of the ε-formula.

The inference steps are turned into inference steps: the inference schema [of modus ponens] into the inference schema; the substitution rule for free atoms as well as quantifier introduction (Schemata (α) and (β) on p. 103f. of [HILBERT & BERNAYS, 1934] or on p. 102f. of [HILBERT & BERNAYS, 1968; 2017b]) into the substitution rule including ε-terms.

Finally, the ε-formula is taken as a new axiom scheme instead of (ε_0) because it has the advantage of being free of quantifiers.

The argumentation of the previous paragraphs is actually part of the proof transformation that constructively proves the first of HILBERT–BERNAYS' two theorems on ε-elimination in first-order logic, the so-called *1st ε-Theorem.* In its sharpened form, this theorem can be stated as follows. Note that the original speaks of "bound variables" instead of "bound atoms" and of "formula variables" instead of "predicate atoms", because what we call (free) "variables" is not part of the formula languages of HILBERT–BERNAYS.

Theorem 4.4 (Sharpened 1st ε-Thm., p.79f. of [HILBERT & BERNAYS, 1939; 1970]) *From a derivation of* $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.\ A$ *(containing no bound atoms besides the ones bound by the prefix* $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.$ *) from the formulas* $P_1, \ldots, P_k$ *(containing neither predicate atoms nor bound atoms) in the predicate calculus (incl. the ε-formula and $=$-substitutability as axiom schemes, plus $=$-reflexivity), we can construct a (finite) disjunction of the form* $\bigvee_{i=0}^{s} A\{x_1^{\mathbb{B}}, \ldots, x_r^{\mathbb{B}} \mapsto t_{i,1}, \ldots, t_{i,r}\}$ *and a derivation of it*

- *in which bound atoms do not occur at all*
- *from* $P_1, \ldots, P_k$ *and $=$-axioms (containing neither predicate atoms nor bound atoms)*
- *in the quantifier-free predicate calculus*
  (i.e. tautologies plus the inference schema [of modus ponens] and the substitution rule).

Note that r, s range over natural numbers including 0*, and that A, $t_{i,j}$, and P_i are ε-free because otherwise they would have to include (additional) bound atoms.*

Moreover, the *2nd ε-Theorem* (in [HILBERT & BERNAYS, 1939; 1970]) states that the ε (just as the ι, cf. [HILBERT & BERNAYS, 1934; 1968]) is a conservative extension of the predicate calculus in the sense that each formal proof of an ε-free formula can be transformed into a formal proof that does not use the ε at all.

For logics different from classical axiomatic first-order predicate logic, however, it is not a conservative extension when we add the ε either with (ε_0), with (ε_1), or with the ε-formula to other first-order logics — may they be weaker such as *intuitionistic* first-order logic, or stronger such as first-order set theories with axiom schemes over arbitrary terms *including the ε*; cf. [WIRTH, 2008, § 3.1.3]. Moreover, even in classical first-order logic there is no translation from the formulas containing the ε to formulas not containing it.

4.7 Our Objective

While the historiographical and technical research on the ε-theorems is still going on and the methods of ε-elimination and ε-substitution did not die with HILBERT's program, this is not our subject here. We are less interested in HILBERT's formal program and the consistency of mathematics than in the powerful use of logic in creative processes. And, instead of the tedious syntactic proof transformations, which easily lose their usefulness and elegance within their technical complexity and which — more importantly — can only refer to an already existing logic, we look for *model-theoretic* means for finding new logics and new applications. And the question that still has to be answered in this field is:

> *What would be a proper semantics for* HILBERT*'s ε?*

4.8 Indefinite and Committed Choice

Just as the ι-symbol is usually taken to be the referential interpretation of the *definite* articles in natural languages, it is our opinion that the ε-symbol should be that of the *indefinite* determiners (articles and pronouns) such as "a(n)" or "some".

Example 4.5 (ε instead of ι again) *(continuing Example 4.1)*
It may well be the case that

$$\mathsf{Holy\,Ghost} \;=\; \varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus}) \quad\wedge\quad \mathsf{Joseph} \;=\; \varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus})$$

i.e. that "The Holy Ghost is a̲ father of Jesus and Joseph is a̲ father of Jesus." But this does not bring us into trouble with the Pope because we do not know whether all fathers of Jesus are equal. This will become clearer when we reconsider this in Example 4.14.

Closely connected to indefinite choice (also called "indeterminism" or "don't care nondeterminism") is the notion of *committed choice*. For example, when we have a new telephone, we typically *don't care* which number we get, but once a number has been chosen for our telephone, we will insist on a *commitment to this choice*, so that our phone number is not changed between two incoming calls.

Example 4.6 (Committed choice)
Suppose we want to prove $\quad \exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$
According to (ε_1) from § 4.6 this reduces to $\quad \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$
Since there is no solution to $x^{\mathbb{B}} \neq x^{\mathbb{B}}$ we can replace
$\varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$ with anything. Thus, the above reduces to $\quad \mathsf{0} \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$
and then, by exactly the same argumentation, to $\quad \mathsf{0} \neq \mathsf{1}$
which is true in the natural numbers.
Thus, we have proved our original formula $\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$, which, however, is false.
What went wrong? Of course, we have to commit to our choice for all occurrences of the ε-term introduced when eliminating the existential quantifier: If we choose $\mathsf{0}$ on the left-hand side, we have to commit to the choice of $\mathsf{0}$ on the right-hand side as well.

4.9 Quantifier Elimination and Subordinate ε-terms

Before we can introduce to our treatment of the ε, we also have to get more acquainted with the ε in general.

The elimination of $\forall$- and $\exists$-quantifiers with the help of ε-terms (cf. § 4.6) may be more difficult than expected when some ε-terms become "subordinate" to others.

Definition 4.7 (Subordinate) An ε-term $\varepsilon v^{\mathbb{B}}.\ B$ (or, more generally, a binder on $v^{\mathbb{B}}$ together with its scope B) is *superordinate* to an (occurrence of an) ε-term $\varepsilon x^{\mathbb{B}}.\ A$ if

1. $\varepsilon x^{\mathbb{B}}.\ A$ is a subterm of B and
2. an occurrence of the bound atom $v^{\mathbb{B}}$ in $\varepsilon x^{\mathbb{B}}.\ A$ is free in B
   (i.e. the binder on $v^{\mathbb{B}}$ binds an occurrence of $v^{\mathbb{B}}$ in $\varepsilon x^{\mathbb{B}}.\ A$).

An (occurrence of an) ε-term $\varepsilon x^{\mathbb{B}}.\ A$ is *subordinate* to an ε-term $\varepsilon v^{\mathbb{B}}.\ B$ (or, more generally, to a binder on $v^{\mathbb{B}}$ together with its scope B) if $\varepsilon v^{\mathbb{B}}.\ B$ is superordinate to $\varepsilon x^{\mathbb{B}}.\ A$.

On p. 24 of [Hilbert & Bernays, 1939; 1970], these subordinate ε-terms, which are responsible for the difficulty to prove the ε-theorems constructively, are called "untergeordnete ε-Ausdrücke". Note that — contrary to Hilbert–Bernays — we do not use a special name for ε-terms with free occurrences of bound atoms here — such as "ε-Ausdrücke" ("ε-expressions" or "quasi ε-terms") instead of "ε-Terme" ("ε-terms") — but simply call them "ε-terms" as well.

Example 4.8 (Quantifier Elimination and Subordinate ε-Terms)
Let us repeat the formulas (ε_1) and (ε_2) from § 4.6 here:

$$\exists x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \tag{ε_1}$$
$$\forall x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg A\} \tag{ε_2}$$

Let us consider the formula

$$\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$$

and apply (ε_1) and (ε_2) to remove the four quantifiers completely.

We introduce the following abbreviations, where $w^{\mathbb{B}}$, $x^{\mathbb{B}}$, $y^{\mathbb{B}}$, $w^{\mathbb{B}}_a$, $x^{\mathbb{B}}_a$, $y^{\mathbb{B}}_a$, $z^{\mathbb{B}}_a$ are bound atoms and w_a, x_a, y_a, z_a are meta-level symbols for functions from terms to terms:

$$\begin{array}{lcl}
z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y^{\mathbb{B}}) & = & \varepsilon z^{\mathbb{B}}_a.\ \neg\mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}}_a) \\
y_a(w^{\mathbb{B}})(x^{\mathbb{B}}) & = & \varepsilon y^{\mathbb{B}}_a.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}_a, z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y^{\mathbb{B}}_a)) \\
x_a(w^{\mathbb{B}}) & = & \varepsilon x^{\mathbb{B}}_a.\ \neg\mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}_a, y_a(w^{\mathbb{B}})(x^{\mathbb{B}}_a), z_a(w^{\mathbb{B}})(x^{\mathbb{B}}_a)(y_a(w^{\mathbb{B}})(x^{\mathbb{B}}_a))), \\
w_a & = & \varepsilon w^{\mathbb{B}}_a.\ \mathsf{P}(w^{\mathbb{B}}_a, x_a(w^{\mathbb{B}}_a), y_a(w^{\mathbb{B}}_a)(x_a(w^{\mathbb{B}}_a)), z_a(w^{\mathbb{B}}_a)(x_a(w^{\mathbb{B}}_a))(y_a(w^{\mathbb{B}}_a)(x_a(w^{\mathbb{B}}_a)))),
\end{array}$$

Innermost rewriting with (ε_1) and (ε_2) results in a unique normal form after at most as many steps as there are quantifiers. Thus, we eliminate inside-out, i.e. we start with the elimination of $\forall z^{\mathbb{B}}$. The transformation is:

$$\begin{array}{ll}
\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}}), \\
\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y^{\mathbb{B}})), \\
\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y_a(w^{\mathbb{B}})(x^{\mathbb{B}}), z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y_a(w^{\mathbb{B}})(x^{\mathbb{B}}))), \\
\exists w^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x_a(w^{\mathbb{B}}), y_a(w^{\mathbb{B}})(x_a(w^{\mathbb{B}})), z_a(w^{\mathbb{B}})(x_a(w^{\mathbb{B}}))(y_a(w^{\mathbb{B}})(x_a(w^{\mathbb{B}})))), \\
 & \mathsf{P}(w_a, x_a(w_a), y_a(w_a)(x_a(w_a)), z_a(w_a)(x_a(w_a))(y_a(w_a)(x_a(w_a)))).
\end{array}$$

Note that the resulting formula is quite deep and has more than one thousand occurrences of the ε-binder. Indeed, in general, n nested quantifiers result in an ε-nesting depth of 2^n-1.

To understand this, let us have a closer look a the resulting formula. Let us write it as

$$\mathsf{P}(w_a, x_b, y_d, z_h) \tag{4.8.1}$$

then (after renaming some bound atoms) we have

$$
\begin{array}{rcll}
z_h &=& \varepsilon z_h^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b, y_d, z_h^{\mathbb{B}}), & (4.8.2)\\
y_d &=& \varepsilon y_d^{\mathbb{B}}.\ \mathsf{P}(w_a, x_b, y_d^{\mathbb{B}}, z_g(y_d^{\mathbb{B}})) & (4.8.3)\\
&& \text{with } z_g(y_d^{\mathbb{B}}) = \varepsilon z_g^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b, y_d^{\mathbb{B}}, z_g^{\mathbb{B}}), & (4.8.4)\\
x_b &=& \varepsilon x_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c(x_b^{\mathbb{B}}), z_f(x_b^{\mathbb{B}})) & (4.8.5)\\
&& \text{with } z_f(x_b^{\mathbb{B}}) = \varepsilon z_f^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c(x_b^{\mathbb{B}}), z_f^{\mathbb{B}}) & (4.8.6)\\
&& \text{and } y_c(x_b^{\mathbb{B}}) = \varepsilon y_c^{\mathbb{B}}.\ \mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e(x_b^{\mathbb{B}})(y_c^{\mathbb{B}})) & (4.8.7)\\
&& \quad \text{with } z_e(x_b^{\mathbb{B}})(y_c^{\mathbb{B}}) = \varepsilon z_e^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e^{\mathbb{B}}), & (4.8.8)\\
w_a &=& \varepsilon w_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b(w_a^{\mathbb{B}}), z_d(w_a^{\mathbb{B}})) & (4.8.9)\\
&& \text{with } z_d(w_a^{\mathbb{B}}) = \varepsilon z_d^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b(w_a^{\mathbb{B}}), z_d^{\mathbb{B}}) & (4.8.10)\\
&& \text{and } y_b(w_a^{\mathbb{B}}) = \varepsilon y_b^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c(w_a^{\mathbb{B}})(y_b^{\mathbb{B}})) & (4.8.11)\\
&& \quad \text{with } z_c(w_a^{\mathbb{B}})(y_b^{\mathbb{B}}) = \varepsilon z_c^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c^{\mathbb{B}}), & (4.8.12)\\
&& x_a(w_a^{\mathbb{B}}) = \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})) & (4.8.13)\\
&& \quad \text{with } z_b(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}) = \varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b^{\mathbb{B}}) & (4.8.14)\\
&& \quad \text{and } y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}) = \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}})) & (4.8.15)\\
&& \qquad \text{with } z_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}}) = \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}}). & (4.8.16)
\end{array}
$$

First of all, note that the bound atoms with free occurrences in the indented ε-terms (i.e., in the order of their appearance, the bound atoms $y_d^{\mathbb{B}}$, $x_b^{\mathbb{B}}$, $y_c^{\mathbb{B}}$, $w_a^{\mathbb{B}}$, $y_b^{\mathbb{B}}$, $x_a^{\mathbb{B}}$, $y_a^{\mathbb{B}}$) are actually bound by the next ε to the left, to which the respective ε-terms thus become subordinate. For example, the ε-term $z_g(y_d^{\mathbb{B}})$ is subordinate to the ε-term y_d binding $y_d^{\mathbb{B}}$.

Moreover, the ε-terms defined by the above equations are exactly those that require a commitment to their choice. This means that each of z_a, z_b, z_c, z_d, z_e, z_f, z_g, z_h, each of y_a, y_b, y_c, y_d, and each of x_a, x_b may be chosen differently without affecting soundness of the equivalence transformation. Note that the variables are strictly nested into each other; so we must choose in the order of

$$z_a,\ y_a,\ z_b,\ x_a,\ z_c,\ y_b,\ z_d,\ w_a,\ z_e,\ y_c,\ z_f,\ x_b,\ z_g,\ y_d,\ z_h.$$

Furthermore, in case of all ε-terms except w_a, x_b, y_d, z_h, we actually have to choose a function instead of a simple object.

In Hilbert–Bernays' view, however, there are neither functions nor objects at all, but only terms (and expressions with free occurrences of bound atoms):

In the standard notation the term $x_a(w_a^{\mathbb{B}})$ reads

$$
\varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}\left(\begin{array}{l}
w_a^{\mathbb{B}},\\
x_a^{\mathbb{B}},\\
\varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}\big(\ w_a^{\mathbb{B}},\ \ x_a^{\mathbb{B}},\ \ y_a^{\mathbb{B}},\ \ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}})\ \big),\\
\varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}\Big(\ w_a^{\mathbb{B}},\ \ x_a^{\mathbb{B}},\ \ \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}\big(\ w_a^{\mathbb{B}},\ \ x_a^{\mathbb{B}},\ \ y_a^{\mathbb{B}},\ \ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}})\ \big),\ \ z_b^{\mathbb{B}}\ \Big)
\end{array}\right).
$$

Moreover, $y_b(w_a^{\mathbb{B}})$ reads

$$\varepsilon y_b^{\mathbb{B}}.\ \neg\mathsf{P}\left(\begin{array}{l} w_a^{\mathbb{B}}, \\ \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}\left(\begin{array}{l} w_a^{\mathbb{B}}, \\ x_a^{\mathbb{B}}, \\ \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(\ w_a^{\mathbb{B}},\ x_a^{\mathbb{B}},\ y_a^{\mathbb{B}},\ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}},x_a^{\mathbb{B}},y_a^{\mathbb{B}},z_a^{\mathbb{B}})\ ), \\ \varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}\big(\ w_a^{\mathbb{B}},\ x_a^{\mathbb{B}},\ \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(\ w_a^{\mathbb{B}},\ x_a^{\mathbb{B}},\ y_a^{\mathbb{B}},\ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}},x_a^{\mathbb{B}},y_a^{\mathbb{B}},z_a^{\mathbb{B}})\ ),\ z_b^{\mathbb{B}}\ \big) \end{array}\right), \\ y_b^{\mathbb{B}}, \\ \varepsilon z_c^{\mathbb{B}}.\ \neg\mathsf{P}\left(\begin{array}{l} w_a^{\mathbb{B}}, \\ \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}\left(\begin{array}{l} w_a^{\mathbb{B}}, \\ x_a^{\mathbb{B}}, \\ \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(\ w_a^{\mathbb{B}},\ x_a^{\mathbb{B}},\ y_a^{\mathbb{B}},\ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}},x_a^{\mathbb{B}},y_a^{\mathbb{B}},z_a^{\mathbb{B}})\ ), \\ \varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}\big(\ w_a^{\mathbb{B}},\ x_a^{\mathbb{B}},\ \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(\ w_a^{\mathbb{B}},\ x_a^{\mathbb{B}},\ y_a^{\mathbb{B}},\ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}},x_a^{\mathbb{B}},y_a^{\mathbb{B}},z_a^{\mathbb{B}})\ ),\ z_b^{\mathbb{B}}\ \big) \end{array}\right), \\ y_b^{\mathbb{B}}, \\ z_c^{\mathbb{B}} \end{array}\right) \end{array}\right).$$

Condensed data on the above terms read as follows:

	ε-nesting depth	number of ε-binders	ACKERMANN rank	ACKERMANN degree
$z_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}})$	1	1	1	undefined
$y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})$	2	2	2	undefined
$z_b(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})$	3	3	1	undefined
$x_a(w_a^{\mathbb{B}})$	4	6	3	undefined
$z_c(w_a^{\mathbb{B}})(y_b^{\mathbb{B}})$	5	7	1	undefined
$y_b(w_a^{\mathbb{B}})$	6	14	2	undefined
$z_d(w_a^{\mathbb{B}})$	7	21	1	undefined
w_a	8	42	4	1
$z_e(y_c^{\mathbb{B}})(w_a^{\mathbb{B}})$	9	43	1	undefined
$y_c(x_b^{\mathbb{B}})$	10	86	2	undefined
$z_f(x_b^{\mathbb{B}})$	11	129	1	undefined
x_b	12	258	3	2
$z_g(y_d^{\mathbb{B}})$	13	301	1	undefined
y_d	14	602	2	3
z_h	15	903	1	4
$\mathsf{P}(w_a, x_b, y_d, z_h)$	15	1805	undefined	undefined

For $\forall w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \forall y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$ instead of $\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$, we get the same exponential growth of nesting depth as in the example above, when we completely eliminate the quantifiers using (ε_2). The only difference is that we get additional occurrences of '$\neg$'. If we have quantifiers of the same kind, however, we had better choose them in parallel; e.g., for $\forall w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \forall y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$, we choose

$$v_a \quad := \quad \varepsilon v^{\mathbb{B}}.\ \neg\mathsf{P}(1^{\mathrm{st}}(v^{\mathbb{B}}), 2^{\mathrm{nd}}(v^{\mathbb{B}}), 3^{\mathrm{rd}}(v^{\mathbb{B}}), 4^{\mathrm{th}}(v^{\mathbb{B}})),$$

and then take $\mathsf{P}(1^{\mathrm{st}}(v_a), 2^{\mathrm{nd}}(v_a), 3^{\mathrm{rd}}(v_a), 4^{\mathrm{th}}(v_a))$ as result of the elimination.

Roughly speaking, in today's automated theorem proving, cf. e.g. [FITTING, 1996], the exponential explosion of term depth of the example above is avoided by an outside-in removal of δ-quantifiers *without removing the quantifiers below ε-binders* and by a replacement of γ-quantified variables with free variables without choice-conditions. For the formula of Example 4.8, this yields $\mathsf{P}(w^{\mathbb{V}}, x_e, y^{\mathbb{V}}, z_e)$ with $x_e = \varepsilon x_e^{\mathbb{B}}.\ \neg\exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{V}}, x_e^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$ and $z_e = \varepsilon z_e^{\mathbb{B}}.\ \neg\mathsf{P}(w^{\mathbb{V}}, x_e, y^{\mathbb{V}}, z_e^{\mathbb{B}})$. Thus, in general, the nesting of binders for the complete elimination of a prenex of n quantifiers does not become deeper than $\frac{1}{4}(n{+}1)^2$.

Moreover, if we are only interested in reduction and not in equivalence transformation of a formula, we can abstract SKOLEM terms from the ε-terms and just reduce to the formula $\mathsf{P}(w^{\mathbb{V}}, x^{\mathbb{A}}(w^{\mathbb{V}}), y^{\mathbb{V}}, z^{\mathbb{A}}(w^{\mathbb{V}})(y^{\mathbb{V}}))$. In non-SKOLEMizing inference systems with variable-conditions we get $\mathsf{P}(w^{\mathbb{V}}, x^{\mathbb{A}}, y^{\mathbb{V}}, z^{\mathbb{A}})$ instead, with $\{(w^{\mathbb{V}}, x^{\mathbb{A}}),\ (w^{\mathbb{V}}, z^{\mathbb{A}}),\ (y^{\mathbb{V}}, z^{\mathbb{A}})\}$ as an extension to the variable-condition. Note that with SKOLEMization or variable-conditions we have no growth of nesting depth at all, and the same will be the case for our approach to ε-terms.

4.10 Do not be afraid of Indefiniteness!

From the discussion in § 4.8, one could get the impression that an indefinite logical treatment of the ε is not easy to find. Indeed, on the first sight, there is the problem that some standard axiom schemes cannot be taken for granted, such as substitutability

$$s=t \qquad \Rightarrow \qquad f(s)=f(t)$$

and reflexivity

$$t=t$$

Note that substitutability is similar to the *extensionality axiom*

$$\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \qquad \Rightarrow \qquad \varepsilon x^{\mathbb{B}}.\ A_0 \quad = \quad \varepsilon x^{\mathbb{B}}.\ A_1 \tag{E2}$$

(cf. § B.1.1) when we take logical equivalence as equality. Moreover, note that

$$\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \quad = \quad \varepsilon x^{\mathbb{B}}.\ \mathsf{true} \tag{Reflex}$$

is an instance of reflexivity.

Thus, it seems that — in case of an indefinite ε — the replacement of a subterm with an equal term is problematic, and so is the equality of syntactically equal terms.

It may be interesting to see that — in computer programs — we are quite used to committed choice and to an indefinite behavior of choosing, and that the violation of substitutability and even reflexivity is no problem there:

Example 4.9 (Violation of Substitutability and Reflexivity in Programs)
In the implementation of the specification of the web-based hypertext system of [Mattick & Wirth, 1999], we needed a function that chooses an element from a set implemented as a list. Its ML code is:

```
fun choose s = case s of Set (i :: _) => i | _ => raise Empty;
```

And, of course, it simply returns the first element of the list. For another set that is equal — but where the list may have another order — the result may be different. Thus, the behavior of the function `choose` is indefinite for a given set, but any time it is called for an implemented set, it chooses a particular element and *commits to this choice,* i.e.: when called again, it returns the same value. In this case we have `choose s` = `choose s`, but `s` = `t` does not imply `choose s` = `choose t`. In an implementation where some parallel reordering of lists may take place, even `choose s` = `choose s` may be wrong.

From this example we may learn that the question of `choose s` = `choose s` may be indefinite until the choice steps have actually been performed. *This is exactly how we will treat our ε.* The steps that are performed in logic are related to proving: Reductive inference steps that make proof trees grow toward the leaves, and choice steps that instantiate variables and atoms for various purposes.

Thus, on the one hand, when we want to prove

$$\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \ = \ \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$$

we can choose $\mathsf{0}$ for both occurrences of $\varepsilon x^{\mathbb{B}}.\ \mathsf{true}$, get $\mathsf{0}=\mathsf{0}$, and the proof is successful.

On the other hand, when we want to prove

$$\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \ \neq \ \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$$

we can choose $\mathsf{0}$ for one occurrence and $\mathsf{1}$ for the other, get $\mathsf{0}\neq\mathsf{1}$, and the proof is successful again.

This procedure may seem wondrous again, but is very similar to something quite common for free variables with empty choice-conditions:

On the one hand, when we want to prove

$$x^{\mathbb{V}}=y^{\mathbb{V}}$$

we can choose $\mathsf{0}$ to replace both $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$, get $\mathsf{0}=\mathsf{0}$, and the proof is successful.

On the other hand, when we want to prove

$$x^{\mathbb{V}}\neq y^{\mathbb{V}}$$

we can choose $\mathsf{0}$ to replace $x^{\mathbb{V}}$ and $\mathsf{1}$ to replace $y^{\mathbb{V}}$, get $\mathsf{0}\neq\mathsf{1}$, and the proof is successful again.

4.11 Replacing ε-terms with Free Variables

There is an important difference between the inequations $\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \neq \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$ and $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ at the end of § 4.10: The latter does not violate the reflexivity axiom! And we are going to cure the violation of the former immediately with the help of our free variables, but now with non-empty choice-conditions. Instead of $\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \neq \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$ we write $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ and remember what these free variables stand for by storing this into a function C, called a *choice-condition*:

$$C(x^{\mathbb{V}}) \quad := \quad \varepsilon x^{\mathbb{B}}.\ \mathsf{true},$$

$$C(y^{\mathbb{V}}) \quad := \quad \varepsilon x^{\mathbb{B}}.\ \mathsf{true}.$$

For a first step, suppose that our ε-terms are not subordinate to any outside binder (cf. Definition 4.7). Then, we can replace an ε-term $\varepsilon z^{\mathbb{B}}.\ A$ with a fresh free variable $z^{\mathbb{V}}$ and extend the partial function C by

$$C(z^{\mathbb{V}}) \quad := \quad \varepsilon z^{\mathbb{B}}.\ A.$$

By this procedure we can eliminate all ε-terms without loosing any syntactic information.

As a first consequence of this elimination, the substitutability and reflexivity axioms are immediately regained, and the problems discussed in § 4.10 disappear.

A second reason for replacing the ε-terms with free variables is that the latter can solve the question whether a committed choice is required: We can express

committed choice by repeatedly using the same free variable, and

choice without commitment by using several variables with the same choice-condition.

Indeed, this also solves our problems with committed choice of Example 4.6 of § 4.8: Now, again using (ε_1), $\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$ reduces to $x^{\mathbb{V}} \neq x^{\mathbb{V}}$ with

$$C(x^{\mathbb{V}}) \quad := \quad \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$

and the proof attempt immediately fails because of the now regained reflexivity axiom.

As the second step, we still have to explain what to do with subordinate ε-terms. If the ε-term $\varepsilon v_l^{\mathbb{B}}.\ A$ contains free occurrences of exactly the distinct bound atoms $v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}$, then we have to replace this ε-term with the application term $z^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})$ of the same type as $v_l^{\mathbb{B}}$ (for a fresh free variable $z^{\mathbb{V}}$) and to extend the choice-condition C by

$$C(z^{\mathbb{V}}) \quad := \quad \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ A.$$

Example 4.10 (Higher-Order Choice-Condition) *(continuing Example 4.8 of § 4.9)*
In our framework, the complete elimination of ε-terms in (4.8.1) of Example 4.8 results in

$$\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}}) \qquad \text{(cf. (4.8.1)!)}$$

with the following higher-order choice-condition:

$$\begin{array}{lcrl}
C(z_h^{\mathbb{V}}) & := & \varepsilon z_h^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{B}}) & \text{(cf. (4.8.2)!)} \\
C(y_d^{\mathbb{V}}) & := & \varepsilon y_d^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{B}}, z_c^{\mathbb{V}}(y_d^{\mathbb{B}})) & \text{(cf. (4.8.3)!)} \\
C(z_g^{\mathbb{V}}) & := & \lambda y_d^{\mathbb{B}}.\ \varepsilon z_g^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{B}}, z_g^{\mathbb{B}}) & \text{(cf. (4.8.4)!)} \\
C(x_b^{\mathbb{V}}) & := & \varepsilon x_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{V}}(x_b^{\mathbb{B}}), z_f^{\mathbb{V}}(x_b^{\mathbb{B}})) & \text{(cf. (4.8.5)!)} \\
C(z_f^{\mathbb{V}}) & := & \lambda x_b^{\mathbb{B}}.\ \varepsilon z_f^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{V}}(x_b^{\mathbb{B}}), z_f^{\mathbb{B}}) & \text{(cf. (4.8.6)!)} \\
C(y_c^{\mathbb{V}}) & := & \lambda x_b^{\mathbb{B}}.\ \varepsilon y_c^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e^{\mathbb{V}}(x_b^{\mathbb{B}})(y_c^{\mathbb{B}})) & \text{(cf. (4.8.7)!)} \\
C(z_e^{\mathbb{V}}) & := & \lambda x_b^{\mathbb{B}}.\ \lambda y_c^{\mathbb{B}}.\ \varepsilon z_e^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e^{\mathbb{B}}) & \text{(cf. (4.8.8)!)} \\
C(w_a^{\mathbb{V}}) & := & \varepsilon w_a^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{V}}(w_a^{\mathbb{B}}), z_d^{\mathbb{V}}(w_a^{\mathbb{B}})) & \text{(cf. (4.8.9)!)} \\
C(z_d^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \varepsilon z_d^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{V}}(w_a^{\mathbb{B}}), z_d^{\mathbb{B}}) & \text{(cf. (4.8.10)!)} \\
C(y_b^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \varepsilon y_b^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c^{\mathbb{V}}(w_a^{\mathbb{B}})(y_b^{\mathbb{B}})) & \text{(cf. (4.8.11)!)} \\
C(z_c^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda y_b^{\mathbb{B}}.\ \varepsilon z_c^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c^{\mathbb{B}}) & \text{(cf. (4.8.12)!)} \\
C(x_a^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})) & \text{(cf. (4.8.13)!)} \\
C(z_b^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda x_a^{\mathbb{B}}.\ \varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b^{\mathbb{B}}) & \text{(cf. (4.8.14)!)} \\
C(y_a^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda x_a^{\mathbb{B}}.\ \varepsilon y_a^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}})) & \text{(cf. (4.8.15)!)} \\
C(z_a^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda x_a^{\mathbb{B}}.\ \lambda y_a^{\mathbb{B}}.\ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}}) & \text{(cf. (4.8.16)!)}
\end{array}$$

Note that this representation of (4.8.1) is smaller and easier to understand than all previous ones. Indeed, by combination of λ-abstraction and term sharing via free variables, in our framework the ε becomes practically feasible.

All in all, by this procedure we can replace all ε-terms in all formulas and sequents. The only place where the ε still occurs is the range of the choice-condition C; and also there it is not essential because, instead of

$$C(z^{\mathbb{V}}) \;=\; \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ A,$$

we could write

$$C(z^{\mathbb{V}}) \;=\; \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ A\{v_l^{\mathbb{B}} \mapsto z^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$$

as we have actually done in [WIRTH, 2004; 2006; 2008; 2012b; 2012c].

4.12 Instantiating Free Variables ("ε-Substitution")

Having already realized Requirement I (Indication of Commitment) of § 4.2 in § 4.11, we are now going to explain how to satisfy Requirement II (Reasoning). To this end, we have to explain how to replace free variables with terms that satisfy their choice-conditions.

The first thing to know about free variables with choice-conditions is: Just like the free variables without choice-conditions (introduced by γ-rules e.g.) and contrary to free atoms, the free variables with choice-conditions (introduced by δ^+-rules e.g.) are *rigid* in the sense that the only way to replace a free variable is to do it *globally*, i.e. in all formulas and all choice-conditions with the same term in an atomic transaction.

In *reductive* theorem proving, such as in sequent, tableau, matrix, or indexed-formula-tree calculi, we are in the following situation: While a free variable without choice-condition can be replaced with nearly everything, the replacement of a free variable with a choice-condition requires some proof work, and a free atom cannot be instantiated at all.

Contrariwise, when formulas are used as tools instead of tasks, free atoms can indeed be replaced — and this even locally (i.e. non-rigidly) and repeatedly. This is the case not only for purely *generative* calculi (such as resolution and paramodulation calculi) and HILBERT-style calculi (such as the predicate calculus of [HILBERT & BERNAYS, 1934; 1939; 1968; 1970]), but also for the lemma and induction hypothesis application in the otherwise reductive calculi of [WIRTH, 2004], cf. [WIRTH, 2004, § 2.5.2].

More precisely — again considering *reductive* theorem proving, where formulas are proof tasks — a free variable without choice-condition may be instantiated with any term (of appropriate type) that does not violate the current variable-condition, cf. § 5.7 for details. The instantiation of a free variable with choice-condition additionally requires some proof work depending on the current choice-condition, cf. Definition 5.13 for the formal details. In general, if a substitution σ replaces the free variable $y^{\mathbb{V}}$ in the domain of the choice-condition C, then — to know that the global instantiation of the entire proof forest with σ is correct — we have to prove $(Q_C(y^{\mathbb{V}}))\sigma$, where Q_C is given as follows:

Definition 4.11 (Q_C)
Q_C is the function that maps every $z^{\mathbb{V}} \in \mathrm{dom}(C)$ with $C(z^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ (for some bound atoms $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}$ and some formula B) to the single-formula sequent

$$\forall v_0^{\mathbb{B}}.\ \ldots \forall v_{l-1}^{\mathbb{B}}.\ \left(\ \exists v_l^{\mathbb{B}}.\ B \quad \Rightarrow \quad B\{v_l^{\mathbb{B}} \mapsto z^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}\ \right),$$

and is otherwise undefined.

Note that $Q_C(y^{\mathbb{V}})$ is nothing but a formulation of HILBERT–BERNAYS' axiom (ε_0) in our framework. (See our § 4.5 for (ε_0)).

Moreover, Lemma 5.19 will state the validity of $Q_C(y^{\mathbb{V}})$. Therefore, the commitment to a choice comes only with the substitution σ. Indeed, regarding the σ-instance of $Q_C(y^{\mathbb{V}})$ whose provability is required, it is only the *arbitrariness* of the substitution σ that realizes the *indefiniteness* of the choice for the ε.

Now, as an example for Q_C, we can replay Example 3.1 and use it for a discussion of the δ^+-rule instead of the δ^--rule:

Example 4.12 (Soundness of δ^+-rule)
The formula
$$\exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}}\!=\!x^{\mathbb{B}})$$
is not universally valid. We can start a reductive proof attempt as follows:

γ-step: $\quad \forall x^{\mathbb{B}}.\ (y^{\mathbb{V}}\!=\!x^{\mathbb{B}}),\quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}}\!=\!x^{\mathbb{B}})$

δ^+-step: $\quad (y^{\mathbb{V}}\!=\!x^{\mathbb{V}}),\quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}}\!=\!x^{\mathbb{B}})$

Now, if the free variable $y^{\mathbb{V}}$ could be replaced with the free variable $x^{\mathbb{V}}$, then we would get the tautology $(x^{\mathbb{V}}\!=\!x^{\mathbb{V}})$, i.e. we would have proved an invalid formula. To prevent this, as indicated to the lower right of the bar of the first of the δ^+-rules in § 3.4 on Page 13, the δ^+-step has to record
$$\mathbb{VA}(\forall x^{\mathbb{B}}.\ (y^{\mathbb{V}}\!=\!x^{\mathbb{B}}))\times\{x^{\mathbb{V}}\}\quad=\quad\{(y^{\mathbb{V}},x^{\mathbb{V}})\}$$
in a positive variable-condition, where $(y^{\mathbb{V}},x^{\mathbb{V}})$ means that "$x^{\mathbb{V}}$ positively depends on $y^{\mathbb{V}}$" (or that "$y^{\mathbb{V}}$ is a subterm of the description of $x^{\mathbb{V}}$"), so that we may never instantiate the free variable $y^{\mathbb{V}}$ with a term containing the free variable $x^{\mathbb{V}}$, because this instantiation would result in cyclic dependencies (or in a cyclic term).

Contrary to Example 3.1, we have a further opportunity here to complete this proof attempt into a successful proof: If the the substitution $\sigma := \{x^{\mathbb{V}}\mapsto y^{\mathbb{V}}\}$ could be applied, then we would get the tautology $(y^{\mathbb{V}}\!=\!y^{\mathbb{V}})$, i.e. we would have proved an invalid formula. To prevent this — as indicated to the upper right of the bar of the first of the δ^+-rules in § 3.4 on Page 13 — the δ^+-step has to record
$$\big(\ x^{\mathbb{V}},\ \ \varepsilon x^{\mathbb{B}}.\ \neg(y^{\mathbb{V}}\!=\!x^{\mathbb{B}})\ \big)$$
in the choice-condition C. If we take this pair as an equation, then the intuition behind the above statement that $y^{\mathbb{V}}$ is somehow a subterm of the description of $x^{\mathbb{V}}$ becomes immediately clear. If we take it as element of the graph of the function C, however, then we can compute $(Q_C(x^{\mathbb{V}}))\sigma$ and try to prove it. $Q_C(x^{\mathbb{V}})$ is
$$\exists x^{\mathbb{B}}.\ \neg(y^{\mathbb{V}}\!=\!x^{\mathbb{B}})\quad\Rightarrow\quad\neg(y^{\mathbb{V}}\!=\!x^{\mathbb{V}});$$
so $(Q_C(x^{\mathbb{V}}))\sigma$ is
$$\exists x^{\mathbb{B}}.\ \neg(y^{\mathbb{V}}\!=\!x^{\mathbb{B}})\quad\Rightarrow\quad\neg(y^{\mathbb{V}}\!=\!y^{\mathbb{V}}).$$
In classical logic with equality this is equivalent to $\exists x^{\mathbb{B}}.\ \neg(y^{\mathbb{V}}\!=\!x^{\mathbb{B}})\ \Rightarrow\ \mathsf{false}$, and then to $\forall x^{\mathbb{B}}.\ (y^{\mathbb{V}}\!=\!x^{\mathbb{B}})$. If we were able to show the truth of this formula, then it would be sound to apply the substitution σ to prove the above sequent resulting from the γ-step. That sequent, however, already includes this formula as an element of its disjunction. Thus, no progress is possible by means of the δ^+-rules here; and so this example is not a counterexample to the soundness of the δ^+-rules.

Example 4.13 (Predecessor Function)

Suppose that our domain is natural numbers and that y^{V} has the choice-condition

$$C(y^{\mathrm{V}}) \quad = \quad \lambda v_0^{\mathbb{B}}.\ \varepsilon v_1^{\mathbb{B}}.\ \big(\ v_0^{\mathbb{B}} = v_1^{\mathbb{B}} + 1\ \big).$$

Then, before we may instantiate y^{V} with the symbol p for the predecessor function specified by

$$\forall x^{\mathbb{B}}.\ \big(\ \mathsf{p}(x^{\mathbb{B}}{+}1) = x^{\mathbb{B}}\ \big),$$

we have to prove the single-formula sequent $(Q(y^{\mathrm{V}}))\{y^{\mathrm{V}} \mapsto \mathsf{p}\}$, which reads

$$\forall v_0^{\mathbb{B}}.\ \Big(\ \exists v_1^{\mathbb{B}}.\ \big(\ v_0^{\mathbb{B}} = v_1^{\mathbb{B}} + 1\ \big) \Rightarrow \big(\ v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}}) + 1\ \big)\ \Big).$$

In fact, the single formula of this sequent immediately follows from the specification of p.

Note that the fact that $\mathsf{p}(0)$ is not specified here is no problem in this ε-substitution because $\varepsilon v_1^{\mathbb{B}}.\ (0 = v_1^{\mathbb{B}} + 1)$ is not specified by (ε_0) either.

Example 4.14 (Canossa 1077) *(continuing Example 4.5)*

(See [Fried, 2012] if you want to look behind the omnipresent legend and find out what really seems to have happened at Canossa in January 1077.)

The situation of Example 4.5 now reads

$$\mathsf{Holy\ Ghost} \ = \ z_0^{\mathrm{V}} \quad \wedge \quad \mathsf{Joseph} \ = \ z_1^{\mathrm{V}} \tag{4.14.1}$$

with $C(z_0^{\mathrm{V}}) = \varepsilon z_0^{\mathbb{B}}.\ \mathsf{Father}(z_0^{\mathbb{B}}, \mathsf{Jesus})$,
and $C(z_1^{\mathrm{V}}) = \varepsilon z_1^{\mathbb{B}}.\ \mathsf{Father}(z_1^{\mathbb{B}}, \mathsf{Jesus})$.

This does not bring us into the old trouble with the Pope because nobody knows whether $z_0^{\mathrm{V}} = z_1^{\mathrm{V}}$ holds or not.

On the one hand, knowing (4.1.2) from Example 4.1 of § 4.4, we can prove (4.14.1) as follows: Let us replace z_0^{V} with $\mathsf{Holy\ Ghost}$ because, for $\sigma_0 := \{z_0^{\mathrm{V}} \mapsto \mathsf{Holy\ Ghost}\}$, from $\mathsf{Father}(\mathsf{Holy\ Ghost}, \mathsf{Jesus})$ we conclude

$$\exists z_0^{\mathbb{B}}.\ \mathsf{Father}(z_0^{\mathbb{B}}, \mathsf{Jesus}) \quad \Rightarrow \quad \mathsf{Father}(\mathsf{Holy\ Ghost}, \mathsf{Jesus}),$$

which is nothing but the required $(Q_C(z_0^{\mathrm{V}}))\sigma_0$.

Analogously, we replace z_1^{V} with Joseph because, for $\sigma_1 := \{z_1^{\mathrm{V}} \mapsto \mathsf{Joseph}\}$, from (4.1.2) we conclude the required $(Q_C(z_1^{\mathrm{V}}))\sigma_1$. After these replacements, (4.14.1) becomes the tautology

$$\mathsf{Holy\ Ghost} \ = \ \mathsf{Holy\ Ghost} \quad \wedge \quad \mathsf{Joseph} \ = \ \mathsf{Joseph}$$

On the other hand, if we want to have trouble, we can apply the substitution

$$\sigma' \ = \ \{z_0^{\mathrm{V}} \mapsto \mathsf{Joseph},\ z_1^{\mathrm{V}} \mapsto \mathsf{Joseph}\}$$

to (4.14.1) because both $(Q_C(z_0^{\mathrm{V}}))\sigma'$ and $(Q_C(z_1^{\mathrm{V}}))\sigma'$ are equal to $(Q_C(z_1^{\mathrm{V}}))\sigma_1$ up to renaming of bound atoms. Then our task is to show

$$\mathsf{Holy\ Ghost} \ = \ \mathsf{Joseph} \quad \wedge \quad \mathsf{Joseph} \ = \ \mathsf{Joseph}.$$

Note that this course of action is stupid, even under the aspect of theorem proving alone.

5 Formal Presentation of Our Semantics

To satisfy Requirement III (Semantics) of § 4.2, we will now present our novel semantics for HILBERT's ε formally. This is required for precision and consistency. As consistency of our new semantics is not trivial at all, technical rigor cannot be avoided. From §§ 2 and 4, the reader should have a good intuition of our intended representation and semantics of HILBERT's ε, free variables, atoms, and choice-conditions in our framework.

5.1 Organization of § 5

After some preliminary subsections, we formalize variable-conditions and their consistency (§ 5.5) and discuss alternatives to the design decisions in the formalization of variable-conditions (§ 5.6).

Moreover, we explain how to deal with free variables syntactically (§ 5.7) and semantically (§§ 5.8 and 5.9).

After formalizing choice-conditions and their compatibility (§ 5.10), we define our notion of validity and discuss some examples (§ 5.11). One of these examples is especially interesting because we show that — with our new more careful treatment of negative information in our *positive/negative* variable-conditions — we can now model HENKIN quantification directly.

Our interest goes beyond soundness in that we want to have *"preservation of solutions".* By this we mean the following: All *closing substitutions* for the free variables — i.e. all solutions that transform a proof attempt (to which a proposition has been reduced) into a closed proof — are also solutions of the original proposition. This is similar to a proof in PROLOG (cf. [KOWALSKI, 1974], [CLOCKSIN & MELLISH, 2003]), computing answers to a query proposition that contains free variables. Therefore, we discuss this solution-preserving notion of *reduction* (§ 5.15), in particular under the aspect of extensions of variable-conditions and choice-conditions (§ 5.12), and under the aspect of global instantiation of free variables with choice-conditions ("ε-substitution") (§ 5.13).

Finally, in § 5.16, we show soundness, safeness, and solution-preservation for our γ-, δ^-, and δ^+-rules of §§ 3.2, 3.3, and 3.4.

All in all, we extend and simplify the presentation of [WIRTH, 2008], which already simplifies and extends the presentation of [WIRTH, 2004] and which is extended with additional linguistic applications in [WIRTH, 2012c]. Note, however, that [WIRTH, 2004] additionally contains some comparative discussions and compatible extensions for *descente infinie*, which also apply to our new version here.

5.2 Basic Notions and Notation

'$\mathbf{N}$' denotes the set of natural numbers and '$\prec$' the ordering on $\mathbf{N}$. Let $\mathbf{N}_+ := \{\, n \in \mathbf{N} \mid 0 \neq n \,\}$. We use '$\uplus$' for the union of disjoint classes and 'id' for the identity function. For classes R, A, and B we define:

$\mathrm{dom}(R)$	$:= \{\, a \mid \exists b.\ (a,b) \in R \,\}$	*domain*
${}_A\!\upharpoonleft\! R$	$:= \{\, (a,b) \in R \mid a \in A \,\}$	*(domain-) restriction to* A
$\langle A \rangle R$	$:= \{\, b \mid \exists a \in A.\ (a,b) \in R \,\}$	*image of* A, i.e. $\langle A \rangle R = \mathrm{ran}({}_A\!\upharpoonleft\! R)$

And the dual ones:

$\mathrm{ran}(R)$	$:= \{\, b \mid \exists a.\ (a,b) \in R \,\}$	*range*
$R\!\upharpoonright_B$	$:= \{\, (a,b) \in R \mid b \in B \,\}$	*range-restriction to* B
$R\langle B \rangle$	$:= \{\, a \mid \exists b \in B.\ (a,b) \in R \,\}$	*reverse-image of* B, i.e. $R\langle B \rangle = \mathrm{dom}(R\!\upharpoonright_B)$

Furthermore, we use '$\emptyset$' to denote the empty set as well as the empty function. Functions are (right-) unique relations, and so the meaning of "$f \circ g$" is extensionally given by $(f \circ g)(x) = g(f(x))$. The *class of total functions from* A *to* B is denoted as $A \to B$. The *class of (possibly) partial functions from* A *to* B is denoted as $A \leadsto B$. Both $\to$ and $\leadsto$ associate to the right, i.e. $A \leadsto B \to C$ reads $A \leadsto (B \to C)$.

Let R be a binary relation. R is said to be a relation *on* A if $\mathrm{dom}(R) \cup \mathrm{ran}(R) \subseteq A$. R is *irreflexive* if $\mathrm{id} \cap R = \emptyset$. It is A-*reflexive* if ${}_A\!\upharpoonleft\!\mathrm{id} \subseteq R$. Speaking of a *reflexive* relation we refer to the largest A that is appropriate in the local context, and referring to this A we write R^0 to ambiguously denote ${}_A\!\upharpoonleft\!\mathrm{id}$. With $R^1 := R$, and $R^{n+1} := R^n \circ R$ for $n \in \mathbf{N}_+$, R^m denotes the m-step relation for R. The *transitive closure* of R is $R^+ := \bigcup_{n \in \mathbf{N}_+} R^n$. The *reflexive transitive closure* of R is $R^* := \bigcup_{n \in \mathbf{N}} R^n$. A relation R (on A) is *well-founded* if every non-empty class B ($\subseteq A$) has an R-minimal element, i.e. $\exists a \in B.\ \neg \exists a' \in B.\ a' R\, a$.

5.3 Choice Functions

To be useful in context with Hilbert's ε, the notion of a "choice function" must be generalized: We need a *total* function on the power-set of any universe. Thus, a value must be supplied even for the empty set:

Definition 5.1 ([Generalized] [Function-] Choice Function)
f is a *choice function* [*on* A] if f is a function with [$A \subseteq \mathrm{dom}(f)$ and]
$$f : \mathrm{dom}(f) \to \bigcup(\mathrm{dom}(f)) \text{ and } \forall Y \in \mathrm{dom}(f).\ \big(\ f(Y) \in Y\ \big).$$
f is a *generalized choice function* [*on* A] if f is a function with [$A \subseteq \mathrm{dom}(f)$ and]
$$f : \mathrm{dom}(f) \to \bigcup(\mathrm{dom}(f)) \text{ and } \forall Y \in \mathrm{dom}(f).\ \big(\ f(Y) \in Y \ \vee\ Y = \emptyset\ \big).$$
f is a *function-choice function for a function* F if f is a function with $\mathrm{dom}(F) \subseteq \mathrm{dom}(f)$
$$\text{and } \forall x \in \mathrm{dom}(F).\ \big(\ f(x) \in F(x)\ \big).$$

Corollary 5.2
The empty function $\emptyset$ *is both a choice function and a generalized choice function.*
If $\mathrm{dom}(f) = \{\emptyset\}$, *then* f *is neither a choice function nor a generalized choice function.*
If $\emptyset \notin \mathrm{dom}(f)$, *then* f *is a generalized choice function if and only if* f *is a choice function.*
If $\emptyset \in \mathrm{dom}(f)$, *then* f *is a generalized choice function if and only if*
there is a choice function f' *and an* $x \in \bigcup(\mathrm{dom}(f'))$ *such that* $f = f' \uplus \{(\emptyset, x)\}$.

5.4 Variables, Atoms, Constants, and Substitutions

We assume the following sets of symbols to be disjoint:

$\mathbb{V}$ (free) (rigid) *variables*, which serve as unknowns or
the free variables of [FITTING, 1990; 1996]

$\mathbb{A}$ *(free) atoms*, which serve as parameters and must not be bound

$\mathbb{B}$ *bound atoms*, which may be bound

Σ *constants*, i.e. the function and predicate symbols from the signature

We define:

$$\mathbb{VA} := \mathbb{V} \uplus \mathbb{A}$$

$$\mathbb{VAB} := \mathbb{V} \uplus \mathbb{A} \uplus \mathbb{B}$$

By slight abuse of notation, for $S \in \{\mathbb{V}, \mathbb{A}, \mathbb{B}, \mathbb{VA}, \mathbb{VAB}\}$, we write "$S(\Gamma)$" to denote the set of symbols from S that have free occurrences in Γ.

Let σ be a substitution.

σ is a *substitution on V* if $\mathrm{dom}(\sigma) \subseteq V$.

The following indented statement (as simple as it is) will require some discussion.

> We denote with "$\Gamma\sigma$" the result of replacing each (free) occurrence of a symbol $x \in \mathrm{dom}(\sigma)$ in Γ with $\sigma(x)$; possibly after renaming in Γ some symbols that are bound in Γ, in particular because a capture of their free occurrences in $\sigma(x)$ must be avoided.

Note that such a renaming of symbols that are bound in Γ will hardly be required for the following reason: We will bind only symbols from the set $\mathbb{B}$ of bound atoms. And — unless explicitly stated otherwise — we tacitly assume that all occurrences of bound atoms from $\mathbb{B}$ in a term or formula or in the range of a substitution are *bound occurrences* (i.e. that a bound atom $x^{\mathbb{B}} \in \mathbb{B}$ occurs only in the scope of a binder on $x^{\mathbb{B}}$). Thus, in standard situations, even without renaming, no additional occurrences can become bound (i.e. captured) when applying a substitution.

Only if we want to exclude the binding of a bound atom within the scope of another binding of the same bound atom (e.g. for the sake of readability and in the tradition of HILBERT–BERNAYS), then we may still have to rename some of the bound atoms in Γ. For example, for Γ being the formula $\forall x^{\mathbb{B}}.\ (x^{\mathbb{B}} = y^{\mathbb{V}})$ and σ being the substitution $\{y^{\mathbb{V}} \mapsto \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = x^{\mathbb{B}})\}$, we may want the result of $\Gamma\sigma$ to be something like $\forall z^{\mathbb{B}}.\ (z^{\mathbb{B}} = \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = x^{\mathbb{B}}))$ instead of $\forall x^{\mathbb{B}}.\ (x^{\mathbb{B}} = \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = x^{\mathbb{B}}))$.

Moreover — unless explicitly stated otherwise — in this paper we will use only substitutions on subsets of $\mathbb{V}$. Thus, also the occurrence of "(free)" in the statement indented above is hardly of any relevance here, because we will never bind elements of $\mathbb{V}$.

5.5 Consistent Positive/Negative Variable-Conditions

Variable-conditions are binary relations on free variables and free atoms. They put conditions on the possible instantiation of free variables, and on the dependencies of their valuations. In this paper, for clarity of presentation, a variable-condition is formalized as a pair (P, N) of binary relations, which we will call a "positive/negative variable-condition":

- The first component (P) of such a pair is a binary relation that is meant to express *positive* dependencies. It comes with the intention of transitivity, although it will typically not be closed up to transitivity for reasons of presentation and efficiency.

  The overall idea is that the occurrence of a pair $(x^{\mathbb{VA}}, y^{\mathbb{V}})$ in this positive relation means something like

  "the value of $y^{\mathbb{V}}$ may well depend on $x^{\mathbb{VA}}$"

  or

  "the value of $y^{\mathbb{V}}$ is described in terms of $x^{\mathbb{VA}}$".

- The second component (N), however, is meant to capture *negative* dependencies.

  The overall idea is that the occurrence of a pair $(x^{\mathbb{V}}, y^{\mathbb{A}})$ in this negative relation means something like

  "the value of $x^{\mathbb{V}}$ has to be fixed before the value of $y^{\mathbb{A}}$ can be determined"

  or

  "the value of $x^{\mathbb{V}}$ must not depend on $y^{\mathbb{A}}$"

  or

  "$y^{\mathbb{A}}$ is fresh for $x^{\mathbb{V}}$".

  Relations similar to this negative relation (N) already occurred as the only component of a variable-condition in [Wirth, 1998], and later — with a completely different motivation — as "*freshness* conditions" also in [Gabbay & Pitts, 2002].

Definition 5.3 (Positive/Negative Variable-Condition)
A *positive/negative variable-condition* is a pair (P, N) with

$$P \subseteq \mathbb{VA} \times \mathbb{V}$$

and

$$N \subseteq \mathbb{V} \times \mathbb{A}.$$

A relation exactly like this positive relation (P) was the only component of a variable-condition as defined and used identically throughout [Wirth, 2002; 2004; 2006; 2008; 2012b; 2012c]. Note, however, that, in these publications, we had to admit this single positive relation to be a subset of $\mathbb{VA}\times\mathbb{VA}$ (instead of the restriction to $\mathbb{VA}\times\mathbb{V}$ of Definition 5.3 in this paper), because it had to simulate the negative relation (N) in addition; thereby losing some expressive power as compared to our positive/negative variable-conditions here (cf. Example A.1).

In the following definition, the well-foundedness guarantees that all dependencies can be traced back to independent symbols and that no variable may transitively depend on itself, whereas the irreflexivity makes sure that no contradictious dependencies can occur.

Definition 5.4 (Consistency)
A pair (P, N) is *consistent* if

$$P \text{ is well-founded}$$

and

$$P^+ \circ N \text{ is irreflexive.}$$

Let (P, N) be a positive/negative variable-condition. Let us think of our (binary) relations P and N as edges of a directed graph whose vertices are the symbols for atoms and variables currently in use. Then, $P^+ \circ N$ is irreflexive if and only if there is no cycle in $P \cup N$ that contains exactly one edge from N. Moreover, in practice, a positive/negative variable-condition (P, N) can always be chosen to be finite in both its components. In the case that P is finite, P is well-founded if and only if P is acyclic. Thus we get:

Corollary 5.5
Let (P, N) be a positive/negative variable-condition with $|P| \in \mathbf{N}$.
(P, N) is consistent if and only if
each cycle in the directed graph of $P \uplus N$ contains more than one edge from N.
In case of $|N| \in \mathbf{N}$, the right-hand side of this equivalence can be effectively tested with an asymptotic time complexity of $|P| + |N|$.

Note that, in the finite case, the test of Corollary 5.5 seems to be both the most efficient and the most human-oriented way to represent the question of consistency of positive/negative variable-conditions.

5.6 Further Discussion of our Formalization of Variable-Conditions

Note that the two relations P and N of a positive/negative variable-condition (P, N) are always disjoint because their ranges must be disjoint according to Definition 5.3. Thus, from a technical point of view, we could merge P and N into a single relation, but we prefer to have two relations for the two different functions (the positive and the negative one) of the variable-conditions in this paper, instead of the one relation for one function of [WIRTH, 2002; 2004; 2006; 2008; 2012b; 2012c], which realized the negative function only with a significant loss of relevant information. Our main reason to have two different relations is that it makes sense to relax the restriction on the negative relation in future publications to

$$N \;\subseteq\; \mathbb{V} \times \mathbb{VA},$$

cf. e.g. Example A.1.

Moreover, in Definition 5.3, we have excluded the possibility that two atoms $a^{\mathbb{A}}$, $b^{\mathbb{A}} \in \mathbb{A}$ may be related to each other in any of the two components of a positive/negative variable-condition (P, N):

- $y^{\mathbb{VA}}\ P\ a^{\mathbb{A}}$ is excluded for intentional reasons: An atom $a^{\mathbb{A}}$ cannot depend on any other symbol $y^{\mathbb{VA}}$. In this sense an atom is indeed atomic and can be seen as a black box.

- $b^{\mathbb{A}}\ N\ a^{\mathbb{A}}$, however, is excluded for technical reasons only.
  Two distinct atoms $a^{\mathbb{A}}$, $b^{\mathbb{A}}$ in nominal terms [URBAN &AL., 2004] are indeed always fresh for each other: $a^{\mathbb{A}} \# b^{\mathbb{A}}$. In our notation, this would read: $b^{\mathbb{A}}\ N\ a^{\mathbb{A}}$.
  The reason why we did not include $(\mathbb{A}{\times}\mathbb{A}) \setminus {}_{\mathbb{A}}{\upharpoonleft}\mathrm{id}$ into the negative component N is simply that we want to be close to the data structures of a both efficient and human-oriented graph implementation.
  Furthermore, consistency of a positive/negative variable-condition (P, N) is equivalent to consistency of $\big(\ P,\ \ N \uplus ((\mathbb{A}{\times}\mathbb{A}) \setminus {}_{\mathbb{A}}{\upharpoonleft}\mathrm{id})\ \big)$.
  Indeed, if we added $(\mathbb{A}{\times}\mathbb{A}) \setminus {}_{\mathbb{A}}{\upharpoonleft}\mathrm{id}$ to N, the result of the acyclicity test of Corollary 5.5 would not be changed: If there were a cycle with a single edge from $(\mathbb{A}{\times}\mathbb{A}) \setminus {}_{\mathbb{A}}{\upharpoonleft}\mathrm{id}$, then its previous edge would have to be one of the original edges of N; and so this cycle would have more than one edge from $N \uplus ((\mathbb{A}{\times}\mathbb{A}) \setminus {}_{\mathbb{A}}{\upharpoonleft}\mathrm{id})$, and thus would not count as a counterexample to consistency.

Furthermore, we could remove the set $\mathbb{B}$ of bound atoms from our sets of symbols and consider its elements to be elements of the set $\mathbb{A}$ of atoms. Besides some additional care on free occurrences of atoms in § 5.4, an additional price we would have to pay for this removal is that we would have to include $\mathbb{V}{\times}\mathbb{B}$ as a subset into the second component (N) of all our positive/negative variable-conditions (P, N). The reason for this inclusion is that we must guarantee that it is not possible that a bound atom $b^{\mathbb{B}}$ can be read by some variable $x^{\mathbb{V}}$, in particular after an elimination of binders. Then, by this inclusion, in case of $b^{\mathbb{B}}\ P^{+}\ x^{\mathbb{V}}$, we would get a cycle $b^{\mathbb{B}}\ P^{+}\ x^{\mathbb{V}}\ N\ b^{\mathbb{B}}$ with only one edge from N. Although, in practical contexts, we can always get along with a finite subset of $\mathbb{V}{\times}\mathbb{B}$, the essential pairs of this subset would still be quite many and would be most confusing already in small examples. For instance, for the higher-order choice-condition of Example 4.10, almost four dozen pairs from $\mathbb{V}{\times}\mathbb{B}$ are technically required, compared to only a good dozen pairs that are actually relevant to the problem (cf. Example 5.14(a)).

5.7 Extensions, σ-Updates, and (P, N)-Substitutions

Within a progressing reasoning process, positive/negative variable-conditions may be subject to only one kind of transformation, which we simply call an "extension".

Definition 5.6 ([Weak] Extension)
(P', N') is an [*weak*] *extension of* (P, N) if
(P', N') is a positive/negative variable-condition, $P \subseteq P'$ [or at least $P \subseteq (P')^{+}$], and $N \subseteq N'$.

As an immediate corollary of Definitions 5.6 and 5.4 we get:

Corollary 5.7
If (P', N') *is a consistent positive/negative variable-condition and an [weak] extension of* (P, N), *then* (P, N) *is a consistent positive/negative variable-condition as well.*

A σ-update is a special form of an extension:

Definition 5.8 (σ-Update, Dependence Relation)
Let (P, N) be a positive/negative variable-condition and σ be a substitution on $\mathbb{V}$.
The *dependence relation of* σ is

$$D \quad := \quad \{ \ (z^{\mathbb{VA}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\sigma) \wedge z^{\mathbb{VA}} \in \mathbb{VA}(\sigma(x^{\mathbb{V}})) \ \}.$$

The σ-*update of* (P, N) is $(P \cup D, N)$. [5]

Definition 5.9 ((P, N)-Substitution)
Let (P, N) be a positive/negative variable-condition. σ is a (P, N)-*substitution* if
σ is a substitution on $\mathbb{V}$ and the σ-update of (P, N) is consistent.

Syntactically, $(x^{\mathbb{V}}, a^{\mathbb{A}}) \in N$ is to express that a (P, N)-substitution σ must not replace $x^{\mathbb{V}}$ with a term in which $a^{\mathbb{A}}$ could ever occur; i.e. that $a^{\mathbb{A}}$ is fresh for $x^{\mathbb{V}}$: $a^{\mathbb{A}} \ \# \ x^{\mathbb{V}}$. This is indeed guaranteed if any σ-update (P', N') of (P, N) is again required to be consistent, and so on. We can see this as follows: For $z^{\mathbb{V}} \in \mathbb{V}(\sigma(x^{\mathbb{V}}))$, we get

$$z^{\mathbb{V}} \ \ P' \ \ x^{\mathbb{V}} \ \ N' \ \ a^{\mathbb{A}}.$$

If we now try to apply a second substitution σ' with $a^{\mathbb{A}} \in \mathbb{A}(\sigma'(z^{\mathbb{V}}))$ (so that $a^{\mathbb{A}}$ occurs in $(x^{\mathbb{V}}\sigma)\sigma'$, contrary to what we initially expressed as our freshness intention), then σ' is not a (P', N')-substitution because, for the σ'-update (P'', N'') of (P', N'), we have

$$a^{\mathbb{A}} \ \ P'' \ \ z^{\mathbb{V}} \ \ P'' \ \ x^{\mathbb{V}} \ \ N'' \ \ a^{\mathbb{A}};$$

so $(P'')^{+} \circ N''$ is not irreflexive. All in all, the positive/negative variable-condition

- (P', N') blocks any instantiation of $(x^{\mathbb{V}}\sigma)$ resulting in a term containing $a^{\mathbb{A}}$, just as
- (P, N) blocked $x^{\mathbb{V}}$ before the application of σ.

5.8 Semantic Presuppositions

Instead of defining truth from scratch, we require some abstract properties typically holding in two-valued model semantics.

Truth is given relative to a Σ-structure $\mathcal{S}$, which provides some *non-empty set* as the universe (or "carrier", "domain") (for each type). Moreover, we assume that every Σ-structure $\mathcal{S}$ is not only defined on the predicate and function symbols of the signature Σ, but is defined also on the symbols $\forall$ and $\exists$ such that $\mathcal{S}(\exists)$ serves as a function-choice function for the universe function $\mathcal{S}(\forall)$ in the sense that, for each type α of Σ, the universe for the type α is denoted by $\mathcal{S}(\forall)_\alpha$ and

$$\mathcal{S}(\exists)_\alpha \;\in\; \mathcal{S}(\forall)_\alpha\,.$$

For $\mathrm{X} \subseteq \mathbb{VAB}$, we denote the set of total $\mathcal{S}$-valuations of X (i.e. functions mapping atoms and variables in X to objects of the universe of $\mathcal{S}$) with

$$\mathrm{X} \to \mathcal{S}\,,$$

and the set of (possibly) partial $\mathcal{S}$-valuations of X with

$$\mathrm{X} \rightsquigarrow \mathcal{S}\,.$$

Here we expect types to be respected in the sense that, for each $\delta : \mathrm{X} \to \mathcal{S}$ and for each $x^{\mathbb{VAB}} \in \mathrm{X}$ with $x^{\mathbb{VAB}} : \alpha$ (i.e. $x^{\mathbb{VAB}}$ has type α), we have $\delta(x^{\mathbb{VAB}}) \in \mathcal{S}(\forall)_\alpha$.

For $\delta : \mathrm{X} \to \mathcal{S}$, we denote with "$\mathcal{S} \uplus \delta$" the extension of $\mathcal{S}$ to X. More precisely, we assume some evaluation function "eval" such that $\mathrm{eval}(\mathcal{S}\uplus\delta)$ maps every term whose free-occurring symbols are from $\Sigma\uplus\mathrm{X}$ into the universe of $\mathcal{S}$ (respecting types). Moreover, $\mathrm{eval}(\mathcal{S}\uplus\delta)$ maps every formula B whose free-occurring symbols are from $\Sigma\uplus\mathrm{X}$ to TRUE or FALSE, such that:

$$B \text{ is true in } \mathcal{S}\uplus\delta \quad \text{iff} \quad \mathrm{eval}(\mathcal{S}\uplus\delta)(B) = \mathsf{TRUE}.$$

We leave open what our formulas and what our Σ-structures exactly are. The latter can range from first-order Σ-structures to higher-order modal Σ-models; provided that the following three properties — which (explicitly or implicitly) belong to the standard of most logic textbooks — hold for every term or formula B, every Σ-structure $\mathcal{S}$, and every $\mathcal{S}$-valuation $\delta : \mathbb{VAB} \rightsquigarrow \mathcal{S}$.

EXPLICITNESS LEMMA
The value of the evaluation of B depends only on the valuation of those variables and atoms that actually have free occurrences in B; i.e., for $\mathrm{X} := \mathbb{VAB}(B)$, if $\mathrm{X} \subseteq \mathrm{dom}(\delta)$, then:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(B) \quad = \quad \mathrm{eval}(\mathcal{S} \;\uplus\; {}_{\mathrm{X}}{\upharpoonright}\delta)(B).$$

SUBSTITUTION [VALUE] LEMMA
Let σ be a substitution on $\mathbb{VAB}$. If $\mathbb{VAB}(B\sigma) \subseteq \mathrm{dom}(\delta)$, then:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) \quad = \quad \mathrm{eval}\Big(\mathcal{S} \;\uplus\; \big((\, \sigma \;\uplus\; {}_{\mathbb{VAB}\setminus\mathrm{dom}(\sigma)}{\upharpoonright}\mathrm{id}\,) \circ \mathrm{eval}(\mathcal{S} \uplus \delta) \,\big) \Big)\Big(B \Big).$$

VALUATION LEMMA
The evaluation function treats application terms from $\mathbb{VAB}$ straightforwardly in the sense that for every $v_0^{\mathbb{VAB}}, \ldots, v_{l-1}^{\mathbb{VAB}}, y^{\mathbb{VAB}} \in \mathrm{dom}(\delta)$ with $v_0^{\mathbb{VAB}} : \alpha_0, \ \ldots, \ v_{l-1}^{\mathbb{VAB}} : \alpha_{l-1}$, $y^{\mathbb{VAB}} : \alpha_0 \to \cdots \to \alpha_{l-1} \to \alpha_l$ for some types $\alpha_0, \ldots, \alpha_{l-1}, \alpha_l$, we have:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(y^{\mathbb{VAB}}(v_0^{\mathbb{VAB}}) \cdots (v_{l-1}^{\mathbb{VAB}})) \ = \ \delta(y^{\mathbb{VAB}})(\delta(v_0^{\mathbb{VAB}})) \cdots (\delta(v_{l-1}^{\mathbb{VAB}})).$$

Note that we need the case of the VALUATION LEMMA where $y^{\mathbb{VAB}}$ is a higher-order symbol (i.e. the case of $l \succ 0$) only when higher-order choice-conditions are required. Besides this, the basic language of the general reasoning framework, however, may well be first-order and does not have to include function application.

Moreover, in the few cases where we explicitly refer to quantifiers, implication, or negation, such as in our inference rules of §§ 3.2, 3.3, and 3.4. or in our version of axiom (ε_0) (cf. Definition 4.11), and in the lemmas and theorems that refer to these (namely Lemmas 5.19 and 5.24, Theorem 5.26(6), and Theorem 5.27),[6] we have to know that the quantifiers, the implication, and the negation show the standard semantic behavior of classical logic:

∀-LEMMA
Assume $\mathbb{VAB}(\forall x^{\mathbb{B}}.\ A) \subseteq \mathrm{dom}(\delta)$. The following two are equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(\forall x^{\mathbb{B}}.\ A) = \mathsf{TRUE}$
- $\mathrm{eval}(\mathcal{S} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}\!\upharpoonright\!\delta \uplus \chi)(A) = \mathsf{TRUE}$ for every $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$

∃-LEMMA
Assume $\mathbb{VAB}(\exists x^{\mathbb{B}}.\ A) \subseteq \mathrm{dom}(\delta)$. The following two are equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(\exists x^{\mathbb{B}}.\ A) = \mathsf{TRUE}$,
- $\mathrm{eval}(\mathcal{S} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}\!\upharpoonright\!\delta \uplus \chi)(A) = \mathsf{TRUE}$ for some $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$

⇒-LEMMA
Assume $\mathbb{VAB}(A \Rightarrow B) \subseteq \mathrm{dom}(\delta)$. The following two are equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(A \Rightarrow B) = \mathsf{TRUE}$
- $\mathrm{eval}(\mathcal{S} \uplus \delta)(A) = \mathsf{FALSE}$ or $\mathrm{eval}(\mathcal{S} \uplus \delta)(B) = \mathsf{TRUE}$

¬-LEMMA
Assume $\mathbb{VAB}(A) \subseteq \mathrm{dom}(\delta)$. The following two are equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(A) = \mathsf{TRUE}$
- $\mathrm{eval}(\mathcal{S} \uplus \delta)(\neg A) = \mathsf{FALSE}$

5.9 Semantic Relations and $\mathcal{S}$-Raising-Valuations

We now come to some technical definitions required for our semantic (model-theoretic) counterparts of our syntactic (P, N)-substitutions.

Let $\mathcal{S}$ be a Σ-structure. An *$\mathcal{S}$-raising-valuation* π plays the rôle of a *raising function*, a dual of a SKOLEM function as defined in [MILLER, 1992]. This means that π does not simply map each variable directly to an object of $\mathcal{S}$ (of the same type), but may additionally read the values of some atoms under an $\mathcal{S}$-valuation $\tau : \mathbb{A} \to \mathcal{S}$. More precisely, we assume that π takes some restriction of τ as a second argument, say $\tau' : \mathbb{A} \rightsquigarrow \mathcal{S}$ with $\tau' \subseteq \tau$. In short:

$$\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}.$$

Moreover, for each variable $x^{\mathbb{V}}$, we require that the set $\mathrm{dom}(\tau')$ of atoms read by $\pi(x^{\mathbb{V}})$ is identical for all τ. This identical set will be denoted with $S_\pi\langle\!\{x^{\mathbb{V}}\}\!\rangle$ below. Technically, we require that there is some "semantic relation" $S_\pi \subseteq \mathbb{A}\times\mathbb{V}$ such that for all $x^{\mathbb{V}} \in \mathbb{V}$:

$$\pi(x^{\mathbb{V}}) \ : \ (S_\pi\langle\!\{x^{\mathbb{V}}\}\!\rangle \to \mathcal{S}) \to \mathcal{S}.$$

This means that $\pi(x^{\mathbb{V}})$ can read the τ-value of $y^{\mathbb{A}}$ if and only if $(y^{\mathbb{A}}, x^{\mathbb{V}}) \in S_\pi$. Note that, for each $\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$, at most one such semantic relation exists, namely the one of the following definition.

Definition 5.10 (Semantic Relation (S_π))
The *semantic relation for* π is

$$S_\pi \ := \ \{ \ (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathbb{V} \wedge y^{\mathbb{A}} \in \mathrm{dom}(\textstyle\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}})))) \ \}.$$

Definition 5.11 ($\mathcal{S}$-Raising-Valuation)
Let $\mathcal{S}$ be a Σ-structure. π is an *$\mathcal{S}$-raising-valuation* if

$$\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$$

and, for all $x^{\mathbb{V}} \in \mathrm{dom}(\pi)$:

$$\pi(x^{\mathbb{V}}) : (S_\pi\langle\!\{x^{\mathbb{V}}\}\!\rangle \to \mathcal{S}) \to \mathcal{S}.$$

Finally, we need the technical means to turn an $\mathcal{S}$-raising-valuation π together with an $\mathcal{S}$-valuation τ of the atoms into an $\mathcal{S}$-valuation $\mathsf{e}(\pi)(\tau)$ of the variables:

Definition 5.12 (e)
We define the function $\mathsf{e} : \ (\mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \ \to \ (\mathbb{A} \to \mathcal{S}) \ \to \ \mathbb{V} \ \rightsquigarrow \ \mathcal{S}$
for $\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}, \quad \tau : \mathbb{A} \to \mathcal{S}, \ x^{\mathbb{V}} \in \mathbb{V}$
by $\mathsf{e}(\pi)(\tau)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})({}_{S_\pi\langle\!\{x^{\mathbb{V}}\}\!\rangle}\!\upharpoonleft\tau).$

The "e" stands for "<u>e</u>valuation" and replaces an "ϵ" used in previous publications, which was too easily confused with HILBERT's ε.

5.10 Choice-Conditions and Compatibility

In the following definition, we define choice-conditions as syntactic objects. They influence our semantics by a compatibility requirement, which will be described in Definition 5.15.

Definition 5.13 (Choice-Condition, Return Type)
C is a (P,N)-*choice-condition* if

- (P,N) is a consistent positive/negative variable-condition and
- C is a partial function from $\mathbb{V}$ into the set of higher-order ε-terms

such that, for every $y^{\mathbb{V}} \in \mathrm{dom}(C)$, the following items hold for some types $\alpha_0,\ldots,\alpha_l$:

1. The value $C(y^{\mathbb{V}})$ is of the form
$$\lambda v_0^{\mathbb{B}}.\ \ldots\ \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$$
for some formula B and for some mutually distinct bound atoms $v_0^{\mathbb{B}},\ldots,v_l^{\mathbb{B}} \in \mathbb{B}$ with $\mathbb{B}(B) \subseteq \{v_0^{\mathbb{B}},\ldots,v_l^{\mathbb{B}}\}$ and $v_0^{\mathbb{B}} : \alpha_0,\ \ldots,\ v_l^{\mathbb{B}} : \alpha_l$.
2. $y^{\mathbb{V}} : \alpha_0 \rightarrow \cdots \rightarrow \alpha_{l-1} \rightarrow \alpha_l$.
3. $z^{\mathbb{VA}}\ P^+\ y^{\mathbb{V}}$ for all $z^{\mathbb{VA}} \in \mathbb{VA}(C(y^{\mathbb{V}}))$.

In the situation described, α_l is *the return type of* $C(y^{\mathbb{V}})$.
β is *a return type of* C if there is a $z^{\mathbb{V}} \in \mathrm{dom}(C)$ such that β is the return type of $C(z^{\mathbb{V}})$.

Example 5.14 (Choice-Condition) *(continuing Example 4.10)*

(a) If (P,N) is a consistent positive/negative variable-condition that satisfies
$$z_a^{\mathbb{V}}\ P\ y_a^{\mathbb{V}}\ P\ z_b^{\mathbb{V}}\ P\ x_a^{\mathbb{V}}\ P\ z_c^{\mathbb{V}}\ P\ y_b^{\mathbb{V}}\ P\ z_d^{\mathbb{V}}\ P\ w_a^{\mathbb{V}}\ P\ z_e^{\mathbb{V}}\ P\ y_c^{\mathbb{V}}\ P\ z_f^{\mathbb{V}}\ P\ x_b^{\mathbb{V}}\ P\ z_g^{\mathbb{V}}\ P\ y_d^{\mathbb{V}}\ P\ z_h^{\mathbb{V}},$$
then the C of Example 4.10 is a (P,N)-choice-condition, indeed.

(b) If some clever person tried to do the entire quantifier elimination of Example 4.10 by
$$\begin{array}{rcl} C'(z_h^{\mathbb{V}}) & := & \varepsilon z_h^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{B}}) \\ C'(y_d^{\mathbb{V}}) & := & \varepsilon y_d^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{B}}, z_h^{\mathbb{V}}) \\ C'(x_b^{\mathbb{V}}) & := & \varepsilon x_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}}) \\ C'(w_a^{\mathbb{V}}) & := & \varepsilon w_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}}) \end{array}$$
then he would — among other constraints — have to satisfy $z_h^{\mathbb{V}}\ P^+\ y_d^{\mathbb{V}}\ P^+\ z_h^{\mathbb{V}}$, because of item 3 of Definition 5.13 and the values of C' at $y_d^{\mathbb{V}}$ and $z_h^{\mathbb{V}}$. This would make P non-well-founded. Thus, this C' cannot be a (P,N)-choice-condition for any (P,N), because the consistency of (P,N) is required in Definition 5.13. Note that the choices required by C' for $y_d^{\mathbb{V}}$ and $z_h^{\mathbb{V}}$ are in an unsolvable conflict, indeed.

(c) For a more elementary example, take
$$C''(x^{\mathbb{V}})\ :=\ \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = y^{\mathbb{V}}) \qquad\qquad C''(y^{\mathbb{V}})\ :=\ \varepsilon y^{\mathbb{B}}.\ (x^{\mathbb{V}} \neq y^{\mathbb{B}})$$
Then $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$ form a vicious circle of conflicting choices for which no valuation can be found that is compatible with C''. But C'' is no choice-condition at all because there is no *consistent* positive/negative variable-condition (P,N) such that C'' is a (P,N)-choice-condition.

Definition 5.15 (Compatibility)
Let C be a (P,N)-choice-condition. Let $\mathcal{S}$ be a Σ-structure.
π is *$\mathcal{S}$-compatible with* $(C,(P,N))$ if the following items hold:

1. π is an $\mathcal{S}$-raising-valuation (cf. Definition 5.11) and $(P \cup S_\pi, N)$ is consistent (cf. Definitions 5.4 and 5.10).

2. For every $y^{\mathbb{V}} \in \mathrm{dom}(C)$ with $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots\ \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ for some formula B, and for every $\tau : \mathbb{A} \to \mathcal{S}$, and for every $\chi : \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\} \to \mathcal{S}$:

   If B is true in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$,
   then $B\{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$ is true in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$ as well.

   (For e, see Definition 5.12.)

To understand item 2 of Definition 5.15, let us consider a (P,N)-choice-condition

$$C := \{(y^{\mathbb{V}},\ \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B)\},$$

which restricts the value of $y^{\mathbb{V}}$ according to the higher-order ε-term $\lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$. Then, roughly speaking, this choice-condition C requires that whenever there is a χ-value of $v_l^{\mathbb{B}}$ such that B is true in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$, the π-value of $y^{\mathbb{V}}$ is chosen in such a way that $B\{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$ becomes true in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$ as well. Note that the free variables of the formula $B\{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$ cannot read the χ-value of any of the bound atoms $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}$, because free variables can never depend on the value of any bound atoms.

Moreover, item 2 of Definition 5.15 is closely related to HILBERT's ε-operator in the sense that — roughly speaking — $y^{\mathbb{V}}$ must be given one of the values admissible for

$$\lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B.$$

As the choice for $y^{\mathbb{V}}$ depends on the symbols that have a free occurrence in that higher-order ε-term, we included these dependencies into the positive relation P of the consistent positive/negative variable-condition (P,N) in item 3 of Definition 5.13. By this inclusion, conflicts like the one shown in Example 5.14(c) are obviated.

Let (P,N) be a consistent positive/negative variable-condition. Then the empty function $\emptyset$ is a (P,N)-choice-condition. Moreover, each $\pi : \mathbb{V} \to \{\emptyset\} \to \mathcal{S}$ is $\mathcal{S}$-compatible with $(\emptyset,(P,N))$ because of $S_\pi = \emptyset$. Furthermore, assuming an adequate principle of choice on the meta level, a compatible π always exists according to the following lemma. This existence relies on item 3 of Definition 5.13 and on the well-foundedness of P.

Lemma 5.16 *Let C be a (P,N)-choice-condition. Let $\mathcal{S}$ be a Σ-structure. Assume that, for every return type α of C, there is a generalized choice function on the power-set of $\mathcal{S}(\forall)_\alpha$.*
[Let ρ be an $\mathcal{S}$-raising-valuation with $S_\rho \subseteq P^+$.]
Then there is an $\mathcal{S}$-raising-valuation π such that the following hold:

- *π is $\mathcal{S}$-compatible with $(C,(P,N))$.*
- *$S_\pi = {}_{\mathbb{A}}{\upharpoonright}(P^+)$.*

[• ${}_{\mathbb{V}\setminus\mathrm{dom}(C)}{\upharpoonright}\pi = {}_{\mathbb{V}\setminus\mathrm{dom}(C)}{\upharpoonright}\rho$.]

5.11 $(C,(P,N))$-Validity

Definition 5.17 ($(C,(P,N))$-Validity, K)

Let C be a (P,N)-choice-condition. Let G be a set of sequents.
Let $\mathcal{S}$ be a Σ-structure. Let $\delta : \mathbb{VA} \rightsquigarrow \mathcal{S}$ be an $\mathcal{S}$-valuation.

G is $(C,(P,N))$-*valid in* $\mathcal{S}$ if
G is $(\pi,\mathcal{S})$-valid for some π that is $\mathcal{S}$-compatible with $(C,(P,N))$.

G is $(\pi,\mathcal{S})$-*valid* if G is true in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau$ for every $\tau : \mathbb{A} \to \mathcal{S}$.

G is *true in* $\mathcal{S}\uplus\delta$ if Γ is true in $\mathcal{S}\uplus\delta$ for all $\Gamma \in G$.
A sequent Γ is *true in* $\mathcal{S}\uplus\delta$ if there is some formula listed in Γ that is true in $\mathcal{S}\uplus\delta$.

Validity in a class of Σ-structures is understood as validity in each of the Σ-structures of that class. If we omit the reference to a special Σ-structure we mean validity in some fixed class K of Σ-structures, such as the class of all Σ-structures or the class of HERBRAND Σ-structures.

Example 5.18 ($(\emptyset,(P,N))$-Validity)

For $x^{\mathbb{V}} \in \mathbb{V}$, $y^{\mathbb{A}} \in \mathbb{A}$, the single-formula sequent $x^{\mathbb{V}} = y^{\mathbb{A}}$ is $(\emptyset,(\emptyset,\emptyset))$-valid in any Σ-structure $\mathcal{S}$ because we can choose $S_\pi := \mathbb{A}\times\mathbb{V}$ and $\pi(x^{\mathbb{V}})(\tau) := \tau(y^{\mathbb{A}})$ for $\tau : \mathbb{A} \to \mathcal{S}$, resulting in

$$\mathsf{e}(\pi)(\tau)(x^{\mathbb{V}}) = \pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau) = \pi(x^{\mathbb{V}})({}_{\mathbb{A}}{\upharpoonright}\tau) = \pi(x^{\mathbb{V}})(\tau) = \tau(y^{\mathbb{A}}).$$

This means that $(\emptyset,(\emptyset,\emptyset))$-validity of $x^{\mathbb{V}} = y^{\mathbb{A}}$ is equivalent to validity of

$$\forall y_0^{\mathbb{B}}.\ \exists x_0^{\mathbb{B}}.\ (x_0^{\mathbb{B}} = y_0^{\mathbb{B}}). \tag{1}$$

Moreover, note that $\mathsf{e}(\pi)(\tau)$ has access to the τ-value of $y^{\mathbb{A}}$ just as a raising function $x_1^{\mathbb{B}}$ for $x_0^{\mathbb{B}}$ has access to $y_0^{\mathbb{B}}$ in the raised (i.e. dually SKOLEMized) form $\exists x_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_1^{\mathbb{B}}(y_0^{\mathbb{B}}) = y_0^{\mathbb{B}})$ of (1).

Contrary to this, for $P := \emptyset$ and $N := \mathbb{V}\times\mathbb{A}$, the same single-formula sequent $x^{\mathbb{V}} = y^{\mathbb{A}}$ is not $(\emptyset,(P,N))$-valid in general, because then the required consistency of $(P \cup S_\pi, N)$ implies $S_\pi = \emptyset$; otherwise $P \cup S_\pi \cup N$ has a cycle of length 2 with exactly one edge from N. Thus, the value of $x^{\mathbb{V}}$ cannot depend on $\tau(y^{\mathbb{A}})$ anymore:

$$\pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau) = \pi(x^{\mathbb{V}})({}_{\emptyset}{\upharpoonright}\tau) = \pi(x^{\mathbb{V}})(\emptyset).$$

This means that $(\emptyset,(\emptyset,\mathbb{V}\times\mathbb{A}))$-validity of $x^{\mathbb{V}} = y^{\mathbb{A}}$ is equivalent to validity of

$$\exists x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_0^{\mathbb{B}} = y_0^{\mathbb{B}}). \tag{2}$$

Moreover, note that $\mathsf{e}(\pi)(\tau)$ has no access to the τ-value of $y^{\mathbb{A}}$ just as a raising function $x_1^{\mathbb{B}}$ for $x_0^{\mathbb{B}}$ has no access to $y_0^{\mathbb{B}}$ in the raised form $\exists x_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_1^{\mathbb{B}}() = y_0^{\mathbb{B}})$ of (2).

For a more general example let $G = \{\ A_{i,0} \ldots A_{i,n_i-1} \mid i \in I\ \}$, where, for $i \in I$ and $j \prec n_i$, the $A_{i,j}$ are formulas with variables from $\boldsymbol{v}$ and atoms from $\boldsymbol{a}$.
Then $(\emptyset,(\emptyset,\mathbb{V}\times\mathbb{A}))$-validity of G means validity of $\exists \boldsymbol{v}.\ \forall \boldsymbol{a}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$
whereas $(\emptyset,(\emptyset,\emptyset))$-validity of G means validity of $\forall \boldsymbol{a}.\ \exists \boldsymbol{v}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$

Ignoring the question of γ-multiplicity, also any other sequence of universal and existential quantifiers can be represented by a consistent positive/negative variable-condition (P,N), simply by starting from the consistent positive/negative variable-condition $(\emptyset, \emptyset)$ and applying the γ- and δ-rules from §§ 3.2, 3.3, and 3.4. A reverse translation of a positive/negative variable-condition (P, N) into a sequence of quantifiers, however, may require a strengthening of dependencies, in the sense that a subsequent backward translation would result in a *more restrictive* consistent positive/negative variable-condition (P', N') with $P \subseteq P'$ and $N \subseteq N'$. This means that our framework can express quantificational dependencies more fine-grained than standard quantifiers; cf. Example A.1.

For a further example on validity, see Example A.1, which treats HENKIN quantification and IF-logic quantifiers and which we have put into the appendix because of its length.

As already noted in § 4.12, the single-formula sequent $Q_C(y^{\mathrm{v}})$ of Definition 4.11 is a formulation of axiom (ε_0) of § 4.6 in our framework.

Lemma 5.19 ($(C, (P, N))$-Validity of $Q_C(y^{\mathrm{v}})$)
Let C be a (P, N)-choice-condition. Let $y^{\mathrm{v}} \in \mathrm{dom}(C)$. Let $\mathcal{S}$ be a Σ-structure.

1. *$Q_C(y^{\mathrm{v}})$ is $(\pi, \mathcal{S})$-valid for every π that is $\mathcal{S}$-compatible with $(C, (P, N))$.*
2. *$Q_C(y^{\mathrm{v}})$ is $(C, (P, N))$-valid in $\mathcal{S}$; provided that for every return type α of C (cf. Definition 5.13), there is a generalized choice function on the power-set of $\mathcal{S}(\forall)_\alpha$.*

5.12 Extended Extensions

Just like the positive/negative variable-condition (P, N), the (P, N)-choice-condition C may be extended during proofs. This kind of extension together with a simple soundness condition plays an important rôle in inference:

Definition 5.20 (Extended Extension)
$(C', (P', N'))$ is an *extended extension of* $(C, (P, N))$ if

- C is a (P, N)-choice-condition (cf. Definition 5.13),
- C' is a (P', N')-choice-condition,
- (P', N') is an extension of (P, N) (cf. Definition 5.6), and
- $C \subseteq C'$.

Lemma 5.21 (Extended Extension)
Let $(C', (P', N'))$ be an extended extension of $(C, (P, N))$.
If π is $\mathcal{S}$-compatible with $(C', (P', N'))$, then π is $\mathcal{S}$-compatible with $(C, (P, N))$ as well.

5.13 Extended σ-Updates

After global application of a (P,N)-substitution σ, we now have to update both (P,N) and C:

Definition 5.22 (Extended σ-Update)
Let C be a (P,N)-choice-condition and let σ be a substitution on $\mathbb{V}$.
The *extended σ-update* $(C',(P',N'))$ *of* $(C,(P,N))$ is given as follows:

$$C' \;:=\; \{\ (x^{\mathrm{V}}, B\sigma) \mid (x^{\mathrm{V}}, B) \in C \wedge x^{\mathrm{V}} \notin \mathrm{dom}(\sigma)\ \},$$

$$(P',N') \text{ is the } \sigma\text{-update of } (P,N) \text{ (cf. Definition 5.8).}$$

Note that a σ-update (cf. Definition 5.8) is an extension (cf. Definition 5.6), whereas an extended σ-update is not an extended extension in general, because entries of the choice-condition may be modified or even deleted, such that we may have $C \nsubseteq C'$. The remaining properties of an extended extension, however, are satisfied:

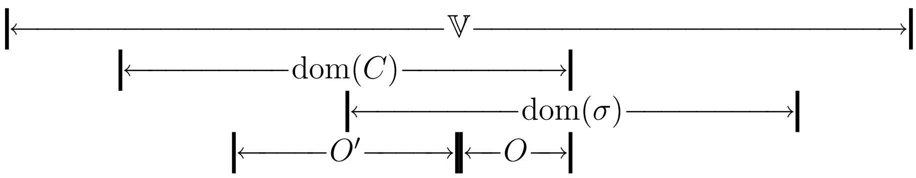
Lemma 5.23 (Extended σ-Update) *Let C be a (P,N)-choice-condition. Let σ be a (P,N)-substitution. Let $(C',(P',N'))$ be the extended σ-update of $(C,(P,N))$. Then C' is a (P',N')-choice-condition.*

5.14 The Main Lemma

Lemma 5.24 ((P,N)-Substitutions and $(C,(P,N))$-Validity)
Let (P,N) be a positive/negative variable-condition.
Let C be a (P,N)-choice-condition. Let σ be a (P,N)-substitution.
Let $(C',(P',N'))$ be the extended σ-update of $(C,(P,N))$. Let $\mathcal{S}$ be a Σ-structure.
Let π' be an $\mathcal{S}$-raising-valuation that is $\mathcal{S}$-compatible with $(C',(P',N'))$.
Let O and O' be two disjoint sets with $O \subseteq \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$ and $O' \subseteq \mathrm{dom}(C) \setminus O$.
Moreover, assume that σ respects C on O in the given semantic context in the sense that $(\langle O \rangle Q_C)\sigma$ is $(\pi',\mathcal{S})$-valid (cf. Definition 4.11 for Q_C).
Furthermore, regarding the set O' (where σ may disrespect C), assume the following items to hold:

- *O' covers the variables in $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$ besides O in the sense of*

$$\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) \;\subseteq\; O' \uplus O.$$



- *O' satisfies the closure condition $\langle O' \rangle P^+ \cap \mathrm{dom}(C) \subseteq O'$.*
- *For every $y^{\mathbb{V}} \in O'$, for α being the return type of $C(y^{\mathbb{V}})$ (cf. Definition 5.13), there is a generalized choice function on the power-set of $\mathcal{S}(\forall)_\alpha$.*

Then there is an $\mathcal{S}$-raising-valuation π that is $\mathcal{S}$-compatible with $(C,(P,N))$ and that satisfies the following:

1. *For every term or formula B with $O' \cap \mathbb{V}(B) = \emptyset$ and possibly with some unbound occurrences of bound atoms from a set $W \subseteq \mathbb{B}$, and for every $\tau : \mathbb{A} \to \mathcal{S}$ and every $\chi : W \to \mathcal{S}$:* $\quad \mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi')(\tau) \uplus \tau \uplus \chi)(B\sigma) \;=\; \mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi)(B)$.
2. *For every set of sequents G with $O' \cap \mathbb{V}(G) = \emptyset$ we have:*

$$G\sigma \text{ is } (\pi',\mathcal{S})\text{-valid iff } G \text{ is } (\pi,\mathcal{S})\text{-valid.}$$

In Lemma 5.24, we illustrate the subclass relation with a Lambert diagram [Lambert, 1764, Dianoiologie, §§ 173–194], similar to a Venn diagram. In general, a Lambert diagram expresses nothing but the following: If — in vertical projection — each point of the overlap of the lines for classes $A_1,\ldots,A_m$ is covered by a line of the classes $B_1,\ldots,B_n$ then $A_1 \cap \cdots \cap A_m \subseteq B_1 \cup \cdots \cup B_n$; moreover, the points not covered by a line for A are considered to be covered by a line for the complement $\overline{A}$.

Note that Lemma 5.24 gets a lot simpler when we require the entire (P,N)-substitution σ to respect the (P,N)-choice-condition C by setting $O := \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$ and $O' := \emptyset$; in particular all requirements on O' are trivially satisfied then. Moreover, note that the (still quite long) proof of Lemma 5.24 is more than a factor of 2 shorter than the proof of the analogous Lemma B.5 in [Wirth, 2004] (together with Lemma B.1, its additionally required sub-lemma).

5.15 Reduction

Reduction is the reverse of consequence. It is the backbone of logical reasoning, especially of abduction and goal-directed deduction. In our case, a reduction step does not only reduce a set of problems to another set of problems, but also guarantees that the solutions of the latter also solve the former; here "solutions" means those $\mathcal{S}$-raising-valuations of the (rigid) (free) variables from $\mathbb{V}$ which are $\mathcal{S}$-compatible with $(C,(P,N))$ for the positive/negative variable-condition (P,N) and the (P,N)-choice-condition C given by the context of the reduction step.

Definition 5.25 (Reduction)
Let (P,N) be a positive/negative variable-condition. Let C be a (P,N)-choice-condition. Let G_0 and G_1 be sets of sequents. Let $\mathcal{S}$ be a Σ-structure.
G_0 $(C,(P,N))$*-reduces to* G_1 *in* $\mathcal{S}$ if for every π that is $\mathcal{S}$-compatible with $(C,(P,N))$:

If G_1 is $(\pi,\mathcal{S})$-valid, then G_0 is $(\pi,\mathcal{S})$-valid as well.

Theorem 5.26 (Reduction)
Let (P,N) be a positive/negative variable-condition. Let C be a (P,N)-choice-condition. Let G_0, G_1, G_2, and G_3 be sets of sequents. Let $\mathcal{S}$ be a Σ-structure.

1. (Validity) *If G_0 $(C,(P,N))$-reduces to G_1 in $\mathcal{S}$ and G_1 is $(C,(P,N))$-valid in $\mathcal{S}$, then G_0 is $(C,(P,N))$-valid in $\mathcal{S}$, too.*

2. (Reflexivity) *In case of $G_0 \subseteq G_1$: G_0 $(C,(P,N))$-reduces to G_1 in $\mathcal{S}$.*

3. (Transitivity) *If G_0 $(C,(P,N))$-reduces to G_1 in $\mathcal{S}$
and G_1 $(C,(P,N))$-reduces to G_2 in $\mathcal{S}$,
then G_0 $(C,(P,N))$-reduces to G_2 in $\mathcal{S}$.*

4. (Additivity) *If G_0 $(C,(P,N))$-reduces to G_2 in $\mathcal{S}$
and G_1 $(C,(P,N))$-reduces to G_3 in $\mathcal{S}$,
then $G_0 \cup G_1$ $(C,(P,N))$-reduces to $G_2 \cup G_3$ in $\mathcal{S}$.*

5. (Monotonicity) *For $(C',(P',N'))$ being an extended extension of $(C,(P,N))$:*

(a) If G_0 is $(C',(P',N'))$-valid in $\mathcal{S}$, then G_0 is also $(C,(P,N))$-valid in $\mathcal{S}$.

(b) If G_0 $(C,(P,N))$-reduces to G_1 in $\mathcal{S}$, then G_0 also $(C',(P',N'))$-reduces to G_1 in $\mathcal{S}$.

6. (Instantiation of Free Variables) *Let σ be a (P,N)-substitution.
Let $(C',(P',N'))$ be the extended σ-update of $(C,(P,N))$.
Set $M := \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$. Choose some $V \subseteq \mathbb{V}$ with $\mathbb{V}(G_0, G_1) \subseteq V$.
Set $O := M \cap P^*\langle V \rangle$. Set $O' := \mathrm{dom}(C) \cap \langle M \backslash O \rangle P^*$.
Assume that for every $y^{\mathbb{V}} \in O'$, for α being the return type of $C(y^{\mathbb{V}})$ (cf. Definition 5.13), there is a generalized choice function on the power-set of $\mathcal{S}(\forall)_\alpha$.*

(a) If $G_0\sigma \cup (\langle O \rangle Q_C)\sigma$ is $(C',(P',N'))$-valid in $\mathcal{S}$, then G_0 is $(C,(P,N))$-valid in $\mathcal{S}$.

*(b) If G_0 $(C,(P,N))$-reduces to G_1 in $\mathcal{S}$,
then $G_0\sigma$ $(C',(P',N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$ in $\mathcal{S}$.*

7. (Instantiation of Free Atoms) *Let ν be a substitution on $\mathbb{A}$.
If $\mathbb{V}(G_0) \times \mathrm{dom}(\nu) \subseteq N$, then $G_0\nu$ $(C,(P,N))$-reduces to G_0 in $\mathcal{S}$.*

5.16 Soundness, Safeness, and Solution-Preservation

Soundness of inference rules has the global effect that if we reduce a set of sequents to an empty set, then we know that the original set is valid. Soundness is an essential property of inference rules.

Safeness of inference rules has the global effect that if we reduce a set of sequents to an invalid set, then we know that already the original set was invalid. Safeness is helpful in rejecting false assumptions and in patching failed proof attempts.

As explained before, for a reduction step in our framework, we are not contend with soundness: We want *solution-preservation* in the sense that an $\mathcal{S}$-raising-valuation π that makes the set of sequents of the reduced proof state $(\pi, \mathcal{S})$-valid is guaranteed to do the same for the original input proposition, provided that π is $\mathcal{S}$-compatible with $(C, (P, N))$ for the positive/negative variable-condition (P, N) and the (P, N)-choice-condition C of the reduced proof state.

All our inference rules of § 3 have all of these properties. This is obvious for the trivial α- and β-rules. For the inference rules where this is not obvious, i.e. our γ- and δ^-- and δ^+-rules of §§ 3.2, 3.3, and 3.4, we state these properties in the following theorem.

Theorem 5.27

Let (P, N) be a positive/negative variable-condition. Let C be a (P, N)-choice-condition. Let us consider any of the γ-, δ^--, and δ^+-rules of §§ 3.2, 3.3, and 3.4.
Let G_0 and G_1 be the sets of the sequent above and of the sequents below the bar of that rule, respectively.
Let C'' be the set of the pair indicated to the upper right of the bar if there is any (which is the case only for the δ^+-rules) or the empty set otherwise.
Let V be the relation indicated to the lower right of the bar if there is any (which is the case only for the δ^-- and δ^+-rules) or the empty set otherwise.
Let us weaken the informal requirement "Let $x^{\mathbb{A}}$ be a fresh free atom" of the δ^--rules to its technical essence "$x^{\mathbb{A}} \in \mathbb{A} \setminus \big(\ \mathrm{dom}(P) \cup \mathbb{A}(\mathit{\Gamma}, A, \mathit{\Pi})\ \big)$".
Let us weaken the informal statement "Let $x^{\mathbb{V}}$ be a fresh free variable" of the δ^+-rules to its technical essence "$x^{\mathbb{V}} \in \mathbb{V} \setminus \big(\ \mathrm{dom}(C \cup P \cup N) \cup \mathbb{V}(A)\ \big)$".
Let us set $C' := C \cup C''$, $P' := P \cup V\restriction_{\mathbb{V}}$, $N' := N \cup V\restriction_{\mathbb{A}}$.

Then $(C', (P', N'))$ is an extended extension of $(C, (P, N))$ (cf. Definition 5.20).

Moreover, the considered inference rule is sound, safe, and solution-preserving in the sense that G_0 and G_1 mutually $(C', (P', N'))$-reduce to each other in every Σ-structure $\mathcal{S}$.

6 Summary and Discussion

6.1 Positive/Negative Variable-Conditions

We take a *sequent* to be a list of formulas which denotes the disjunction of these formulas. In addition to the standard frameworks of two-valued logics, our formulas may contain free atoms and variables with a context-independent semantics: While we admit explicit quantification to bind only *bound atoms* (written $x^{\mathbb{B}}$), our *free atoms* (written $x^{\mathbb{A}}$) are implicitly universally quantified. Moreover, free variables (written $x^{\mathbb{V}}$) are implicitly existentially quantified. The structure of this implicit form of quantification without quantifiers and without binders is represented globally in a *positive/negative variable-condition* (P, N), which can be seen as a directed graph on free atoms and variables whose edges are elements of either P or N.

Without loss of generality in practice, let us assume that P is finite. Then, a positive/negative variable-condition (P, N) is *consistent* if each cycle of the directed graph has more than one edge from N.

Roughly speaking, on the one hand, a *free variable* $y^{\mathbb{V}}$ is put into the scope of another free variable or atom $x^{\mathbb{VA}}$ by an edge $(x^{\mathbb{VA}}, y^{\mathbb{V}})$ in P; and, on the other hand, a *free atom* $y^{\mathbb{A}}$ is put into the scope of another free variable or atom $x^{\mathbb{VA}}$ by an edge $(x^{\mathbb{VA}}, y^{\mathbb{A}})$ in N.

On the one hand, an edge $(x^{\mathbb{VA}}, y^{\mathbb{V}})$ *must* be put into P

- if $y^{\mathbb{V}}$ is introduced in a δ^{+}-step where $x^{\mathbb{VA}}$ occurs in the principal[2] formula, and also
- if $y^{\mathbb{V}}$ is globally replaced with a term in which $x^{\mathbb{VA}}$ occurs.

On the other hand, an edge $(x^{\mathbb{VA}}, y^{\mathbb{A}})$ *must* be put into N

- if $x^{\mathbb{VA}}$ is actually a free *variable*, and $y^{\mathbb{A}}$ is introduced in a δ^{-}-step where $x^{\mathbb{VA}}$ occurs in the sequent (either in the principal formula or in the parametric formulas)[2]

Furthermore, such edges *may* always be added to the positive/negative variable-condition, as long as it remains consistent. Such an unforced addition of edges might be appropriate especially in the formulation of a new proposition:

- partly, because we may need this for modeling the intended semantics by representing the intended quantificational structure for the free variables and atoms of the new proposition;
- partly, because we may need this for enabling induction in the form of FERMAT's *descente infinie* on the free atoms of the proposition; cf. [WIRTH, 2004, §§ 2.5.2 and 3.3]. (This is closely related to the satisfaction of the condition on N in Theorem 5.26(7).)

6.2 Semantics of Positive/Negative Variable-Conditions

The value assigned to a free variable $y^{\mathbb{V}}$ by an *$\mathcal{S}$-raising-valuation* π may depend on the value assigned to an atom $x^{\mathbb{A}}$ by an *$\mathcal{S}$-valuation*. In that case, the *semantic relation* S_{π} contains an edge $(x^{\mathbb{A}}, y^{\mathbb{V}})$. Moreover, π is enforced to obey the quantificational structure by the requirement that $(P \cup S_{\pi}, N)$ must be consistent; cf. Definitions 5.10 and 5.15.

6.3 Replacing ε-Terms with Free Variables

Suppose that an ε-term $\varepsilon z^{\mathbb{B}}.\ B$ has free occurrences of exactly the bound atoms $v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}$ which are not free atoms of our framework, but are actually bound in the syntactic context in which this ε-term occurs. Then we can replace it in this context with the application term $z^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})$ for a fresh free variable $z^{\mathbb{V}}$ and set the value of a global function C (called the *choice-condition*) at $z^{\mathbb{V}}$ according to

$$C(z^{\mathbb{V}}) \quad := \quad \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon z^{\mathbb{B}}.\ B,$$

and augment P with an edge $(y^{\mathbb{VA}}, z^{\mathbb{V}})$ for each free variable or free atom $y^{\mathbb{VA}}$ occurring in B.

6.4 Semantics of Choice-Conditions

A free variable $z^{\mathbb{V}}$ in the domain of the global choice-condition C must take a value that makes $C(z^{\mathbb{V}})$ true — if such a choice is possible. This can be formalized as follows. Let "eval" be the standard evaluation function. Let $\mathcal{S}$ be any of the semantic structures (or models) under consideration. Let δ be a valuation of the free variables and free atoms (resulting from an $\mathcal{S}$-raising-valuation of the variables and an $\mathcal{S}$-valuation of the atoms). Let χ be an arbitrary $\mathcal{S}$-valuation of the bound atoms $v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}, z^{\mathbb{B}}$. Then $\delta(z^{\mathbb{V}})$ must be a function that chooses a value that makes B true whenever possible, in the sense that $\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi)(B) = \mathsf{TRUE}$ implies $\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi)(B\mu) = \mathsf{TRUE}$ for

$$\mu := \{z^{\mathbb{B}} \mapsto z^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}.$$

6.5 Substitution of Free Variables ("ε-Substitution")

The kind of logical inference we essentially need is (problem-) *reduction,* the backbone of abduction and goal-directed deduction; cf. § 5.15. In a tree of reduction steps our free variables and free atoms show the following behavior with respect to their instantiation:

Atoms behave as constant parameters. A free variable $y^{\mathbb{V}}$, however, may be globally instantiated with any term by application of a substitution σ; unless, of course, in case $y^{\mathbb{V}}$ is in the domain of the global choice-condition C, in which case σ must additionally satisfy $C(y^{\mathbb{V}})$, in a sense to be explained below.

In addition, the applied substitution σ must always be an (P, N)-*substitution.* This means that the current positive/negative variable-condition (P, N) remains consistent when we extend it to its so-called σ-*update*, which augments P with the edges from the free variables and free atoms in $\sigma(z^{\mathbb{V}})$ to $z^{\mathbb{V}}$, for each free variable $z^{\mathbb{V}}$ in the domain $\mathrm{dom}(\sigma)$.

Moreover, the global choice-condition C must be updated by removing $z^{\mathbb{V}}$ from its domain $\mathrm{dom}(C)$ and by applying σ to the C-values of the free variables remaining in $\mathrm{dom}(C)$.

Now, in case of a free variable $z^{\mathbb{V}} \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$, σ satisfies the current choice-condition C if $(Q_C(z^{\mathbb{V}}))\sigma$ is valid in the context of the updated variable-condition and choice-condition. Here, for a choice-condition $C(z^{\mathbb{V}})$ given as above, $Q_C(z^{\mathbb{V}})$ denotes the formula

$$\forall v_0^{\mathbb{B}}.\ \ldots \forall v_{l-1}^{\mathbb{B}}.\ (\ \exists z^{\mathbb{B}}.\ B \Rightarrow B\mu\ ),$$

which is nothing but our version of HILBERT's axiom (ε_0); cf. Definition 4.11. Under these conditions, the invariance of reduction under substitution is stated in Theorem 5.26(6b).

Finally, note that $Q_C(z^{\mathbb{V}})$ itself is always valid in our framework; cf. Lemma 5.19.

6.6 Where have all the ε-Terms gone?

After the replacement described in § 6.3 and, in more detail, in § 4.11, the ε-symbol occurs neither in our terms, nor in our formulas, but only in the range of the current choice-condition, where its occurrences are inessential, as explained at the end of § 4.11.

As a consequence of this removal, our formulas are much more readable than in the standard approach of in-line presentation of ε-terms, which always was nothing but a theoretical presentation because in practical proofs the ε-terms would have grown so large that the mere size of them made them inaccessible to human inspection. To see this, compare our presentation in Example 4.10 to the one in Example 4.8, and note that the latter is still hard to read although we have invested some efforts in finding a readable form of presentation.

From a mathematical point of view, however, the original ε-terms are still present in our approach; up to isomorphism and with the exception of some irrelevant term sharing. To make these ε-terms explicit in a formula A for a given (P, N)-choice-condition C, we just have to do the following:

Step 1: Let us consider the relation C not as a function, but as a ground term rewriting system: This means that we read $\left(\ z^{\mathbb{V}},\quad \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon z^{\mathbb{B}}_{.}B\ \right) \in C$ as a rewrite rule saying that we may replace the free variable $z^{\mathbb{V}}$ (the left-hand side of the rule, which is not a variable but a constant w.r.t. the rewriting system) with the right-hand side $\lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon z^{\mathbb{B}}.\ B$ in any given context as long as we want.

By Definition 5.13(3), we know that all variables in B are smaller than $z^{\mathbb{V}}$ in P^+. By the consistency of our positive/negative variable-condition (P, N) (according to Definition 5.13), we know that P^+ is a well-founded ordering. Thus its multi-set extension is a well-founded ordering as well. Moreover, the multi-set of the free variable $z^{\mathbb{V}}$ of the left-hand side is bigger than the multi-set of the free-variable occurrences in the right-hand side in the well-founded multi-set extension of P^+. Thus, if we rewrite a formula, the multi-set of the free-variable occurrences in the rewritten formula is smaller than the multi-set of the free-variable occurrences in the original formula.

Therefore, normalization of any formula A with these rewrite rules terminates with a formula A'.

Step 2: As typed $\lambda\alpha\beta$-reduction is also terminating, we can apply it to remove the λ-terms introduced to A' by the rewriting of Step 1, resulting in a formula A''.

Then — with the proper semantics for the ε-binder — the formulas A' and A'' are equivalent to A, but do not contain any free variables that are in the domain of C. This means that A'' is equivalent to A, but does not contain ε-constrained free variables anymore.

Moreover, if the free variables in A resulted from the elimination of ε-terms as described in §§ 4.11 and 6.3, then all λ-terms that were not already present in A are provided with arguments and are removed by the rewriting of Step 2. Therefore, no λ-symbol occurs in the formula A'' if the formula A resulted from a first-order formula.

For example, if we normalize $\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}})$ with respect to the rewriting system given by the (P, N)-choice-condition C of of Example 4.10, and then by $\lambda\alpha\beta$-reduction, we end up in a normal form which is the first-order formula (4.8.1) of Example 4.8, with the exception of the renaming of some bound atoms that are bound by ε.

If each element z^{V} in the domain of C binds a unique bound atom z^{B} by the ε in the higher-order ε-term $C(z^{\mathrm{V}})$, then the normal form A'' can even preserve our information on committed choice when we consider any ε-term binding an occurrence of a bound atom of the same name to be committed to the same choice. In this sense, the representation given by the normal form is equivalent to our original one given by $\mathsf{P}(w_a^{\mathrm{V}}, x_b^{\mathrm{V}}, y_d^{\mathrm{V}}, z_h^{\mathrm{V}})$ and C.

6.7 Are we breaking with the Traditional Treatment of HILBERT's ε?

Our new semantic free-variable framework was actually developed to meet the requirements analysis for the combination of mathematical induction in the liberal style of FERMAT's *descente infinie* with state-of-the-art logical deduction. The framework provides a formal system in which a working mathematician can straightforwardly develop his proofs supported by powerful automation; cf. [WIRTH, 2004].

If traditionalism meant restriction to the expressional means of the past — say the first half of the 20th century with its foundational crisis and special emphasis on constructivism, intuitionism, and finitism — then our approach would not classify as traditional. Although we offer the extras of non-committed choice and a model-theoretic notion of validity, we nevertheless see our framework based on Q_C as a form of (ε_0) (cf. § 4.12) as an upward-compatible extension of HILBERT–BERNAYS' original framework with (ε_0) as the only axiom for the ε. And with its equivalents for the traditional ε-terms (cf. § 6.6) and with its support for the global proof transformation given by the ε-substitution methods (cf. §§ 4.12, 5.15, and 6.5), our framework is indeed deeply rooted in the HILBERT–BERNAYS tradition.

Note that the fear of inconsistency should have been soothed anyway in the meantime by WITTGENSTEIN, cf. e.g. [DIAMOND, 1976]. The main disadvantage of an exclusively axiomatic framework as compared to one that also offers a model-theoretic semantics is the following: Constructive proofs of practically relevant theorems easily become too huge and too tedious, whereas semantic proofs are of a better manageable size. More important is the possibility to invent *new and more suitable logics for new applications* with semantic means, whereas proof transformations can refer only to already existing logics (cf. § 4.7).

We intend to pass the heritage of HILBERT's ε on to new generations interested in computational linguistics, automated theorem proving, and mathematics assistance systems; fields in which — with very few exceptions — the overall common opinion still is (the wrong one) that the ε hardly can be of any practical benefit.

The differences, however, between our free-variable framework for the ε and HILBERT's original underspecified ε-operator, in the order of increasing importance, are the following:

1. The term-sharing of ε-terms with the help of free variables improves the readability of our formulas considerably.
2. We do not have the requirement of globally committed choice for any ε-term: Different free variables with the same choice-condition may take different values. Nevertheless, *ε-substitution* works at least as well as in the original framework.
3. Opposed to all other classical validities for the ε (including the semantics of [ASSER, 1957], [HERMES, 1965], and [LEISENRING, 1969]), the implicit quantification over the choice of our free variables is existential instead of universal. This change simplifies formal reasoning in all relevant contexts, because we have to consider only an arbitrary single solution (or choice, substitution) instead of checking all of them.

7 Conclusion

Our more flexible semantics for HILBERT's ε and our novel free-variable framework presented in this paper were developed to solve the difficult soundness problems arising in the combination of mathematical induction in the liberal style of FERMAT's *descente infinie* with state-of-the-art deduction.[7] Thereby, they had passed an evaluation of their usefulness even before they were recognized as a candidate for the semantics that HILBERT's school in logic may have had in mind for their ε. While this is a speculation, it is definite that the semantic framework for HILBERT's ε proposed in this paper has the following advantages:

Indication of Commitment: The requirement of a commitment to a choice is expressed syntactically and most clearly by the sharing of a free variable; cf. § 4.11.

Semantics: The semantics of the ε is simple and straightforward in the sense that the ε-operator becomes similar to the referential use of indefinite articles and determiners in natural languages, cf. [WIRTH, 2012c].

Our semantics for the ε is based on an abstract formal approach that extends a semantics for closed formulas (satisfying only very weak requirements, cf. § 5.8) to a semantics with existentially quantified "free variables" and universally quantified "free atoms", replacing the three kinds of free variables of [WIRTH, 2004; 2006; 2008; 2012b; 2012c], i.e. existential (free γ-variables), universal (free δ^--variables), and ε-constrained (free δ^+-variables). The simplification achieved by the reduction from three to two kinds of free variables results in a remarkable reduction of the complexity of our framework and will make its adaptation to applications much easier.

In spite of this simplification, we have enhanced the expressiveness of our framework by replacing the variable-conditions of [WIRTH, 2002; 2004; 2006; 2008; 2012b; 2012c] with our *positive/negative* variable-conditions here, such that our framework now admits us to represent HENKIN quantification directly; cf. Example A.1. From a philosophical point of view, this clearer differentiation also provides a deep insight into the true nature and the relation of the δ^-- and the δ^+-rules.

Reasoning: Our representation of an ε-term $\varepsilon x^{\mathbb{B}}.\ A$ can be replaced with *any* term t that satisfies the formula $\exists x^{\mathbb{B}}.\ A \ \Rightarrow\ A\{x^{\mathbb{B}}{\mapsto}t\}$, cf. § 4.12. Thus, the correctness of such a replacement is likely to be expressible and verifiable in the original calculus. Our free-variable framework for the ε is especially convenient for developing proofs in the style of a working mathematician, cf. [WIRTH, 2004; 2006; 2012b]. Indeed, our approach makes proof work most simple because we do not have to consider all proper choices t for x (as in all other model-theoretic approaches) but only a single arbitrary one, which is fixed in a global proof transformation step.

Finally, we hope that our new semantic framework will help to solve further practical and theoretical problems with the ε and improve the applicability of the ε as a logic tool for description and reasoning. And already without the ε (i.e. for the case that the choice-condition is empty, cf. e.g. [WIRTH, 2012a; 2014]), our free-variable framework should find a multitude of applications in all areas of computer-supported reasoning.

A Henkin Quantification and IF Logic

In [Wirth, 2012c, § 6.4.1], we showed that Henkin quantification was problematic for the variable-conditions of that paper, which had only one component, namely the positive one of our positive/negative variable-conditions here: Indeed, there the only way to model an example of a Henkin quantification precisely was to increase the order of some variables by raising. Let us consider the same example here again and show that now we can model its Henkin quantification directly with a *consistent* positive/*negative* variable-condition, but *without raising.*

Example A.1 (Henkin Quantification)

In [Hintikka, 1974], quantifiers in first-order logic were found insufficient to give the precise semantics of some English sentences. In [Hintikka, 1996], *IF logic,* i.e. Independence-Friendly logic — a first-order logic with more flexible quantifiers — was presented to overcome this weakness. In [Hintikka, 1974], we find the following sentence:

> Some relative of each villager and
> some relative of each townsman hate each other. (H0)

Let us first change to a lovelier subject:

> Some loved one of each woman and
> some loved one of each man love each other. (H1)

For our purposes here, we consider (H1) to be equivalent to the following sentence, which may be more meaningful and easier to understand:

> We can fix a loved one for each woman and a loved one for each man, such that
> for every pair of woman and man, these loved ones could love each other.

(H1) can be represented by the following Henkin-quantified IF-logic formula:

$$\forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \left(\left(\begin{array}{rl} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array}\right) \Rightarrow\ \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.\ \left(\begin{array}{rl} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{array}\right)\right) \qquad \text{(H2)}$$

Let us refer to the standard game-theoretic semantics for quantifiers (cf. e.g. [Hintikka, 1996]), which is defined as follows: Witnesses have to be picked for the quantified variables outside-in. We have to pick the witnesses for the γ-quantifiers (i.e., in (H2), for the existential quantifiers), and our opponent in the game picks the witnesses for the δ-quantifiers (i.e. for the universal quantifiers in (H2)). We win if the resulting quantifier-free formula evaluates to true. A formula is true if we have a winning strategy.

Then an IF-logic quantifier such as "$\exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.$" in (H2) is a special quantifier, which is a bit different from "$\exists y_1^{\mathbb{B}}.$". Game-theoretically, it has the following semantics: It asks us to pick the loved one $y_1^{\mathbb{B}}$ independently from the choice of the man $y_0^{\mathbb{B}}$ (by our opponent in the game), although the IF-logic quantifier occurs in the scope of the quantifier "$\forall y_0^{\mathbb{B}}.$".

Note that Formula (H2) is already close to anti-prenex form. In fact, if we move its quantifiers closer toward the leaves of the formula tree, this does not admit us to reduce their dependencies. It is more interesting, however, to move the quantifiers of (H2) out — to obtain prenex form — and then to simplify the prenex by using the equivalence of "$\forall y_0^{\mathbb{B}}.\ \exists y_1^{\mathbb{B}}/y_0^{\mathbb{B}}.$" and "$\exists y_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.$", resulting in:

$$\forall x_0^{\mathbb{B}}.\ \exists y_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.\ \left(\begin{pmatrix} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{pmatrix} \Rightarrow \begin{pmatrix} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{pmatrix}\right) \tag{H2$'$}$$

Note that this formula is stronger than the following formula with standard quantifiers:

$$\forall x_0^{\mathbb{B}}.\ \exists y_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}.\ \left(\begin{pmatrix} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{pmatrix} \Rightarrow \begin{pmatrix} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}, x_1^{\mathbb{B}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}, y_1^{\mathbb{B}}) \end{pmatrix}\right) \tag{S2$'$}$$

An alternative way to define the semantics of IF-logic quantifiers is by describing their effect on the equivalent *raised* forms of the formulas in which they occur. *Raising* is a dual of SKOLEMization, cf. [MILLER, 1992]. The raised form of (S2$'$) is the following:

$$\exists y_1^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}.\ \forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \left(\begin{pmatrix} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{pmatrix} \Rightarrow \begin{pmatrix} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}(y_0^{\mathbb{B}}, x_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}(x_0^{\mathbb{B}}), x_1^{\mathbb{B}}(y_0^{\mathbb{B}}, x_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}(y_0^{\mathbb{B}}, x_0^{\mathbb{B}}), y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \end{pmatrix}\right) \tag{S3}$$

For HENKIN-quantified IF-logic formulas, the raised form is defined as usual, besides that a γ-quantifier, say "$\exists x_1^{\mathbb{B}}.$", followed by a slash as in "$\exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.$", is raised such that $x_0^{\mathbb{B}}$ does not appear as an argument to the raising function for $x_1^{\mathbb{B}}$. Accordingly, *mutatis mutandis*, (H2) as well as (H2$'$) are equivalent to their common raised form (H3) below, where $x_0^{\mathbb{B}}$ does not occur as an argument to the raising function $x_1^{\mathbb{B}}$ — contrary to (S3), which is strictly implied by (H3) because we can choose the loved one of the woman differently for different men.

$$\exists y_1^{\mathbb{B}}.\ \exists x_1^{\mathbb{B}}.\ \forall x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ \left(\begin{pmatrix} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{pmatrix} \Rightarrow \begin{pmatrix} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}(y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}(x_0^{\mathbb{B}}), x_1^{\mathbb{B}}(y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}(y_0^{\mathbb{B}}), y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \end{pmatrix}\right) \tag{H3}$$

Now, (H3) looks already very much like the following tentative representation of (H1) in our framework of free atoms and variables:

$$\begin{pmatrix} & \mathsf{Female}(x_0^{\mathbb{A}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{A}}) \end{pmatrix} \Rightarrow \begin{pmatrix} & \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{V}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{V}}, y_1^{\mathbb{V}}) \end{pmatrix} \tag{H1$'$}$$

with choice-condition C given by

$$\begin{array}{rcl} C(y_1^{\mathbb{V}}) & := & \varepsilon y_1^{\mathbb{B}}.\ (\mathsf{Female}(x_0^{\mathbb{A}}) \Rightarrow \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{B}})) \\ C(x_1^{\mathbb{V}}) & := & \varepsilon x_1^{\mathbb{B}}.\ (\mathsf{Male}(y_0^{\mathbb{A}}) \Rightarrow \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{B}})) \end{array}$$

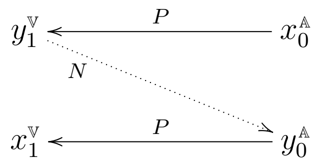
which requires our positive/negative variable-condition (P, N) to contain $(x_0^{\mathbb{A}}, y_1^{\mathbb{V}})$ and $(y_0^{\mathbb{A}}, x_1^{\mathbb{V}})$ in the positive relation P (by item 3 of Definition 5.13).

The concrete form of this choice-condition C was chosen to mirror the structure of the natural-language sentence (H1) as close as possible. Actually, however, we do not need exactly this choice-condition here. Indeed, to find a representation in our framework, we could also work with an empty choice-condition. Crucial for our discussion, however, is that we can have

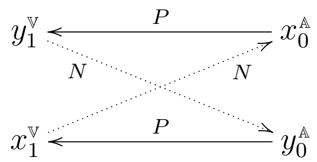
$$(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}), (y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \in P;$$

otherwise the choice of the loved ones could not depend on their lovers.

In any case, we can add $(y_1^{\mathbb{V}}, y_0^{\mathbb{A}})$ to the negative relation N here, namely to express that $y_1^{\mathbb{V}}$ must not read $y_0^{\mathbb{A}}$. Then we obtain:



The same variable-condition is also obtained if we start with the empty variable-condition $(P, N) := (\emptyset, \emptyset)$, remove all quantifiers from (S2′) with our γ- and δ^--rules, and then add $\{(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}), (y_0^{\mathbb{A}}, x_1^{\mathbb{V}})\}$ to P. The corresponding procedure for (H2′), however, has to add also $(x_1^{\mathbb{V}}, x_0^{\mathbb{A}})$ to N as part of the last γ-step that removes the IF-logic quantifier "$\exists x_1^{\mathbb{B}}/x_0^{\mathbb{A}}$." and replaces $x_1^{\mathbb{B}}$ with $x_1^{\mathbb{V}}$. After this procedure, our current positive/negative variable-condition is now given as (P, N) with $P = \{(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}), (y_0^{\mathbb{A}}, x_1^{\mathbb{V}})\}$ and $N = \{(y_1^{\mathbb{V}}, y_0^{\mathbb{A}}), (x_1^{\mathbb{V}}, x_0^{\mathbb{A}})\}$. Thus, we have a single cycle in the graph, namely the following one:



But this cycle necessarily has two edges from the negative relation N. Thus, in spite of this cycle, our positive/negative variable-condition (P, N) is consistent by Corollary 5.5.

With the variable-conditions of [Wirth, 2002; 2004; 2006; 2008; 2012b; 2012c], however, this cycle necessarily destroys the consistency, because they have no distinction between the edges of N and P.

Therefore — if the discussion in [Wirth, 2012c, § 6.4.1] is sound — our new framework of this paper with positive/*negative* variable-conditions is the only one among all approaches suitable for describing the semantics of noun phrases in natural languages that admits us to model IF-logic and Henkin quantifiers without raising.

While the rules for δ-quantifiers of IF logic work just like our normal δ-rules (indeed, the law of the excluded middle fails to hold in IF logic in general), we can now formalize the inference rule for the *γ-quantifiers of IF logic* as follows:

Let $x^{\mathbb{VA}}$ be a free variable or a free atom.
Let t be any term not containing $x^{\mathbb{VA}}$ (i.e. $x^{\mathbb{VA}} \notin \mathbb{VA}(t)$):

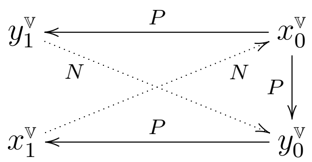
$$\frac{\quad \Gamma \quad \exists y^{\mathbb{B}}/x^{\mathbb{VA}}.\ A \quad \Pi}{A\{y^{\mathbb{B}}\mapsto t\} \quad \Gamma \quad \exists y^{\mathbb{B}}/x^{\mathbb{VA}}.\ A \quad \Pi} \qquad \mathbb{V}(t) \times \{x^{\mathbb{VA}}\}$$

$$\frac{\quad \Gamma \quad \neg\forall y^{\mathbb{B}}/x^{\mathbb{VA}}.\ A \quad \Pi}{\neg A\{y^{\mathbb{B}}\mapsto t\} \quad \Gamma \quad \neg\forall y^{\mathbb{B}}/x^{\mathbb{VA}}.\ A \quad \Pi} \qquad \mathbb{V}(t) \times \{x^{\mathbb{VA}}\}$$

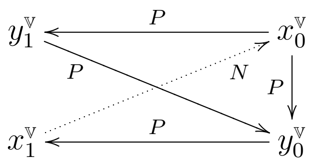
Here, $\mathbb{V}(t)\times\{x^{\mathbb{VA}}\}$ should be added to N, the negative part of the current positive/negative variable-condition (P, N) — no matter whether we have the case $x^{\mathbb{VA}} \in \mathbb{V}$ or actually $x^{\mathbb{VA}} \in \mathbb{A}$. Note that the first of these two cases may violate our range restriction for the negative part given in Definition 5.3, but we already remarked in § 5.6 that this range restriction was only to simplify matters in this paper.

Moreover, note that, because $x^{\mathbb{VA}}$ is not fresh but was typically introduced by a previous application of a δ^-- or δ^+-rule, the application of a γ-rule for IF-logic quantifiers could result in an inconsistent positive/negative variable-condition. Thus, we have to add the requirement for the consistency of the resulting variable-condition as a precondition for the application of these new inference rules.

With these γ-rules for IF-logic quantifiers, we can obtain the cyclic graph above from (H2) or (H2′) just as we obtained the non-cyclic graph above from (S2′). If we replace the two applications of δ^--rules here with two applications of δ^+-rules and start from (H2), then the resulting graph becomes



If we start from (H2′), we obtain



Each of these graphs has the same cycle with only one edge from the negative part N, which means that each of the variable-conditions is inconsistent. Thus, it seems that the application of δ^+-rules to δ-quantifiers with IF-logic γ-quantifiers in their scope is not to be recommended and the δ^--rules should be used instead, just as for outer δ-quantifiers over which we want to do mathematical induction in the style of *descente infinie*. If we always do so, free variables will hardly occur in the second component of IF-logic quantifiers, and then we can get along with the case of $x^{\mathbb{VA}} \in \mathbb{A}$ in the above new γ-rules and do not have to modify our range restriction on the negative part of our positive/negative variable-conditions.

B Semantics for HILBERT's ε in the Literature

Here in § B of the appendix, we will review the literature on the ε's semantics with an emphasis on practical adequacy and the intentions of the HILBERT school in logic.

B.1 Right-Unique Semantics

In contrast to the indefiniteness we suggested in § 4.8, nearly all semantics for HILBERT's ε found elsewhere in the literature are functional, i.e. [*right-*] *unique*; cf. e.g. [LEISENRING, 1969] and the references there.

B.1.1 Extensionality: ACKERMANN's (II,4) = BOURBAKI's (S7) = LEISENRING's (E2)

In [ACKERMANN, 1938] under the label (II,4), in [BOURBAKI, 1939ff.] under the label (S7) (where a τ is written for the ε, which must not be confused with HILBERT's τ-operator[8]), and in [LEISENRING, 1969] under the label (E2), we find the following axiom scheme, which we presented already in § 4.10:

$$\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \qquad \Rightarrow \qquad \varepsilon x^{\mathbb{B}}.\ A_0 \ = \ \varepsilon x^{\mathbb{B}}.\ A_1 \tag{E2}$$

This axiom (E2) must not be confused with the similar formula (E2′) from [WIRTH, 2008, Lemma 31, § 5.6] and [WIRTH, 2012c, Lemma 5.18, § 5.6], which reads in our new framework here as follows:

$$\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \qquad \Rightarrow \qquad x_0^{\mathbb{V}} \ = \ x_1^{\mathbb{V}} \tag{E2′}$$

for two different $x_0^{\mathbb{V}}, x_1^{\mathbb{V}} \in \mathbb{V}\backslash\mathbb{V}(A_1, A_2, \mathrm{dom}(P{\cup}N))$ and for a (P,N)-choice-condition C with $C(x_i^{\mathbb{V}}) = \varepsilon x^{\mathbb{B}}.\ A_i$ for $i \in \{0,1\}$. Our (E2′) can be shown to be $(C,(P,N))$-valid by applying Theorem 5.26(1,5a,6a): Indeed, we can apply the substitution $\{x_1^{\mathbb{V}}{\mapsto}y^{\mathbb{V}}\}$ after an extended extension $(C',(P',N))$ for a fresh variable $y^{\mathbb{V}} \in \mathbb{V} \setminus \mathbb{V}(A_1, A_2, x_0^{\mathbb{V}}, x_1^{\mathbb{V}}, \mathrm{dom}(P{\cup}N))$ with $C'(y^{\mathbb{V}}) = \varepsilon y^{\mathbb{B}}. \left(\begin{array}{ll} & (\ \forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \ \Rightarrow \ y^{\mathbb{B}} = x_0^{\mathbb{V}}\ ) \\ \wedge & (\ \neg\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \ \Rightarrow \ A_1\{x^{\mathbb{B}}{\mapsto}y^{\mathbb{B}}\}\ ) \end{array}\right)$.[9]

Contrary to the valid proposition (E2′), however, (E2) is an axiom that imposes a right-unique behavior for the ε (in the standard framework), depending on the extension of the formula forming the scope of an ε-binder on $x^{\mathbb{B}}$, seen as a predicate on $x^{\mathbb{B}}$. Indeed — from a semantic point of view — the value of $\varepsilon x^{\mathbb{B}}.\ A$ in each Σ-structure $\mathcal{S}$ is functionally dependent on the extension of the formula A, i.e. on $\{\ o \mid \mathrm{eval}(\mathcal{S} \uplus \{x^{\mathbb{B}}{\mapsto}o\})(A)\ \}$.

Therefore, axiomatizations that have (E2) as an axiom or as a consequence of other axioms are called *extensional*.

Note that (E2) has a disastrous effect in intuitionistic logic: The contrapositive of (E2) — together with (ε_0) and say "$0 \neq 1$" — turns every classical validity into an intuitionistic one.[10] For the strong consequences of the ε-formula in intuitionistic logic, see also Note 8.

B.1.2 Weaker than (E2), but still Right-Unique

To overcome this disastrous effect and to get more options for the definition of a semantics of the ε in general, in [Asser, 1957], [Meyer-Viol, 1995], and [Giese & Ahrendt, 1999] the value of $\varepsilon x^{\mathbb{B}}.\,A$ may additionally depend on the *syntax* besides the semantics of the formula in the scope of the ε. The semantics of the ε is then given as a function depending on a Σ-structure and on the syntactic details of the term $\varepsilon x^{\mathbb{B}}.\,A$. In [Giese & Ahrendt, 1999, p.177] we read: "This definition contains no restriction whatsoever on the valuation of ε-terms." This claim, however, is not justified in its universality, because all considered options do still impose the restriction of a right-unique behavior; thereby the claim denies the possibility of an indefinite behavior as given in §§ 4.10 and 4.11. See also § B.2 for an alternative realization of an indefinite semantics.

B.1.3 Overspecification even beyond (E2)

In [Hermes, 1965, p.18], the ε suffers further overspecification in addition to (E2):

$$\varepsilon x.\,\mathsf{false} \;=\; \varepsilon x.\,\mathsf{true} \qquad (\varepsilon_5)$$

Roughly speaking, this axiom sets the value of a generalized choice function on the empty set to its value on the whole universe. For classical logic, we can combine (E2) and (ε_5) into the following axiom of [DeVidi, 1995] for "very extensional" semantics:

$$\forall x.\left(\begin{array}{ll} & (\exists y.\,A_0\{x{\mapsto}y\} \;\Rightarrow\; A_0)\\ \Leftrightarrow & (\exists y.\,A_1\{x{\mapsto}y\} \;\Rightarrow\; A_1)\end{array}\right) \;\Rightarrow\; \varepsilon x.\,A_0 = \varepsilon x.\,A_1 \qquad \text{(vext)}$$

Indeed, (vext) implies (E2) and (ε_5). The other direction, however, does not hold for intuitionistic logic, where, roughly speaking, (vext) additionally implies that if the same elements make A_0 and A_1 as true as possible, then the ε-operator picks the same element of this set, even if the suprema $\exists y.\,A_0\{x{\mapsto}y\}$ and $\exists y.\,A_1\{x{\mapsto}y\}$ (in the complete Heyting algebra) are not equally true.

B.1.4 Strengthening Semantics to Turn Axiomatizations Complete

Although we have been concerned with soundness and safeness of our inference systems, we always accepted their incompleteness as the natural companion of semantics that are sufficiently weak to be useful in practice. Of course, completeness is the theoreticians' favorite puzzle because — as a global property of inference systems — it may be hard to prove, even for inconsistent systems. The objective of completeness gets particularly detached from practical usefulness, if a useful semantics is strengthened to obtain the completeness of a given inference system. Let us look at two examples for this procedure, resulting in practically useless semantics for the ε.

Different possible choices for the value of the generalized choice function on the empty set are discussed in [Leisenring, 1969]. As the consequences of any special choice are quite queer, the only solution that is found to be sufficiently adequate in [Leisenring, 1969] is validity in *all* models given by *all* generalized choice functions on the power-set of the universe. Note, however, that even in this case, in each model, the value of $\varepsilon x.\,A$ is *functionally* dependent on the extension of A.

Roughly speaking, in the textbook [LEISENRING, 1969], the axioms (ε_1) and (ε_2) from § 4.6 and (E2) from § 4.10 are shown to be complete w.r.t. this semantics of the ε in first-order logic.

This completeness makes it unlikely that extensional semantics matches the intentions of HILBERT's school in logic. Indeed, if their intended semantics for the ε could be completely captured by adding the single and straightforward axiom (E2), this axiom would not have been omitted in [HILBERT & BERNAYS, 1939]; it would at least be possible to derive (E2) from some axiomatization in [HILBERT & BERNAYS, 1939].

What makes LEISENRING's notion of validity problematic for theorem proving is that a proof has to consider all appropriate choice functions and cannot just pick an advantageous single one of them. More specifically, when LEISENRING does the step from satisfiability to validity he does the double duality switch from existence of a model and the existence of a choice function to all models and to all choice functions. Our notion of validity in Definition 5.17 does not switch the second duality, but stays with the *existence* of a choice function. Considering the influence that [LEISENRING, 1969] still has today, our avoidance of the universality requirement for choice functions in the definition of validity may be considered our practically most important conceptual contribution to the ε's semantics. If we stuck to LEISENRING's definition of validity, then we would either have to give up the hope of finding proofs in practice, or have to avoid considering validity (beyond truth) in connection with HILBERT's ε, which is HARTLEY SLATER' solution, carefully observed in [SLATER, 1994; 2002; 2007b; 2009; 2011].

This whole misleading procedure of strengthening semantics to obtain completeness for axiomatizations of the ε actually originates in [ASSER, 1957]. The main objective of [ASSER, 1957], however, is to find a semantics such that the basic ε-calculus of [HILBERT & BERNAYS, 1939] — not containing (E2) — is sound and complete for it. This semantics, however, has to depend on the details of the syntactic form of the ε-terms and, moreover, turns out to be necessarily so artificial that ASSER [1957] does not recommend it himself and admits that he thinks that it could not have been intended in [HILBERT & BERNAYS, 1939].

> "Allerdings ist dieser Begriff von Auswahlfunktion so kompliziert, daß sich seine Verwendung in der inhaltlichen Mathematik kaum empfiehlt."
> [ASSER, 1957, p. 59]
>
> "This notion of a choice function, however," (i.e. the type-3 choice function, providing a semantics for the ε-operator) "is so intricate that its application in contentual mathematics is hardly to be recommended."
>
> "Angesichts der Kompliziertheit des Begriffs der Auswahlfunktion dritter Art ergibt sich die Frage, ob bei HILBERT–BERNAYS (" ... ") wirklich beabsichtigt war, diesen Begriff von Auswahlfunktion axiomatisch zu beschreiben. Aus der Darstellung bei HILBERT–BERNAYS glaube ich entnehmen zu können, daß das nicht der Fall ist," [ASSER, 1957, p. 65]
>
> "The intricacy of the notion of the type-3 choice function puts up the question whether the intention in [HILBERT & BERNAYS, 1939] (" ... ") really was to describe this notion of choice function axiomatically. I believe I can draw from the presentation in [HILBERT & BERNAYS, 1939] that that is not the case,"

B.1.5 Roots of the Misunderstanding of a Right-Uniqueness Requirement

The described prevalence of the right-uniqueness requirement may have its historical justification in the fact that, if we expand the dots "..." in the quotation preceding Example 4.2 in § 4.6, the full quotation on p.12 of [Hilbert & Bernays, 1939; 1970] reads:

> "Das ε-Symbol bildet somit eine Art der Verallgemeinerung des μ-Symbols für einen beliebigen Individuenbereich. Der Form nach stellt es eine Funktion eines variablen Prädikates dar, welches außer demjenigen Argument, auf welches sich die zu dem ε-Symbol gehörige gebundene Variable bezieht, noch freie Variable als Argumente ("Parameter") enthalten kann. Der Wert dieser Funktion für ein bestimmtes Prädikat A (bei Festlegung der Parameter) ist ein Ding des Individuenbereichs, und zwar ist dieses Ding gemäß der inhaltlichen Übersetzung der Formel (ε_0) *ein solches, auf das jenes Prädikat A zutrifft, vorausgesetzt, daß es überhaupt auf ein Ding des Individuenbereichs zutrifft.*"

> "Thus, the ε-symbol forms a kind of generalization of the μ-symbol for an arbitrary domain of individuals. According to its form, it constitues a function of a variable predicate, which may contain free variables as arguments ("parameters") in addition to the argument to which the bound variable of the ε-symbol refers. The value of this function for a given predicate A (for fixed parameters) is a thing of the domain of individuals for which — according to the contentual translation of the formula (ε_0) — *the predicate A holds, provided that A holds for any thing of the domain of individuals at all.*"

Here the word "function" could be misunderstood in its narrower mathematical sense, namely to denote a (right-) unique relation. It is stated to be a function, however, only "according to its form", which — in the vernacular that becomes obvious from reading [Hilbert & Bernays, 2017b] — means nothing but "with respect to the process of the formation of formulas". Thus, Hilbert–Bernays' notation of the ε takes the syntactic form of a function. This syntactic weakness was not bothering the work of the Hilbert school in the field of proof theory. With our more practical intentions, the ε's form of a function turns out as a problem even regarding syntax alone, cf. §§ 4.10 and 4.11. And we are not the only ones who have seen this applicational problem: For instance, in [Heusinger, 1997], an index was introduced to the ε to overcome right-uniqueness.

If we nevertheless read "function" as a right-unique relation in the above quotation, what kind of function could be meant but a choice function, choosing an element from the set of objects that satisfy A, i.e. from its extension $\{\ o \mid \mathrm{eval}(\mathcal{S} \uplus \{x^{\mathbb{B}}{\mapsto}o\})(A)\ \}$. Accordingly, in the earlier publication [Hilbert, 1928], we read (p. 68):

> "Darüber hinaus hat das ε die Rolle der Auswahlfunktion, d. h. im Falle, wo $A\,a$ auf mehrere Dinge zutreffen kann, ist $\varepsilon\,A$ *irgendeines* von den Dingen a, auf welche $A\,a$ zutrifft."

> "Beyond that, the ε has the rôle of the choice function, i.e., if $A\,a$ may hold for several objects, $\varepsilon\,A$ is *an arbitrary one* of the things a for which $A\,a$ holds."

Regarding the notation in this quotation, the syntax of the ε is not that of a binder here, but a functional $\varepsilon : (i \rightarrow o) \rightarrow i$, applied to $A : i \rightarrow o$.

The meaning of having "the rôle of the choice function" is defined by the text that follows in the quotation. Thus, it is obvious that HILBERT wants to state the arbitrariness of choice as given by an arbitrary choice function, and that the word "function" does not refer to a requirement of right-uniqueness here.

Moreover, note that the definite article in "the choice function" (instead of the indefinite one) is in conflict with an interpretation as a mathematical function in the narrower sense as well.

Furthermore, DAVID HILBERT was sometimes pretty sloppy with the usage of choice functions in general: For instance, he may well have misinterpreted the consequences of the ε on the Axiom of Choice (cf. [RUBIN & RUBIN, 1985], [HOWARD & RUBIN, 1998]) in the one but last paragraph of [HILBERT, 1923a]. Let us therefore point out the following: Although the ε supplies us with a syntactic means for expressing an *indefinite universal (generalized) choice function* (cf. § 5.2), the axioms (E2), (ε_0), (ε_1), and (ε_2) do not imply the Axiom of Choice in set theories, unless the axiom schemes of Replacement (Collection) and Comprehension (Separation, Subset) also range over expressions containing the ε; cf. [LEISENRING, 1969, § IV 4.4].

HILBERT's school in logic may well have wanted to express what we call *"committed choice"* today, but they simply used the word "function" for the following three reasons:

1. They were not too much interested in semantics anyway.
2. The technical term "committed choice" did not exist at their time.
3. Last but not least, right-uniqueness conveniently serves as a global commitment to any choice and thereby avoids the problem illustrated in Example 4.6 of § 4.8.

B.2 Indefinite Semantics in the Literature

The only occurrence of an indefinite semantics for HILBERT's ε in the literature seems to be [BLASS & GUREVICH, 2000] (and the references there), unless we count the indexed ε of [HEUSINGER, 1997] for indefinite indices as such a semantics as well. The right-uniqueness is actually so prevalent in the literature that a "δ" is written instead of an "ε" in [BLASS & GUREVICH, 2000], because there the right-unique behavior is considered to be essential for the ε.

Consider the formula $\varepsilon x.\,(x{=}x) \ = \ \varepsilon x.\,(x{=}x)$ from [BLASS & GUREVICH, 2000] or the even simpler $\varepsilon x.\,\mathsf{true} \ = \ \varepsilon x.\,\mathsf{true}$ (discussed already in § 4.10), which may be valid or not, depending on the question whether the same object is taken on both sides of the equation or not. In natural language this like "Something is equal to something.", whose truth is indefinite. If you do not think so, consider $\varepsilon x.\,\mathsf{true} \neq \varepsilon x.\,\mathsf{true}$ in addition, i.e. "Something is unequal to something.", and notice that the two sentences seem to be contradictory.

In [Blass & Gurevich, 2000], Kleene's strong three-valued logic is taken as a mathematically elegant means to solve the problems with indefiniteness. In spite of the theoretical significance of this solution, however, Kleene's strong three-valued logic severely restricts its applicability from a practical point of view: In applications, a logic is not an object of investigation but a meta-logical tool, and logical arguments are never made explicit because the presence of logic is either not realized at all or taken to be trivial, even by academics (unless they are formalists); see, for instance, [Pinkal &al., 2001, p.14f.] for Wizard of Oz studies with young students.

Therefore, regarding applications, we had better stick to our common meta-logic, which in the western world is a subset of (modal) classical logic: A western court may accept that Lee Harvey Oswald killed John F. Kennedy as well as that he did not — but cannot accept a third possibility, a *tertium,* as required for Kleene's strong three-valued logic, and especially not the interpretation given in [Blass & Gurevich, 2000], namely that he *both* did and did not kill him, which contradicts any common sense.

C The Proofs

Proof of Lemma 5.16

Under the given assumptions, set $\lhd := P^+$ and $S_\pi := {}_{\mathbb{A}}\!\upharpoonright\!\lhd$.

Claim A: $\lhd = P^+ = (P \cup S_\pi)^+$ is a well-founded ordering.

Claim B: $(P \cup S_\pi, N)$ is a consistent positive/negative variable-condition.

Claim C: $S_\rho \subseteq {}_{\mathbb{A}}\!\upharpoonright\!\lhd = S_\pi \subseteq \lhd$.

Claim D: $S_\pi \circ \lhd \subseteq S_\pi$.

Proof of Claims A, B, C, and D: (P, N) is consistent because C is a (P, N)-choice-condition. Thus, P is well-founded and $\lhd = P^+ = (P \cup S_\pi)^+$ is a well-founded ordering. Moreover, we have $S_\rho, S_\pi, P \subseteq \lhd$. Thus, (P, N) is a weak extension of $(P \cup S_\pi, N)$. Thus, by Corollary 5.7, $(P \cup S_\pi, N)$ is a consistent positive/negative variable-condition. Finally, $S_\pi \circ \lhd = {}_{\mathbb{A}}\!\upharpoonright\!\lhd \circ \lhd \subseteq {}_{\mathbb{A}}\!\upharpoonright\!\lhd = S_\pi$. Q.e.d. (Claims A, B, C, and D)

By recursion on $y^{\mathbb{V}} \in \mathbb{V}$ in $\lhd$, we can define $\pi(y^{\mathbb{V}}) : (S_\pi\langle\{y^{\mathbb{V}}\}\rangle \to \mathcal{S}) \to \mathcal{S}$ as follows.

Let $\tau' : S_\pi\langle\{y^{\mathbb{V}}\}\rangle \to \mathcal{S}$ be arbitrary.

$y^{\mathbb{V}} \in \mathbb{V}\backslash\mathrm{dom}(C)$: If an $\mathcal{S}$-raising-valuation ρ is given, then we set

$$\pi(y^{\mathbb{V}})(\tau') := \rho(y^{\mathbb{V}})({}_{S_\rho\langle\{y^{\mathbb{V}}\}\rangle}\!\upharpoonright\!\tau');$$

which is well-defined according to Claim C. Otherwise, we choose an arbitrary value for $\pi(y^{\mathbb{V}})(\tau')$ from the universe of $\mathcal{S}$ (of the appropriate type). Note that $\mathcal{S}$ is assumed to provide some choice function $\mathcal{S}(\exists)$ for the universe function $\mathcal{S}(\forall)$ according to § 5.8.

$y^{\mathbb{V}} \in \mathrm{dom}(C)$: In this case, we have the following situation: $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ for some formula B and some $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}} \in \mathbb{B}$ with $v_0^{\mathbb{B}} : \alpha_0,\ \ldots,\ v_l^{\mathbb{B}} : \alpha_l$, $y^{\mathbb{V}} : \alpha_0 \to \ldots \to \alpha_{l-1} \to \alpha_l$, and $z^{\mathbb{VA}} \lhd y^{\mathbb{V}}$ for all $z^{\mathbb{VA}} \in \mathbb{VA}(B)$, because C is a (P, N)-choice-condition. In particular, by Claim A, $y^{\mathbb{V}} \notin \mathbb{V}(B)$.

In this case, with the help of the assumed generalized choice function on the power-set of the universe of $\mathcal{S}$ of the sort α_l, we let $\pi(y^{\mathbb{V}})(\tau')$ be the function f that for $\chi : \{v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}\} \to \mathcal{S}$ chooses a value from the universe of $\mathcal{S}$ of type α_l for $f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))$, such that,

$$\text{if possible, } B \text{ is true in } \mathcal{S} \uplus \delta' \uplus \chi',$$

for $\delta' := \mathsf{e}(\pi)(\tau' \uplus \tau'') \uplus \tau' \uplus \tau'' \uplus \chi$ for an arbitrary $\tau'' : (\mathbb{A}\backslash\mathrm{dom}(\tau')) \to \mathcal{S}$, and for $\chi' := \{v_l^{\mathbb{B}} \mapsto f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))\}$.

Note that the point-wise definition of f is correct: by the Explicitness Lemma and because of $y^{\mathbb{V}} \notin \mathbb{V}(B)$, the definition of the value of $f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))$ does not depend on the values of $f(\chi''(v_0^{\mathbb{B}})) \cdots (\chi''(v_{l-1}^{\mathbb{B}}))$ for a different $\chi'' : \{v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}\} \to \mathcal{S}$. Therefore, the function f is well-defined, because it also does not depend on τ'' according to the Explicitness Lemma and Claim 1 below. Finally, π is well-defined by induction on $\lhd$ according to Claim 2 below.

Claim 1: For $z^{\mathbb{VA}} \lhd y^{\mathbb{V}}$, the application term $(\delta' \uplus \chi')(z^{\mathbb{VA}})$ has the the value $\tau'(z^{\mathbb{VA}})$ in case of $z^{\mathbb{VA}} \in \mathbb{A}$, and the value $\pi(z^{\mathbb{VA}})({}_{S_\pi\langle\{z^{\mathbb{VA}}\}\rangle}\!\upharpoonright\!\tau')$ in case of $z^{\mathbb{VA}} \in \mathbb{V}$.

Claim 2: The definition of $\pi(y^{\mathbb{V}})(\tau')$ depends only on such values of $\pi(v^{\mathbb{V}})$ with $v^{\mathbb{V}} \lhd y^{\mathbb{V}}$, and does not depend on τ'' at all.

<u>Proof of Claim 1:</u> For $z^{\mathbb{VA}} \in \mathbb{A}$ the application term has the value $\tau'(z^{\mathbb{VA}})$ because of $z^{\mathbb{VA}} \in S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle$. Moreover, for $z^{\mathbb{VA}} \in \mathbb{V}$, we have $S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle \subseteq S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle$ by Claim D, and therefore the applicative term has the value $\pi(z^{\mathbb{VA}})({}_{S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle}{\upharpoonleft}(\tau' \uplus \tau'')) = \pi(z^{\mathbb{VA}})({}_{S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle}{\upharpoonleft}\tau')$. <u>Q.e.d. (Claim 1)</u>

<u>Proof of Claim 2:</u> In case of $y^{\mathbb{V}} \notin \mathrm{dom}(C)$, the definition of $\pi(y^{\mathbb{V}})(\tau')$ is immediate and independent. Otherwise, we have $z^{\mathbb{VA}} \lhd y^{\mathbb{V}}$ for all $z^{\mathbb{VA}} \in \mathbb{VA}(C(y^{\mathbb{V}}))$. Thus, Claim 2 follows from the EXPLICITNESS LEMMA and Claim 1. <u>Q.e.d. (Claim 2)</u>

Moreover, $\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$ is obviously an $\mathcal{S}$-raising-valuation. Thus, item 1 of Definition 5.15 is satisfied for π by Claim B.

To show that also item 2 of Definition 5.15 is satisfied, let us assume $y^{\mathbb{V}} \in \mathrm{dom}(C)$ and $\tau : \mathbb{A} \to \mathcal{S}$ to be arbitrary with $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$, and let us then assume to the contrary of item 2 that, for some $\chi : \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\} \to \mathcal{S}$ and for $\delta := \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$ and $\sigma := \{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$, we have $\mathrm{eval}(\mathcal{S} \uplus \delta)(B) = \mathsf{TRUE}$ and $\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) = \mathsf{FALSE}$.

Set $\tau' := {}_{S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonleft}\tau$ and $\tau'' := {}_{\mathbb{A}\setminus\mathrm{dom}(\tau')}{\upharpoonleft}\tau$.

Set $\delta' := {}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\delta$ and $f := \pi(y^{\mathbb{V}})(\tau')$.

Set $\chi' := \{v_l^{\mathbb{B}} \mapsto f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))\}$.

Then $\delta' = \mathsf{e}(\pi)(\tau' \uplus \tau'') \uplus \tau' \uplus \tau'' \uplus {}_{\{v_0,\ldots,v_{l-1}\}}{\upharpoonleft}\chi$. Moreover, by the EXPLICITNESS LEMMA, we have $\delta' = {}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id} \circ \mathrm{eval}(\mathcal{S} \uplus \delta)$.

By the VALUATION LEMMA we have

$$\begin{array}{rl} & \mathrm{eval}(\mathcal{S} \uplus \delta)(y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})) \\ = & \delta(y^{\mathbb{V}})(\delta(v_0^{\mathbb{B}})) \cdots (\delta(v_{l-1}^{\mathbb{B}})) \\ = & \mathsf{e}(\pi)(\tau)(y^{\mathbb{V}})(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}})) \\ = & \pi(y^{\mathbb{V}})(\tau')(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}})) \\ = & f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}})). \end{array}$$

Thus, $\chi' = \sigma \circ \mathrm{eval}(\mathcal{S} \uplus \delta)$.

Thus, $\delta' \uplus \chi' = ({}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id} \uplus \sigma) \circ \mathrm{eval}(\mathcal{S} \uplus \delta)$.

Thus, we have, on the one hand,

$$\begin{array}{rl} & \mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi')(B) \\ = & \mathrm{eval}(\mathcal{S} \uplus (({}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id} \uplus \sigma) \circ \mathrm{eval}(\mathcal{S} \uplus \delta)))(B) \\ = & \mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) \\ = & \mathsf{FALSE}, \end{array}$$

where the second equation holds by the SUBSTITUTION [VALUE] LEMMA.

Moreover, on the other hand, we have

$$\begin{array}{rl} & \mathrm{eval}(\mathcal{S} \uplus \delta' \uplus {}_{\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\chi)(B) \\ = & \mathrm{eval}(\mathcal{S} \uplus \delta)(B) \\ = & \mathsf{TRUE}. \end{array}$$

This means that a value (such as $\chi(v_l^{\mathbb{B}})$) could have been chosen for $f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))$ to make B true in $\mathcal{S} \uplus \delta' \uplus \chi'$, but it was not. This contradicts the definition of f.

Q.e.d. (Lemma 5.16)

Proof of Lemma 5.19
Let $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots\lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ for a formula B. Set $\sigma := \{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}})\cdots(v_{l-1}^{\mathbb{B}})\}$. Then we have $Q_C(y^{\mathbb{V}}) = \forall v_0^{\mathbb{B}}.\ \ldots\forall v_{l-1}^{\mathbb{B}}.\ \big(\ \exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\sigma\ \big)$. Let π be $\mathcal{S}$-compatible with $(C,(P,N))$; namely, in the case of item 1, the π mentioned in the lemma, or, in the case of item 2, the π that exists according to Lemma 5.16. Let $\tau : \mathbb{A} \to \mathcal{S}$ be arbitrary. It now suffices to show $\mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau)(Q_C(y^{\mathbb{V}})) = \mathsf{TRUE}$. By the backward direction of the $\forall$-Lemma, it suffices to show $\mathrm{eval}(\mathcal{S} \uplus \delta)(\exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\sigma) = \mathsf{TRUE}$ for an arbitrary $\chi : \{v_0^{\mathbb{B}},\ldots,v_{l-1}^{\mathbb{B}}\} \to \mathcal{S}$, setting $\delta := \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$. By the backward direction of the $\Rightarrow$-Lemma, it suffices to show $\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) = \mathsf{TRUE}$ under the assumption of $\mathrm{eval}(\mathcal{S} \uplus \delta)(\exists v_l^{\mathbb{B}}.\ B) = \mathsf{TRUE}$. From the latter, by the forward direction of the $\exists$-Lemma, there is a $\chi' : \{v_l^{\mathbb{B}}\} \to \mathcal{S}$ such that $\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi')(B) = \mathsf{TRUE}$. By item 2 of Definition 5.15, we get $\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi')(B\sigma) = \mathsf{TRUE}$. By the Explicitness Lemma, we get $\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) = \mathsf{TRUE}$. **Q.e.d. (Lemma 5.19)**

Proof of Lemma 5.21
Let us assume that π is $\mathcal{S}$-compatible with $(C',(P',N'))$. Then, by item 1 of Definition 5.15, $\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$ is an $\mathcal{S}$-raising-valuation and $(P' \cup S_\pi, N')$ is consistent. As (P',N') is an extension of (P,N), we have $P \subseteq P'$ and $N \subseteq N'$. Thus, $(P' \cup S_\pi, N')$ is an extension of $(P \cup S_\pi, N)$. Thus, $(P \cup S_\pi, N)$ is consistent by Corollary 5.7. For π to be $\mathcal{S}$-compatible with $(C,(P,N))$, it now suffices to show item 2 of Definition 5.15. As this property does not depend on the positive/negative variable-conditions anymore, it suffices to note that it actually holds because it holds for C' by assumption and we also have $C \subseteq C'$ by assumption. **Q.e.d. (Lemma 5.21)**

Proof of Lemma 5.23

By assumption, $(C',(P',N'))$ is the extended σ-update of $(C,(P,N))$. Thus, (P',N') is the σ-update of (P,N). Thus, because σ is a (P,N)-substitution, (P',N') is a consistent positive/negative variable-condition by Definition 5.9. Moreover, C is a (P,N)-choice-condition. Thus, C is a partial function from $\mathbb{V}$ into the set of higher-order ε-terms, such that Items 1, 2, and 3 of Definition 5.13 hold. Thus, C' is a partial function from $\mathbb{V}$ into the set of higher-order ε-terms satisfying items 1 and 2 of Definition 5.13 as well. For C' to satisfy also item 3 of Definition 5.13, it now suffices to show the following Claim 1.

Claim 1: Let $y^{\mathbb{V}} \in \mathrm{dom}(C')$ and $z^{\mathbb{VA}} \in \mathbb{VA}(C'(y^{\mathbb{V}}))$. Then we have $z^{\mathbb{VA}}\,(P')^+\,y^{\mathbb{V}}$.

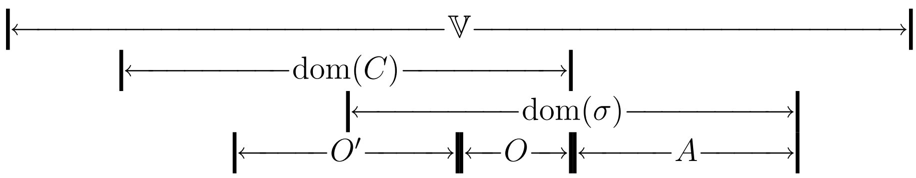
Proof of Claim 1: By the definition of C', we have $z^{\mathbb{VA}} \in \mathbb{VA}(C(y^{\mathbb{V}}))$ or else there is some $x^{\mathbb{V}} \in \mathrm{dom}(\sigma) \cap \mathbb{V}(C(y^{\mathbb{V}}))$ with $z^{\mathbb{VA}} \in \mathbb{VA}(\sigma(x^{\mathbb{V}}))$. Thus, as C is a (P,N)-choice-condition, we have either $z^{\mathbb{VA}}\,P^+\,y^{\mathbb{V}}$ or else $x^{\mathbb{V}}\,P^+\,y^{\mathbb{V}}$ and $z^{\mathbb{VA}} \in \mathbb{VA}(\sigma(x^{\mathbb{V}}))$. Then, as (P',N') is the σ-update of (P,N), by Definition 5.8, we have either $z^{\mathbb{VA}}\,(P')^+\,y^{\mathbb{V}}$ or else $x^{\mathbb{V}}\,(P')^+\,y^{\mathbb{V}}$ and $z^{\mathbb{VA}}\,P'\,x^{\mathbb{V}}$. Thus, in any case, $z^{\mathbb{VA}}\,(P')^+\,y^{\mathbb{V}}$. Q.e.d. (Claim 1)

Q.e.d. (Lemma 5.23)

Proof of Lemma 5.24

Let us assume the situation described in the lemma.

We set $A := \mathrm{dom}(\sigma)\setminus(O'\uplus O)$. As σ is a substitution on $\mathbb{V}$, we have $\mathrm{dom}(\sigma) \subseteq O'\uplus O\uplus A \subseteq \mathbb{V}$.



Note that C' is a (P',N')-choice-condition by Lemma 5.23.

As π' is $\mathcal{S}$-compatible with $(C',(P',N'))$, we know that $(P'\cup S_{\pi'},N')$ s a consistent positive/negative variable-condition. Thus, $\lhd := (P'\cup S_{\pi'})^{+}$ is a well-founded ordering.

Let D be the dependence relation of σ. Set $S_\pi := {}_{\mathbb{A}}{\upharpoonright}{\lhd}$.

<u>Claim 1:</u> We have $P', S_{\pi'}, P, D, S_\pi \subseteq \lhd$ and $(P'\cup S_{\pi'},N')$ is a weak extension of $(P\cup S_\pi,N)$ and of $(\lhd,N)$ (cf. Definition 5.6).

<u>Proof of Claim 1:</u> As (P',N') is the σ-update of (P,N), we have $P' = P\cup D$ and $N' = N$. Thus, $P', S_{\pi'}, P, D, S_\pi \subseteq (P'\cup S_{\pi'})^{+} = \lhd$. <u>Q.e.d. (Claim 1)</u>

<u>Claim 2:</u> $(P\cup S_\pi,N)$ and $(\lhd,N)$ are consistent positive/negative variable-conditions.

<u>Proof of Claim 2:</u> This follows from Claim 1 by Corollary 5.7. <u>Q.e.d. (Claim 2)</u>

<u>Claim 3:</u> ${}_{O'}{\upharpoonright}C$ is an $(\lhd,N)$-choice-condition.

<u>Proof of Claim 3:</u> By Claims 1 and 2 and the assumption that C is a (P,N)-choice-condition. <u>Q.e.d. (Claim 3)</u>

The plan for defining the $\mathcal{S}$-raising-valuation π (which we have to find) is to give $\pi(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau)$ a value as follows:

(α) For $y^{\mathbb{V}} \in \mathbb{V}\setminus(O'\uplus O\uplus A)$, we take this value to be
$$\pi'(y^{\mathbb{V}})({}_{S_{\pi'}\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau).$$
This is indeed possible because of $S_{\pi'} \subseteq {}_{\mathbb{A}}{\upharpoonright}{\lhd} = S_\pi$, so ${}_{S_{\pi'}\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau \subseteq {}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau$.

(β) For $y^{\mathbb{V}} \in O\uplus A$, we take this value to be
$$\mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi')(\tau) \uplus \tau)(\sigma(y^{\mathbb{V}})).$$
Note that, in case of $y^{\mathbb{V}} \in O$, we know that $(Q_C(y^{\mathbb{V}}))\sigma$ is $(\pi',\mathcal{S})$-valid by assumption of the lemma. Moreover, the case of $y^{\mathbb{V}} \in A$ is unproblematic because of $y^{\mathbb{V}} \notin \mathrm{dom}(C)$. Again, π is well-defined in this case because the only part of τ that is accessed by the given value is ${}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau$. Indeed, this can be seen as follows: By Claim 1 and the transitivity of $\lhd$, we have: ${}_{\mathbb{A}}{\upharpoonright}D \cup S_{\pi'}{\circ}D \subseteq {}_{\mathbb{A}}{\upharpoonright}{\lhd} = S_\pi$.

(γ) For $y^{\mathbb{V}} \in O'$, however, we have to take care of $\mathcal{S}$-compatibility with $(C,(P,N))$ explicitly in an $\lhd$-recursive definition on the basis a function ρ implementing (α) and (β). This disturbance does not interfere with the semantic invariance stated in the lemma because occurrences of variables from O' are explicitly excluded in the relevant terms and formulas and, according to the statement of lemma, O' satisfies the appropriate closure condition.

Set $S_\rho := S_\pi$. Let ρ be defined by $(y^{\mathbb{V}} \in \mathbb{V},\ \tau : \mathbb{A} \to \mathcal{S})$

$$\rho(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) := \begin{cases} \pi'(y^{\mathbb{V}})({}_{S_{\pi'}\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) & \text{if } y^{\mathbb{V}} \in \mathbb{V}\backslash(O\uplus A) \\ \mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi')(\tau) \uplus \tau)(\sigma(y^{\mathbb{V}})) & \text{if } y^{\mathbb{V}} \in O\uplus A \end{cases}$$

Let π be the $\mathcal{S}$-raising-valuation that exists according to Lemma 5.16 for the $\mathcal{S}$-raising-valuation ρ and the $(\lhd, N)$-choice-condition ${}_{O'}{\upharpoonleft}C$ (cf. Claim 3). Note that the assumptions of Lemma 5.16 are indeed satisfied here and that the resulting semantic relation S_π of Lemma 5.16 is indeed identical to our pre-defined relation of the same name, thereby justifying our abuse of notation: Indeed, by assumption of Lemma 5.24, for every return type α of ${}_{O'}{\upharpoonleft}C$, there is a generalized choice function on the power-set of the universe of $\mathcal{S}$ for the type α; and we have

$$S_\rho \;=\; S_\pi \;=\; {}_{\mathbb{A}}{\upharpoonleft}\lhd \;=\; {}_{\mathbb{A}}{\upharpoonleft}(\lhd^+).$$

Because of $\mathrm{dom}({}_{O'}{\upharpoonleft}C) = O'$, according to Lemma 5.16, we then have

$${}_{\mathbb{V}\backslash O'}{\upharpoonleft}\pi \;=\; {}_{\mathbb{V}\backslash O'}{\upharpoonleft}\rho$$

and π is $\mathcal{S}$-compatible with $({}_{O'}{\upharpoonleft}C, (\lhd, N))$.

<u>Claim 4:</u> For all $y^{\mathbb{V}} \in O\uplus A$ and $\tau : \mathbb{A} \to \mathcal{S}$, when we set $\delta' := \mathsf{e}(\pi')(\tau) \uplus \tau$:

$$\mathsf{e}(\pi)(\tau)(y^{\mathbb{V}}) = \mathrm{eval}(\mathcal{S} \uplus \delta')(\sigma(y^{\mathbb{V}})).$$

<u>Proof of Claim 4:</u> We have $O\uplus A \subseteq \mathbb{V}\backslash O'$. Thus, Claim 4 follows immediately from the definition of ρ. <u>Q.e.d. (Claim 4)</u>

<u>Claim 5:</u> For all $y^{\mathbb{V}} \in \mathbb{V}\backslash(O'\uplus O\uplus A)$ and $\tau : \mathbb{A} \to \mathcal{S}$: $\mathsf{e}(\pi)(\tau)(y^{\mathbb{V}}) = \mathsf{e}(\pi')(\tau)(y^{\mathbb{V}})$.

<u>Proof of Claim 5:</u> For $y^{\mathbb{V}} \in \mathbb{V}\backslash(O'\uplus O\uplus A)$, we have $y^{\mathbb{V}} \in \mathbb{V}\backslash O'$ and $y^{\mathbb{V}} \in \mathbb{V}\backslash(O\uplus A)$. Thus, $\mathsf{e}(\pi)(\tau)(y^{\mathbb{V}}) = \pi(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) = \rho(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) = \pi'(y^{\mathbb{V}})({}_{S_{\pi'}\langle\{y^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) = \mathsf{e}(\pi')(\tau)(y^{\mathbb{V}})$. <u>Q.e.d. (Claim 5)</u>

<u>Claim 6:</u> For any term or formula B (possibly with some unbound occurrences of bound atoms from the set $W \subseteq \mathbb{B}$) with $O' \cap \mathbb{V}(B) = \emptyset$, and for every $\tau : \mathbb{A} \to \mathcal{S}$ and every $\chi : W \to \mathcal{S}$, when we set $\delta := \mathsf{e}(\pi)(\tau) \uplus \tau$ and $\delta' := \mathsf{e}(\pi')(\tau) \uplus \tau$, we have

$$\mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi)(B\sigma) = \mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi)(B).$$

<u>Proof of Claim 6:</u> $\mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi)(B\sigma) =$ (by the SUBSTITUTION [VALUE] LEMMA)
$\mathrm{eval}(\mathcal{S} \uplus (\sigma \uplus {}_{\mathbb{VAB}\backslash\mathrm{dom}(\sigma)}{\upharpoonleft}\mathrm{id}) \circ \mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi))(B) =$
(by the EXPLICITNESS LEMMA and the VALUATION LEMMA (for the case of $l = 0$))
$\mathrm{eval}(\mathcal{S} \uplus (\sigma \circ \mathrm{eval}(\mathcal{S} \uplus \delta')) \uplus {}_{\mathbb{VA}\backslash\mathrm{dom}(\sigma)}{\upharpoonleft}\delta' \uplus \chi)(B) =$
(by $O\uplus A \subseteq \mathrm{dom}(\sigma) \subseteq O'\uplus O\uplus A$, $O'\cap\mathbb{V}(B) = \emptyset$, and the EXPLICITNESS LEMMA)
$\mathrm{eval}(\mathcal{S} \uplus {}_{O\uplus A}{\upharpoonleft}\sigma \circ \mathrm{eval}(\mathcal{S} \uplus \delta') \uplus {}_{\mathbb{VA}\backslash(O'\uplus O\uplus A)}{\upharpoonleft}\delta' \uplus \chi)(B) =$ (by Claim 4 and Claim 5)
$\mathrm{eval}(\mathcal{S} \uplus {}_{O\uplus A}{\upharpoonleft}\delta \uplus {}_{\mathbb{VA}\backslash(O'\uplus O\uplus A)}{\upharpoonleft}\delta \uplus \chi)(B) =$
(by $O'\cap\mathbb{V}(B) = \emptyset$ and the EXPLICITNESS LEMMA)
$\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi)(B)$. <u>Q.e.d. (Claim 6)</u>

<u>Claim 7:</u> For every set of sequents G' (possibly with some unbound occurrences of bound atoms from the set $W \subseteq \mathbb{B}$) with $O' \cap \mathbb{V}(G') = \emptyset$, and for every $\tau : \mathbb{A} \to \mathcal{S}$ and for every $\chi : W \to \mathcal{S}$: Truth of G' in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$ is equivalent to truth of $G'\sigma$ in $\mathcal{S} \uplus \mathsf{e}(\pi')(\tau) \uplus \tau \uplus \chi$.

Proof of Claim 7: This is a trivial consequence of Claim 6. Q.e.d. (Claim 7)

Claim 8: For $y^{\mathrm{v}} \in \mathrm{dom}(C) \setminus O'$, we have $O' \cap \mathbb{V}(C(y^{\mathrm{v}})) = \emptyset$.

Proof of Claim 8: Otherwise there is some $y^{\mathrm{v}} \in \mathrm{dom}(C) \setminus O'$ and some $z^{\mathrm{v}} \in O' \cap \mathbb{V}(C(y^{\mathrm{v}}))$. Then $z^{\mathrm{v}} P^{+} y^{\mathrm{v}}$ because C is a (P, N)-choice-condition, and then, as $\langle O' \rangle P^{+} \cap \mathrm{dom}(C) \subseteq O'$ by assumption of the lemma, we have the contradicting $y^{\mathrm{v}} \in O'$. Q.e.d. (Claim 8)

Claim 9: Let $y^{\mathrm{v}} \in \mathrm{dom}(C)$ and $C(y^{\mathrm{v}}) = \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$. Let $\tau : \mathbb{A} \to \mathcal{S}$ and $\chi : \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\} \to \mathcal{S}$. Set $\delta := \mathsf{e}(\pi)(\tau) \uplus \tau \uplus \chi$. Set $\mu := \{v_l^{\mathbb{B}} \mapsto y^{\mathrm{v}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$. If B is true in $\mathcal{S} \uplus \delta$, then $B\mu$ is true in $\mathcal{S} \uplus \delta$ as well.

Proof of Claim 9: Set $\delta' := \mathsf{e}(\pi')(\tau) \uplus \tau \uplus \chi$.

$y^{\mathrm{v}} \notin O' \uplus O$: In this case, because of $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) \subseteq O' \uplus O$, we have $y^{\mathrm{v}} \notin \mathrm{dom}(\sigma)$. Thus, as $(C', (P', N'))$ is the extended σ-update of $(C, (P, N))$, we have $C'(y^{\mathrm{v}}) = (C(y^{\mathrm{v}}))\sigma$. By Claim 8, we have $O' \cap \mathbb{V}(B) = \emptyset$.
And then, by our case assumption, also $O' \cap \mathbb{V}(B\mu) = \emptyset$.
By assumption of Claim 9, B is true in $\mathcal{S} \uplus \delta$. Thus, by Claim 7, $B\sigma$ is true in $\mathcal{S} \uplus \delta'$. Thus, as π' is $\mathcal{S}$-compatible with $(C', (P', N'))$, we know that $(B\sigma)\mu$ is true in $\mathcal{S} \uplus \delta'$. Because of $y^{\mathrm{v}} \notin \mathrm{dom}(\sigma)$, this means that $(B\mu)\sigma$ is true in $\mathcal{S} \uplus \delta'$. Thus, by Claim 7, $B\mu$ is true in $\mathcal{S} \uplus \delta$.

$y^{\mathrm{v}} \in O$: By Claim 8, we have $O' \cap \mathbb{V}(B) = \emptyset$.
And then, by our case assumption, also $O' \cap \mathbb{V}(B\mu) = \emptyset$.
Moreover, $(Q_C(y^{\mathrm{v}}))\sigma$ is equal to $\forall v_0^{\mathbb{B}}.\ \ldots \forall v_{l-1}^{\mathbb{B}}.\ (\ \exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\mu\ )\sigma$ and $(\pi', \mathcal{S})$-valid by assumption of the lemma. Thus, by the forward direction of the $\forall$-Lemma, $(\ \exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\mu\ )\sigma$ is true in $\mathcal{S} \uplus \delta'$. Thus, by Claim 7, $\exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\mu$ is true in $\mathcal{S} \uplus \delta$. As, by assumption of Claim 9, B is true in $\mathcal{S} \uplus \delta$, by the backward direction of the $\exists$-Lemma, $\exists v_l^{\mathbb{B}}.\ B$ is true in $\mathcal{S} \uplus \delta$ as well. Thus, by the forward direction of the $\Rightarrow$-Lemma, $B\mu$ is true in $\mathcal{S} \uplus \delta$ as well.

$y^{\mathrm{v}} \in O'$: π is $\mathcal{S}$-compatible with $({}_{O'}{\upharpoonright} C, (\lhd, N))$ by definition, as explicitly stated before Claim 4. Q.e.d. (Claim 9)

By Claims 2 and 9, π is $\mathcal{S}$-compatible with $(C, (P, N))$. And then items 1 and 2 of the lemma are trivial consequences of Claims 6 and 7, respectively.

Q.e.d. (Lemma 5.24)

Proof of Theorem 5.26

The first four items are trivial (Validity, Reflexivity, Transitivity, Additivity).

(5a): If G_0 is $(C', (P', N'))$-valid in $\mathcal{S}$, then there is some π that is $\mathcal{S}$-compatible with $(C', (P', N'))$ such that G_0 is $(\pi, \mathcal{S})$-valid. By Lemma 5.21, π is also $\mathcal{S}$-compatible with $(C, (P, N))$. Thus, G_0 is $(C, (P, N))$-valid, in $\mathcal{S}$.

(5b): Suppose that π is $\mathcal{S}$-compatible with $(C', (P', N'))$, and that G_1 is $(\pi, \mathcal{S})$-valid. By Lemma 5.21, π is also $\mathcal{S}$-compatible with $(C, (P, N))$. Thus, since G_0 $(C, (P, N))$-reduces to G_1, also G_0 is $(\pi, \mathcal{S})$-valid as was to be shown.

(6): Assume the situation described in the lemma.

Claim 1: $O' \subseteq \mathrm{dom}(C) \setminus O$.

Proof of Claim 1: By definition, $O' \subseteq \mathrm{dom}(C)$. It remains to show $O' \cap O = \emptyset$. To the contrary, suppose that there is some $y^{\mathbb{V}} \in O' \cap O$. Then, by the definition of O', there is some $z^{\mathbb{V}} \in M \backslash O$ with $z^{\mathbb{V}}\, P^*\, y^{\mathbb{V}}$. By definition of O, however, we have $y^{\mathbb{V}} \in P^*\langle V\rangle$. Thus, $z^{\mathbb{V}} \in P^*\langle V\rangle$. Thus, $z^{\mathbb{V}} \in O$, a contradiction. Q.e.d. (Claim 1)

Claim 2: $\langle O'\rangle P^+ \cap \mathrm{dom}(C) \subseteq O'$.

Proof of Claim 2: Assume $y^{\mathbb{V}} \in O'$ and $z^{\mathbb{V}} \in \mathrm{dom}(C)$ with $y^{\mathbb{V}}\, P^+\, z^{\mathbb{V}}$. It now suffices to show $z^{\mathbb{V}} \in O'$. Because of $y^{\mathbb{V}} \in O'$, there is some $x^{\mathbb{V}} \in M \backslash O$ with $x^{\mathbb{V}}\, P^*\, y^{\mathbb{V}}$. Thus, $x^{\mathbb{V}}\, P^*\, z^{\mathbb{V}}$. Thus, $z^{\mathbb{V}} \in O'$. Q.e.d. (Claim 2)

Claim 3: $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) \subseteq O' \cup O$.

Proof of Claim 3: $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) = \mathrm{dom}(C) \cap M \subseteq O \cup (\mathrm{dom}(C) \cap (M \backslash O)) \subseteq O \cup (\mathrm{dom}(C) \cap \langle M \backslash O\rangle P^*) = O \cup O'$. Q.e.d. (Claim 3)

Claim 4: $O' \cap \mathbb{V}(G_0, G_1) = O' \cap V = \emptyset$.

Proof of Claim 4: Because of $\mathbb{V}(G_0, G_1) \subseteq V$, it suffices to show the second equality. To the contrary of the second equality, suppose that there is some $y^{\mathbb{V}} \in O' \cap V$. Then, by the definition of O', there is some $z^{\mathbb{V}} \in M \backslash O$ with $z^{\mathbb{V}}\, P^*\, y^{\mathbb{V}}$. By definition of O, however, we have $z^{\mathbb{V}} \in O$, a contradiction. Q.e.d. (Claim 4)

(6a): In case that $G_0\sigma \cup (\langle O\rangle Q_C)\sigma$ is $(C', (P', N'))$-valid in $\mathcal{S}$, there is some π' that is $\mathcal{S}$-compatible with $(C', (P', N'))$ such that $G_0\sigma \cup (\langle O\rangle Q_C)\sigma$ is $(\pi', \mathcal{S})$-valid. Then both $G_0\sigma$ and $(\langle O\rangle Q_C)\sigma$ are $(\pi', \mathcal{S})$-valid. By Claims 1, 2, 3, and 4, let π be given as in Lemma 5.24. Then G_0 is $(\pi, \mathcal{S})$-valid. Moreover, as π is $\mathcal{S}$-compatible with $(C, (P, N))$, G_0 is $(C, (P, N))$-valid in $\mathcal{S}$.

(6b): Let π' be $\mathcal{S}$-compatible with $(C', (P', N'))$, and suppose that $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$ is $(\pi', \mathcal{S})$-valid. Then both $G_1\sigma$ and $(\langle O\rangle Q_C)\sigma$ are $(\pi', \mathcal{S})$-valid. By Claims 1, 2, 3, and 4, let π be given as in Lemma 5.24. Then π is $\mathcal{S}$-compatible with $(C, (P, N))$, and G_1 is $(\pi, \mathcal{S})$-valid. By assumption, G_0 $(C, (P, N))$-reduces to G_1. Thus, G_0 is $(\pi, \mathcal{S})$-valid, too. Thus, by Lemma 5.24, $G_0\sigma$ is $(\pi', \mathcal{S})$-valid as was to be shown.

<u>(7):</u> Let π be $\mathcal{S}$-compatible with $(C,(P,N))$, and suppose that G_0 is $(\pi,\mathcal{S})$-valid. Let $\tau : \mathbb{A}\to\mathcal{S}$ be an arbitrary $\mathcal{S}$-valuation. Set $\delta := \mathsf{e}(\pi)(\tau)\uplus\tau$. It suffices to show $\mathrm{eval}(\mathcal{S}\uplus\delta)(G_0\nu) = \mathsf{TRUE}$.

Define $\tau' : \mathbb{A}\to\mathcal{S}$ via $\tau'(y^{\mathbb{A}}) := \left\{\begin{array}{ll}\tau(y^{\mathbb{A}}) & \text{for } y^{\mathbb{A}}\in\mathbb{A}\setminus\mathrm{dom}(\nu)\\ \mathrm{eval}(\mathcal{S}\uplus\delta)(\nu(y^{\mathbb{A}})) & \text{for } y^{\mathbb{A}}\in\mathrm{dom}(\nu)\end{array}\right\}$.

<u>Claim 5:</u> For $v^{\mathbb{V}}\in\mathbb{V}(G_0)$ we have $\mathsf{e}(\pi)(\tau)(v^{\mathbb{V}}) = \mathsf{e}(\pi)(\tau')(v^{\mathbb{V}})$.

<u>Proof of Claim 5:</u> Otherwise there must be some $y^{\mathbb{A}}\in\mathrm{dom}(\nu)$ with $y^{\mathbb{A}}\ S_\pi\ v^{\mathbb{V}}$. Because of $v^{\mathbb{V}}\in\mathbb{V}(G_0)$ and $\mathbb{V}(G_0)\times\mathrm{dom}(\nu)\subseteq N$, we have $v^{\mathbb{V}}\ N\ y^{\mathbb{A}}$. But then $(P\cup S_\pi, N)$ is not consistent, which contradicts π being $\mathcal{S}$-compatible with $(C,(P,N))$. <u>Q.e.d. (Claim 5)</u>

Then we get by the SUBSTITUTION [VALUE] LEMMA (1st equation), the VALUATION LEMMA (for the case of $l=0$) (2nd equation), by definition of τ' and δ (3rd equation), by the EXPLICITNESS LEMMA and Claim 5 (4th equation), and by the $(\pi,\mathcal{S})$-validity of G_0 (5th equation):

$$\begin{array}{rcl}\mathrm{eval}(\mathcal{S}\uplus\delta)(G_0\nu) &=& \mathrm{eval}\Big(\ \mathcal{S}\ \uplus\ \big(\ (\ \nu\ \uplus\ {}_{\mathbb{V}\mathbb{A}\setminus\mathrm{dom}(\nu)}{\upharpoonleft}\mathrm{id}\ )\ \circ\ \mathrm{eval}(\mathcal{S}\uplus\delta)\ \big)\ \Big)\Big(\ G_0\ \Big)\\ &=& \mathrm{eval}\Big(\ \mathcal{S}\ \uplus\ \big(\ \nu\ \circ\ \mathrm{eval}(\mathcal{S}\uplus\delta)\ \big)\ \uplus\ {}_{\mathbb{V}\mathbb{A}\setminus\mathrm{dom}(\nu)}{\upharpoonleft}\delta\ \Big)\Big(\ G_0\ \Big)\\ &=& \mathrm{eval}\Big(\ \mathcal{S}\ \uplus\ \tau'\ \uplus\ \mathsf{e}(\pi)(\tau)\ \Big)\Big(\ G_0\ \Big)\\ &=& \mathrm{eval}\Big(\ \mathcal{S}\ \uplus\ \tau'\ \uplus\ \mathsf{e}(\pi)(\tau')\ \Big)\Big(\ G_0\ \Big)\\ &=& \mathsf{TRUE}\end{array}$$

Q.e.d. (Theorem 5.26)

Proof of Theorem 5.27

To illustrate our techniques, we only treat the first rule of each kind; the other rules can be treated most similarly. In the situation described in the theorem, it suffices to show that C' is a (P',N')-choice-condition (because the other properties of an extended extension are trivial), and that, for every $\mathcal{S}$-raising-valuation π that is $\mathcal{S}$-compatible with $(C',(P',N'))$, the sets G_0 and G_1 of the upper and lower sequents of the inference rule are equivalent w.r.t. their $(\pi,\mathcal{S})$-validity.

<u>γ-rule:</u> In this case we have $(C',(P',N')) = (C,(P,N))$. Thus, C' is a (P',N')-choice-condition by assumption of the theorem. Moreover, for every $\mathcal{S}$-valuation $\tau : \mathbb{A}\to\mathcal{S}$, and for $\delta := \mathsf{e}(\pi)(\tau)\uplus\tau$, the truths of

$$\{\varGamma\quad \exists y^{\mathbb{B}}.\ A\quad \varPi\}\quad\text{and}\quad \{A\{y^{\mathbb{B}}{\mapsto}t\}\quad \varGamma\quad \exists y^{\mathbb{B}}.\ A\quad \varPi\}$$

in $\mathcal{S}{\uplus}\delta$ are indeed equivalent. The implication from left to right is trivial because the former sequent is a sub-sequent of the latter.

For the other direction, assume that $A\{y^{\mathbb{B}}{\mapsto}t\}$ is true in $\mathcal{S}{\uplus}\delta$. Thus, by the SUBSTITUTION [VALUE] LEMMA (second equation) and the VALUATION LEMMA for $l=0$ (third equation):

$$\begin{array}{rcl}\mathsf{TRUE} &=& \mathrm{eval}(\mathcal{S}\uplus\delta)(A\{y^{\mathbb{B}}{\mapsto}t\})\\ &=& \mathrm{eval}(\mathcal{S}\uplus((\{y^{\mathbb{B}}{\mapsto}t\}\uplus{}_{\mathbb{V}\mathbb{A}\mathbb{B}\setminus\{y^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id})\circ\mathrm{eval}(\mathcal{S}{\uplus}\delta)))(A)\\ &=& \mathrm{eval}(\mathcal{S}\uplus\{y^{\mathbb{B}}{\mapsto}\mathrm{eval}(\mathcal{S}{\uplus}\delta)(t)\}\uplus\delta)(A)\end{array}$$

Thus, by the backward direction of the $\exists$-LEMMA, $\exists y^{\mathbb{B}}.\ A$ is true in $\mathcal{S}{\uplus}\delta$. Thus, the upper sequent is true $\mathcal{S}{\uplus}\delta$.

δ⁻-rule: In this case, we have $x^{\mathbb{A}} \in \mathbb{A} \setminus (\mathrm{dom}(P) \cup \mathbb{A}(\mathit{\Gamma}, A, \mathit{\Pi}))$, $C'' = \emptyset$, and $V = \mathbb{V}(\mathit{\Gamma}\ \ \forall x^{\mathbb{B}}.\, A\ \ \mathit{\Pi}) \times \{x^{\mathbb{A}}\}$. Thus, $C' = C$, $P' = P$, and $N' = N \cup V$.

Claim 1: C' is a (P', N')-choice-condition.

Proof of Claim 1: By assumption of the theorem, C is a (P, N)-choice-condition. Thus, (P, N) is a consistent positive/negative variable-condition. By Definition 5.4, P is well-founded and $P^{+}\circ N$ is irreflexive. Since $x^{\mathbb{A}} \notin \mathrm{dom}(P)$, we have $x^{\mathbb{A}} \notin \mathrm{dom}(P^{+})$. Thus, because of $\mathrm{ran}(V) = \{x^{\mathbb{A}}\}$, also $P^{+}\circ N'$ is irreflexive. Thus, (P', N') is a consistent positive/negative variable-condition, and C' is a (P', N')-choice-condition. Q.e.d. (Claim 1)

Now, for the soundness direction, it suffices to show the contrapositive, namely to assume that there is an $\mathcal{S}$-valuation $\tau : \mathbb{A} \to \mathcal{S}$ such that $\{\mathit{\Gamma}\ \ \forall x^{\mathbb{B}}.\, A\ \ \mathit{\Pi}\}$ is false in $\mathcal{S}\uplus\mathsf{e}(\pi)(\tau) \uplus \tau$, and to show that there is an $\mathcal{S}$-valuation $\tau' : \mathbb{A} \to \mathcal{S}$ such that $\{A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\}\ \ \mathit{\Gamma}\ \ \mathit{\Pi}\}$ is false in $\mathcal{S}\uplus\mathsf{e}(\pi)(\tau')\uplus\tau'$. Under this assumption, the sequent $\mathit{\Gamma\Pi}$ is false in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau)\uplus\tau$.

Claim 2: $\mathit{\Gamma\Pi}$ is false in $\mathcal{S}\uplus\mathsf{e}(\pi)(\tau')\uplus\tau'$ for all $\tau' : \mathbb{A} \to \mathcal{S}$ with ${}_{\mathbb{A}\setminus\{x^{\mathbb{A}}\}}{\upharpoonleft}\tau' = {}_{\mathbb{A}\setminus\{x^{\mathbb{A}}\}}{\upharpoonleft}\tau$.

Proof of Claim 2: Because of $x^{\mathbb{A}} \notin \mathbb{A}(\mathit{\Gamma\Pi})$, by the Explicitness Lemma, if Claim 2 did not hold, there would have to be some $u^{\mathbb{V}} \in \mathbb{V}(\mathit{\Gamma\Pi})$ with $x^{\mathbb{A}}\ S_{\pi}\ u^{\mathbb{V}}$. Then we have $u^{\mathbb{V}}\ N'\ x^{\mathbb{A}}$. Thus, we know that $(P' \cup S_{\pi})^{+} \circ N'$ is not irreflexive, which contradicts π being $\mathcal{S}$-compatible with $(C', (P', N'))$. Q.e.d. (Claim 2)

Moreover, under the above assumption, also $\forall x^{\mathbb{B}}.\, A$ is false in $\mathcal{S}\uplus\mathsf{e}(\pi)(\tau)\uplus\tau$. By the backward direction of the ∀-Lemma, this means that there is some object o such that A is false in $\mathcal{S}\uplus\{x^{\mathbb{B}}{\mapsto}o\}\uplus\mathsf{e}(\pi)(\tau)\uplus\tau$. Set $\tau' := {}_{\mathbb{A}\setminus\{x^{\mathbb{A}}\}}{\upharpoonleft}\tau \uplus \{x^{\mathbb{A}}{\mapsto}o\}$. Then, by the Substitution [Value] Lemma (1st equation), by the Valuation Lemma (for $l = 0$) (2nd equation), and by the Explicitness Lemma and $x^{\mathbb{A}} \notin \mathbb{A}(A)$ (3rd equation), we have:

$$\begin{aligned}
\mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau')(A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\}) &= \\
\mathrm{eval}(\mathcal{S} \uplus ((\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id}) \circ \mathrm{eval}(\mathcal{S} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau')))(A) &= \\
\mathrm{eval}(\mathcal{S} \uplus \{x^{\mathbb{B}}{\mapsto}o\} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau')(A) &= \\
\mathrm{eval}(\mathcal{S} \uplus \{x^{\mathbb{B}}{\mapsto}o\} \uplus \mathsf{e}(\pi)(\tau) \uplus \tau)(A) &= \ \mathsf{FALSE}.
\end{aligned}$$

Claim 4: $A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\}$ is false in $\mathcal{S}\uplus\mathsf{e}(\pi)(\tau')\uplus\tau'$.

Proof of Claim 4: Otherwise, there must be some $u^{\mathbb{V}} \in \mathbb{V}(A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\})$ with $x^{\mathbb{A}}\ S_{\pi}\ u^{\mathbb{V}}$. Then we have $u^{\mathbb{V}}\ N'\ x^{\mathbb{A}}$. Thus, we know that $(P' \cup S_{\pi})^{+} \circ N'$ is not irreflexive, which contradicts π being $\mathcal{S}$-compatible with $(C', (P', N'))$. Q.e.d. (Claim 4)

By the Claims 4 and 2, $\{A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\}\ \ \mathit{\Gamma}\ \ \mathit{\Pi}\}$ is false in $\mathcal{S} \uplus \mathsf{e}(\pi)(\tau') \uplus \tau'$, as was to be show for the soundness direction of the proof.

Finally, for the safeness direction, assume that the sequent $\mathit{\Gamma}\ \ \forall x^{\mathbb{B}}.\, A\ \ \mathit{\Pi}$ is $(\pi, \mathcal{S})$-valid. For arbitrary $\tau : \mathbb{A} \to \mathcal{S}$, we have to show that the lower sequent $A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\}\ \mathit{\Gamma}\ \mathit{\Pi}$ is true in $\mathcal{S}\uplus\delta$ for $\delta := \mathsf{e}(\pi)(\tau) \uplus \tau$. If some formula in $\mathit{\Gamma\Pi}$ is true in $\mathcal{S}\uplus\delta$, then the lower sequent is true in $\mathcal{S}\uplus\delta$ as well. Otherwise, $\forall x^{\mathbb{B}}.\, A$ is true in $\mathcal{S}\uplus\delta$. Then, by the forward direction of the ∀-Lemma, this means that A is true in $\mathcal{S}\uplus\chi\uplus\delta$ for all $\mathcal{S}$-valuations $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$. Then, by the Substitution [Value] Lemma (1st equation), and by the Valuation Lemma (for $l = 0$) (2nd equation), we have:

$$\begin{aligned}
\mathrm{eval}(\mathcal{S} \uplus \delta)(A\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\}) &= \\
\mathrm{eval}(\mathcal{S} \uplus ((\{x^{\mathbb{B}}{\mapsto}x^{\mathbb{A}}\} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id}) \circ \mathrm{eval}(\mathcal{S} \uplus \delta)))(A) &= \\
\mathrm{eval}(\mathcal{S} \uplus \{x^{\mathbb{B}}{\mapsto}\delta(x^{\mathbb{A}})\} \uplus \delta)(A) &= \ \mathsf{TRUE}.
\end{aligned}$$

δ^{+}-rule: In this case, we have $x^{\mathrm{V}} \in \mathbb{V} \setminus (\mathrm{dom}(C \cup P \cup N) \cup \mathbb{V}(A))$,
$C'' = \{(x^{\mathrm{V}}, \varepsilon x^{\mathrm{B}}.\ \neg A)\}$, and $V = \mathbb{VA}(\forall x^{\mathrm{B}}.\ A) \times \{x^{\mathrm{V}}\} = \mathbb{VA}(A) \times \{x^{\mathrm{V}}\}$.
Thus, $C' = C \cup \{(x^{\mathrm{V}}, \varepsilon x^{\mathrm{B}}.\ \neg A)\}$, $P' = P \cup V$, and $N' = N$.
By assumption of the theorem, C is a (P, N)-choice-condition. Thus, (P, N) is a consistent positive/negative variable-condition. Thus, by Definition 5.4, P is well-founded and $P^{+}\circ N$ is irreflexive.

Claim 5: P' is well-founded.
Proof of Claim 5: Let B be a non-empty class. We have to show that there is a P'-minimal element in B. Because P is well-founded, there is some P-minimal element in B. If this element is V-minimal in B, then it is a P'-minimal element in B. Otherwise, this element must be x^{V} and there is an element $n^{\mathrm{VA}} \in B \cap \mathbb{VA}(A)$. Set $B' := \{\ b^{\mathrm{VA}} \in B \mid b^{\mathrm{VA}}\ P^{*}\ n^{\mathrm{VA}}\ \}$. Because of $n^{\mathrm{VA}} \in B'$, we know that B' is a non-empty subset of B. Because P is well-founded, there is some P-minimal element m^{VA} in B'. Then m^{VA} is also a P-minimal element in B. Because of $x^{\mathrm{V}} \notin \mathbb{VA}(A) \cup \mathrm{dom}(P)$, we know that $x^{\mathrm{V}} \notin B'$. Thus, $m^{\mathrm{VA}} \neq x^{\mathrm{V}}$. Thus, m^{VA} is also a V-minimal element of B. Thus, m^{VA} is also a P'-minimal element of B. Q.e.d. (Claim 5)

Claim 6: $(P')^{+} \circ N'$ is irreflexive.
Proof of Claim 6: Suppose the contrary. Because $P^{+}\circ N$ is irreflexive, $P^{*}\circ(V \circ P^{*})^{+}\circ N$ must be reflexive. Because of $\mathrm{ran}(V) = \{x^{\mathrm{V}}\}$ and $\{x^{\mathrm{V}}\} \cap \mathrm{dom}(P \cup N) = \emptyset$, we have $V\circ P = \emptyset$ and $V\circ N = \emptyset$. Thus, $P^{*}\circ(V \circ P^{*})^{+}\circ N = P^{*}\circ V^{+}\circ N = \emptyset$. Q.e.d. (Claim 6)

Claim 7: C' is a (P', N')-choice-condition.
Proof of Claim 7: By Claims 5 and 6, (P', N') is a consistent positive/negative variable-condition. As $x^{\mathrm{V}} \in \mathbb{V}\setminus\mathrm{dom}(C)$, we know that C' is a partial function on $\mathbb{V}$ just as C. Moreover, for $y^{\mathrm{V}} \in \mathrm{dom}(C')$, we either have $y^{\mathrm{V}} \in \mathrm{dom}(C)$ and then
$\mathbb{VA}(C'(y^{\mathrm{V}})) \times \{y^{\mathrm{V}}\} = \mathbb{VA}(C(y^{\mathrm{V}})) \times \{y^{\mathrm{V}}\} \subseteq P^{+} \subseteq (P')^{+}$, or $y^{\mathrm{V}} = x^{\mathrm{V}}$ and then
$\mathbb{VA}(C'(y^{\mathrm{V}})) \times \{y^{\mathrm{V}}\} = \mathbb{VA}(\varepsilon x^{\mathrm{B}}.\ \neg A) \times \{x^{\mathrm{V}}\} = V \subseteq P' \subseteq (P')^{+}$. Q.e.d. (Claim 7)

Now it suffices to show that, for each $\tau : \mathbb{A} \to \mathcal{S}$, and for $\delta := \mathsf{e}(\pi)(\tau) \uplus \tau$, the truth of $\{\Gamma\quad \forall x^{\mathrm{B}}.\ A\quad \Pi\}$ in $\mathcal{S} \uplus \delta$ is equivalent that of $\{A\{x^{\mathrm{B}}\mapsto x^{\mathrm{V}}\}\quad \Gamma\quad \Pi\}$.

For the soundness direction, it suffices to show that the former sequent is true in $\mathcal{S}\uplus\delta$ under the assumption that the latter is. If some formula in $\Gamma\Pi$ is true in $\mathcal{S}\uplus\delta$, then the former sequent is true in $\mathcal{S}\uplus\delta$ as well. Otherwise, this means that $A\{x^{\mathrm{B}}\mapsto x^{\mathrm{V}}\}$ is true in $\mathcal{S}\uplus\delta$. Then, by the forward direction of the $\neg$-Lemma, $\neg A\{x^{\mathrm{B}}\mapsto x^{\mathrm{V}}\}$ is false in $\mathcal{S}\uplus\delta$. By the Explicitness Lemma, $\neg A\{x^{\mathrm{B}}\mapsto x^{\mathrm{V}}\}$ is false in $\mathcal{S}\uplus\delta\uplus\chi$ for all $\chi : \{x^{\mathrm{B}}\} \to \mathcal{S}$. Because π is $\mathcal{S}$-compatible with $(C', (P', N'))$ and because of $C'(x^{\mathrm{V}}) = \varepsilon x^{\mathrm{B}}.\ \neg A$, by Item 2 of Definition 5.15, $\neg A$ is false in $\mathcal{S}\uplus\delta\uplus\chi$ for all $\chi : \{x^{\mathrm{B}}\} \to \mathcal{S}$. Then, by the backward direction of the $\neg$-Lemma, A is true in $\mathcal{S}\uplus\delta\uplus\chi$ for all $\chi : \{x^{\mathrm{B}}\} \to \mathcal{S}$. Then, by the backward direction of the $\forall$-Lemma, $\forall x^{\mathrm{B}}.\ A$ is true in $\mathcal{S}\uplus\delta$.

The safeness direction is perfectly analogous to the case of the δ^{-}-rule.

Q.e.d. (Theorem 5.27)

Acknowledgments

I would like to thank Murdoch J. Gabbay for inspiring and encouraging me to write this paper, for reviewing it, and for much more.

Notes

Note 1 (Bound vs. Bindable)
"Bound" atoms (or variables) should actually be called "bindable" instead of "bound", because we will always have to treat some unbound occurrences of "bound" atoms. When the name of the notion of bound variables was coined, however, neither "bindable" nor the German "bindbar" were considered to be proper words of their respective languages, cf. [Hilbert & Bernays, 2017b, § 4].

Note 2 (Principal Formulas, Side Formulas, and Parametric Formulas)
The notions of a *principal formula* (in German: Hauptformel) and a *side formula* (Seitenformel) were introduced in [Gentzen, 1935] and refined in [Schmidt-Samoa, 2006]. Very roughly speaking, the principal formula of an inference rule is the formula that is reduced by that rule, and the side formulas are the resulting pieces replacing the the principle formula. In our reductive inference rules here, the principal formulas are the formulas above the lines except the ones in $\mathit{\Gamma}$, $\mathit{\Pi}$ (which are called *parametric formulas*, in German: Nebenformeln), and the side formulas are the formulas below the lines except the ones in $\mathit{\Gamma}$, $\mathit{\Pi}$.

Note 3 (Are Liberalized δ-Rules Really More Liberal?)
We could object with the following two points to the classification of the δ^{+}-rules as being more "liberal" than the δ^{-}-rules:

- $\mathbb{V}\mathbb{A}(\forall x^{\mathbb{B}}.\,A)$ is not necessarily a subset of $\mathbb{V}(\mathit{\Gamma}\ \ \forall x^{\mathbb{B}}.\,A\ \ \mathit{\Pi})$, because $\mathbb{V}\mathbb{A}(\forall x^{\mathbb{B}}.\,A)$ may include some additional free atoms.
  First note that δ^{-}-rules and the free atoms did not occur in inference systems with δ^{+}-rules before the publication of [Wirth, 2004]; so in the earlier systems with free δ^{+}-rules only, $\mathbb{V}\mathbb{A}(\forall x^{\mathbb{B}}.\,A)$ was indeed a subset of $\mathbb{V}(\mathit{\Gamma}\ \ \forall x^{\mathbb{B}}.\,A\ \ \mathit{\Pi})$.
  Moreover, the additional atoms blocked by the δ^{+}-rules (as compared to the δ^{-}-rules) can hardly block any reductive proofs of formulas without free atoms and variables. This has following reason. If a proof uses only δ^{+}-reductions, then there will be no (free) atoms around and the critical subset relation holds anyway. So a critical variable-condition can only arise if a δ^{+}-step follows a δ^{-}-step on the same branch. With a reasonably minimal positive/negative variable-condition (P, N), the only additional cycles that could occur by the δ^{+}-rule as compared to the alternative application of a δ^{-}-rules are of the form
  $$y^{\mathbb{V}}\ N\ z^{\mathbb{A}}\ P\ x^{\mathbb{V}}\ P^{*}\ w^{\mathbb{V}}\ P\ y^{\mathbb{V}},$$
  resulting from the following scenario: $y^{\mathbb{V}}\ N\ z^{\mathbb{A}}$ results from a δ^{-}-step, $z^{\mathbb{A}}\ P\ x^{\mathbb{V}}$ results from a subsequent δ^{+}-step on the same branch, $x^{\mathbb{V}}\ P^{*}\ w^{\mathbb{V}}$ results from possible further δ^{+}-steps (δ^{-}-steps cannot produce a relevant cycle!) and instantiations of free variables, and $w^{\mathbb{V}}\ P\ y^{\mathbb{V}}$ finally results from an instantiation of $y^{\mathbb{V}}$.
  Let us now see what happens if we replace the δ^{+}-step with a δ^{-}-step with $x^{\mathbb{A}}$ replacing $x^{\mathbb{V}}$, *ceteris paribus*. Note that this is only possible if $x^{\mathbb{V}}$ was never instantiated, which again explains why there must be at least one step of P between $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$. If the free variable $y^{\mathbb{V}}$ occurs in the upper sequent of this changed step, then new proof immediately fails due to the new cycle
  $$y^{\mathbb{V}}\ N\ x^{\mathbb{A}}\ P^{*}\ w^{\mathbb{V}}\ P\ y^{\mathbb{V}}.$$
  Otherwise, $y^{\mathbb{V}}$ was lost on this branch; but then we must ask ourselves why we instantiated it with a term containing $w^{\mathbb{V}}$. If $w^{\mathbb{V}}$ is essentially shared with another branch, on which $y^{\mathbb{V}}$ has survived, then it must occur in the sequent before the original δ^{+}-step, and so we get the cycle
  $$w^{\mathbb{V}}\ N\ x^{\mathbb{A}}\ P^{*}\ w^{\mathbb{V}}.$$
  Otherwise, if $w^{\mathbb{V}}$ is not shared with another branch, we do not see any reason to instantiate $y^{\mathbb{V}}$ with a term containing $w^{\mathbb{V}}$. Indeed, if $w^{\mathbb{V}}$ is only this branch, then there is no reason; if $w^{\mathbb{V}}$ occurs only on another branch, then a good reason for $x^{\mathbb{V}}\ P^{*}\ w^{\mathbb{V}}$ can be rejected just as for $y^{\mathbb{V}}$ before.

- The δ^{+}-rule may contribute an P-edge to a cycle with exactly one edge from N, whereas the analogous δ^{-}-rule would contribute an N-edge instead, so the analogous cycle would then not count as counterexample to the consistency of the positive/negative variable-condition because it has two edges from N.
  Also in this case we conjecture that δ^{-}-rules do not admit any successful proofs that are not possible with the analogous δ^{+}-rules. A proof of this conjecture, however, is not easy: First, it is a global property which requires us to consider the entire inference system. Second, δ^{-}-rules indeed admit some extra (P,N)-substitutions, which have to be shown not to generate essentially additional proofs. E.g., if we want to prove $\forall y^{\mathbb{B}}.\ \mathsf{Q}(a^{\mathbb{V}}, y^{\mathbb{B}}) \wedge \forall x^{\mathbb{B}}.\ \mathsf{Q}(x^{\mathbb{B}}, b^{\mathbb{V}})$, which is true for a reflexive ordering Q with a minimal and a maximal element, β- and δ^{-}-rules reduce this to the two goals $\mathsf{Q}(a^{\mathbb{V}}, y^{\mathbb{A}})$ and $\mathsf{Q}(x^{\mathbb{A}}, b^{\mathbb{V}})$ with positive/negative variable-condition (P,N) given by $P=\emptyset$ and $N=\{(a^{\mathbb{V}}, y^{\mathbb{A}}),\ (b^{\mathbb{V}}, x^{\mathbb{A}})\}$. Then $\sigma_{\mathbb{A}} := \{a^{\mathbb{V}}{\mapsto}x^{\mathbb{A}},\ b^{\mathbb{V}}{\mapsto}y^{\mathbb{A}}\}$ is a (P,N)-substitution. The analogous δ^{+}-rules would have resulted in the positive/negative variable-condition (P',N') given by $P'=\{(a^{\mathbb{V}}, y^{\mathbb{V}}),\ (b^{\mathbb{V}}, x^{\mathbb{V}})\}$ and $N'=\emptyset$. But $\sigma_{\mathbb{V}} := \{a^{\mathbb{V}}{\mapsto}x^{\mathbb{V}},\ b^{\mathbb{V}}{\mapsto}y^{\mathbb{V}}\}$ is not a (P',N')-substitution!

Note 4 (History of PEANO and his ι)
In [PEANO, 1896f.], GUISEPPE PEANO (1858–1932) wrote $\bar{\iota}$ instead of the ι of Example 4.1, and $\bar{\iota}\{\ x \mid A\ \}$ instead of $\iota x.\,A$. (Note that we have changed the class notation to modern standard here. PEANO actually wrote $\overline{x\epsilon}A$ instead of $\{\ x \mid A\ \}$ in [PEANO, 1896f.].)
The bar above the ι (just as the alternative inversion the symbol) were to indicated that $\bar{\iota}$ was implicitly defined as the inverse operator of the operator ι defined by $\iota y := \{y\}$, which occurred already in [PEANO, 1890] and still in [QUINE, 1981].
The definition of $\bar{\iota}$ reads literally [PEANO, 1896f., Definition 22]:

$$a \in K\ .\ \exists a : x, y \in a\ .\ \supset_{x,y}\ .\ x = y : \supset : x = \bar{\iota}a\ .=.\ a = \iota x$$

This straightforwardly translates into more modern notation as follows:

$$\text{For any class } a\text{:}\quad a \neq \emptyset \wedge \forall x, y.\ (x, y \in a \ \Rightarrow\ x = y) \ \Rightarrow\ \forall x.\ (x = \bar{\iota}a \ \Leftrightarrow\ a = \iota x)$$

Giving up the flavor of an explicit definition of "$x = \bar{\iota}a$", this can be simplified to the following equivalent form:

$$\text{For any class } a\text{:}\quad \exists! x.\ x \in a \ \Rightarrow\ \bar{\iota}a \in a \qquad (\bar{\iota}_0)$$

Besides notational difference, this is (ι_0) of our § 4.4.2.

Note 5 (σ-Updates Admitting Variable-Reuse and -Permutation)
For a version of σ-updates that admits variable-reuse and -permutation as explained in Note 10 of [WIRTH, 2004] and executed in Notes 26–30 of [WIRTH, 2004], the σ-update has to forget about the old meaning of the variables in $\mathrm{dom}(\sigma)$. To this end — instead of the simpler $(P \cup D, N)$ — we have to chose a *σ-update admitting variable-reuse and -permutation* to be

$$\Big(\ \big(\ {}_{\mathbb{V}\mathbb{A}\setminus\mathrm{dom}(\sigma)}{\upharpoonleft}P\ \cup\ P' \circ P\ \big){\upharpoonright}_{\mathbb{V}\setminus\mathrm{dom}(\sigma)},\qquad {}_{\mathbb{V}\setminus\mathrm{dom}(\sigma)}{\upharpoonleft}N\ \cup\ {}_{\mathbb{V}}{\upharpoonleft}P' \circ N\ \Big)$$

for $P' \ :=\ D\ \cup\ {}_{\mathbb{V}\mathbb{A}\setminus\mathrm{dom}(\sigma)}{\upharpoonleft}(P{\upharpoonright}_{\mathrm{dom}(\sigma)})^{+}$.
Note that P' can be simplified to D here by taking as the *σ-update admitting* V_{γ}*-reuse and -permutation:*

$$\Big(\ \big(\ {}_{\mathbb{A}\cup\mathrm{V}_{\delta+}\cup(\mathrm{V}_{\gamma}\setminus\mathrm{dom}(\sigma))}{\upharpoonleft}P\ \cup\ D \circ P\ \cup\ D{\upharpoonright}_{\mathrm{V}_{\delta+}}\ \big),\quad {}_{\mathrm{V}_{\delta+}\cup(\mathrm{V}_{\gamma}\setminus\mathrm{dom}(\sigma))}{\upharpoonleft}N\ \cup\ {}_{\mathbb{V}}{\upharpoonleft}D{\upharpoonright}_{\mathrm{V}_{\gamma}\cap\mathrm{dom}(\sigma)} \circ N\ \Big),$$

provided that we partition $\mathbb{V}$ into two sets $\mathrm{V}_{\delta+} \uplus \mathrm{V}_{\gamma}$, use $\mathrm{V}_{\delta+}$ as the possible domain of the choice-conditions, and admit variable-reuse and -permutation only on V_{γ}, similar to what we already did in Note 10 of [WIRTH, 2004]. (The crucial restriction becomes here the following: For a (positive/negative) σ-update (P'', N'') admitting V_{γ}-reuse and -permutation we have $P'' \subseteq \mathbb{V}\mathbb{A} \times \mathrm{V}_{\delta+}$ and $N'' \subseteq \mathbb{V} \times \mathbb{A}$).
Note, however, that it is actually better to work with the more complicated P', simply because it is more general and because the transitive closure will not be computed in practice, but a graph will be updated just as exemplified in Note 10 of [WIRTH, 2004].

Note 6 (Which directions of the equivalences of the $\forall$-, $\exists$-, $\Rightarrow$-, and $\neg$-Lemma are needed where precisely?)
Lemma 5.19 depends on the backward directions of the $\forall$-LEMMA and the $\Rightarrow$-LEMMA, and on the forward direction of the $\exists$-LEMMA. Lemma 5.24 and Theorem 5.26(6) depend on the forward directions of the $\forall$-LEMMA and the $\Rightarrow$-LEMMA, and on the backward direction of the $\exists$-LEMMA. Theorem 5.27 depends on both directions of the $\forall$-LEMMA, of the $\exists$-LEMMA, and of the $\neg$-LEMMA.

Note 7 (Why does FERMAT's *Descente Infinie* require Choice-Conditions?)
The well-foundedness required for the soundness of *descente infinie* gave rise to a notion of reduction which preserves solutions, cf. Definition 5.25. The liberalized δ-rules as found in [FITTING, 1996] do not satisfy this notion. The addition of our choice-conditions finally turned out to be the only way to repair this defect of the liberalized δ-rules. See [WIRTH, 2004] for more details.

Note 8 (Consequences of the ε-Formula in Intuitionistic Logic)
Adding the ε either with (ε_0), with (ε_1), or with the ε-formula (cf. § 4.6) to intuitionistic first-order logic is equivalent on the ε-free fragment to adding PLATO*'s Principle,* i.e. $\exists y^{\mathbb{B}}.\ (\exists x^{\mathbb{B}}.\ A \ \Rightarrow\ A\{x^{\mathbb{B}}{\mapsto}y^{\mathbb{B}}\})$ with $y^{\mathbb{B}}$ not occurring in A, cf. [MEYER-VIOL, 1995, § 3.3].
Moreover, the non-trivial direction of (ε_2) is $\forall x^{\mathbb{B}}.\ A \ \Leftarrow\ A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg A\}$.
Even intuitionistically, this entails its contrapositive $\neg\forall x^{\mathbb{B}}.\ A \ \Rightarrow\ \neg A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg A\}$,
and then, e.g. by the trivial direction of (ε_1) (when A is replaced with $\neg A$)

$$\neg\forall x^{\mathbb{B}}.\ A \quad\Rightarrow\quad \exists x^{\mathbb{B}}.\ \neg A \tag{Q2}$$

which is not valid in intuitionistic logic in general. Thus, in intuitionistic logic, the universal quantifier becomes strictly weaker by the inclusion of (ε_2) or anything similar for the universal quantifier, such as HILBERT's τ-operator (cf. [HILBERT, 1923a]). More specifically, adding

$$\forall x^{\mathbb{B}}.\ A \quad\Leftarrow\quad A\{x^{\mathbb{B}} \mapsto \tau x^{\mathbb{B}}.\ A\} \tag{τ_0}$$

is equivalent on the τ-free theory to adding $\exists y^{\mathbb{B}}.\ (\forall x^{\mathbb{B}}.\ A \ \Leftarrow\ A\{x^{\mathbb{B}}{\mapsto}y^{\mathbb{B}}\})$ with $y^{\mathbb{B}}$ not occurring in A, which again implies (Q2), cf. [MEYER-VIOL, 1995, § 3.4.2].
From a semantic point of view (cf. [GABBAY, 1981]), the intuitionistic $\forall$ may be eliminated, however, by first applying the GÖDEL translation into the modal logic S4 with classical $\forall$ and $\neg$, cf. e.g. [FITTING, 1999], and then adding the ε conservatively, e.g. by avoiding substitutions via λ-abstraction as in [FITTING, 1975].

Note 9 (Proof of $(C,(P,N))$-validity of (E2′) using Theorem 5.26(1,5a,6a))
Let us give a formal proof of (E2′) in our framework on an abstract level by applying Theorem 5.26. We will reduce the set containing the single-formula sequent of the formula (E2′) to a valid set. Be aware of the requirements on occurrence of the variables as described in § B.1.1. We start with an extended extension $(C',(P',N))$ of the current $(C,(P,N))$ for a fresh variable $y^{\mathbb{V}}$ with $C'(y^{\mathbb{V}})$ as given § B.1.1. Of course, to satisfy Definition 5.13(3), here we set

$$P' \ :=\ P \cup \mathbb{VA}(A_0, A_1, x^{\mathbb{V}}_0) \times \{y^{\mathbb{V}}\}.$$

Set $\sigma := \{x^{\mathbb{V}}_1 \mapsto y^{\mathbb{V}}\}$. Let $(C'',(P'',N))$ be the extended σ-update of $(C',(P',N))$; then

$${}_{\{y^{\mathbb{V}},x^{\mathbb{V}}_0,x^{\mathbb{V}}_1\}}{\upharpoonleft} C'' = {}_{\{y^{\mathbb{V}},x^{\mathbb{V}}_0\}}{\upharpoonleft} C' \quad\text{and}\quad P'' = P' \cup \{(y^{\mathbb{V}}, x^{\mathbb{V}}_1)\}.$$

Note that (P'',N) is consistent because every cycle not possible with (P,N) would have to run through the set $\{y^{\mathbb{V}}, x^{\mathbb{V}}_1\}$, which, however, is disjoint from $\mathrm{dom}(N)$, closed under P'', and cycle-free.
Now we apply Theorem 5.26(6a). According to settings for the meta-variables given there, we have $O = M = \mathrm{dom}(C') {\cap} \mathrm{dom}(\sigma) = \{x^{\mathbb{V}}_1\}$ and $O' = \emptyset$. Consider the set with the two single-formula sequents $(\mathrm{E2})'\sigma$ and $(Q_{C'}(x^{\mathbb{V}}_1))\sigma$. The former sequent reads $\forall x^{\mathbb{B}}.\ (A_0{\Leftrightarrow}A_1) \Rightarrow x^{\mathbb{V}}_0 {=} y^{\mathbb{V}}$. According to Definition 4.11, the latter sequent reads $(\exists x^{\mathbb{B}}.\ A_1 \Rightarrow A_1\{x^{\mathbb{B}} \mapsto x^{\mathbb{V}}_1\})\sigma$, i.e. $\exists x^{\mathbb{B}}.\ A_1 \Rightarrow A_1\{x^{\mathbb{B}} \mapsto y^{\mathbb{V}}\}$. Now a simple case analysis on $\forall x^{\mathbb{B}}.\ (A_0{\Leftrightarrow}A_1)$ shows that this two-element set $(C'',(P'',N))$-reduces to

$$\left\{\ \exists x^{\mathbb{B}}.\ A_0 \ \Rightarrow\ A_0\{x^{\mathbb{B}} \mapsto x^{\mathbb{V}}_0\};\quad \left(\begin{array}{c} \exists y^{\mathbb{B}}.\left(\begin{array}{c} (\forall x^{\mathbb{B}}.\ (A_0{\Leftrightarrow}A_1) \Rightarrow y^{\mathbb{B}} {=} x^{\mathbb{V}}_0) \\ \wedge\ \ (\neg\forall x^{\mathbb{B}}.\ (A_0{\Leftrightarrow}A_1) \Rightarrow A_1\{x^{\mathbb{B}} \mapsto y^{\mathbb{B}}\}) \end{array}\right) \\ \Rightarrow\ \left(\begin{array}{c} (\forall x^{\mathbb{B}}.\ (A_0{\Leftrightarrow}A_1) \Rightarrow y^{\mathbb{V}} {=} x^{\mathbb{V}}_0) \\ \wedge\ \ (\neg\forall x^{\mathbb{B}}.\ (A_0{\Leftrightarrow}A_1) \Rightarrow A_1\{x^{\mathbb{B}} \mapsto y^{\mathbb{V}}\}) \end{array}\right) \end{array}\right)\right\},$$

i.e. to $\{Q_{C''}(x^{\mathbb{V}}_0);\ \ Q_{C''}(y^{\mathbb{V}})\}$, which is $(C'',(P'',N))$-valid by Lemma 5.19. Thus, $(\mathrm{E2}')\sigma$ is $(C',(P'',N))$-valid. By (6a) this means that (E2′) is $(C',(P',N))$-valid, and by (5a) also $(C,(P,N))$-valid, as was to be shown.

Note 10 **(** $0\neq 1$, $\varepsilon x^{\mathbb{B}}.\ A_0 \neq \varepsilon x^{\mathbb{B}}.\ A_1 \ \Rightarrow\ \neg(\forall x^{\mathbb{B}}.\ A_0 \wedge \forall x^{\mathbb{B}}.\ A_1) \ \vdash\ B \vee \neg B$ **in Intuitionistic Logic)**
For the proof of the slightly weaker result $0\neq 1$, (E2) $\vdash$ $B \vee \neg B$ for any formula B, cf. [Bell &al., 2001, Proof of Theorem 6.4], which already occurs in more detail in [Bell, 1993a, § 3], and sketched in [Bell, 1993b, § 7].
Note that, for any implication $A \Rightarrow B$, its contrapositive $\neg B \Rightarrow \neg A$ is a consequence of it, and — in intuitionistic logic — a *proper* consequence in general.
Let B be an arbitrary formula. By renaming we may w.l.o.g. assume that the free atom $x^{\mathbb{A}}$ of the ε-formula does not occur in B. We are going to show that $\vdash B \vee \neg B$ holds in intuitionistic logic under the assumptions of reflexivity, symmetry, and transitivity of "=", the ε-formula (or (ε_0)), and of the formulas $0\neq 1$ and $\varepsilon x^{\mathbb{B}}.\ A_0 \neq \varepsilon x^{\mathbb{B}}.\ A_1 \ \Rightarrow\ \neg(\forall x^{\mathbb{B}}.\ A_0 \wedge \forall x^{\mathbb{B}}.\ A_1)$.
Let $x^{\mathbb{B}}$ be a bound atom not occurring in B. Set $A_i := (B \vee x^{\mathbb{B}} = i)$ for $i \in \{0,1\}$.
Now all that we have to show is a trivial consequence of the following Claims 1 and 2,
$\varepsilon x^{\mathbb{B}}.\ A_0 \neq \varepsilon x^{\mathbb{B}}.\ A_1 \ \Rightarrow\ \neg(\forall x^{\mathbb{B}}.\ A_0 \wedge \forall x^{\mathbb{B}}.\ A_1)$, and Claim 3.
Claim 1: $0=0$, $1=1$, $(\varepsilon\text{-formula})\{A\mapsto A_0\}\{x^{\mathbb{A}}\mapsto 0\}$, $(\varepsilon\text{-formula})\{A\mapsto A_1\}\{x^{\mathbb{A}}\mapsto 1\}$
$\vdash\ B \ \vee\ (\varepsilon x^{\mathbb{B}}.\ A_0 = 0 \ \wedge\ \varepsilon x^{\mathbb{B}}.\ A_1 = 1)$.
Claim 2: $\varepsilon x^{\mathbb{B}}.\ A_0 = 0 \ \wedge\ \varepsilon x^{\mathbb{B}}.\ A_1 = 1$, $0\neq 1$, $\forall x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}} \wedge y^{\mathbb{B}} = z^{\mathbb{B}} \Rightarrow x^{\mathbb{B}} = z^{\mathbb{B}})$
$\vdash\ \varepsilon x^{\mathbb{B}}.\ A_0 \neq \varepsilon x^{\mathbb{B}}.\ A_1$.
Claim 3: $\neg(\forall x^{\mathbb{B}}.\ A_0 \wedge \forall x^{\mathbb{B}}.\ A_1) \ \vdash\ \neg B$.
Proof of Claim 1: Because neither $x^{\mathbb{A}}$ nor $x^{\mathbb{B}}$ occur in B, and because $x^{\mathbb{A}}$ does not occur in A_i, the instances of the ε-formulas read $(B \vee i=i) \Rightarrow (B \vee \varepsilon x^{\mathbb{B}}.\ A_i = i)$. Thus, from $i=i$, we get $B \vee \varepsilon x^{\mathbb{B}}.\ A_i = i$. Thus, we get $(B \vee \varepsilon x^{\mathbb{B}}.\ A_0 = 0) \wedge (B \vee \varepsilon x^{\mathbb{B}}.\ A_1 = 1)$, thus $B \ \vee\ (\varepsilon x^{\mathbb{B}}.\ A_0 = 0 \ \wedge\ \varepsilon x^{\mathbb{B}}.\ A_1 = 1)$ by distributivity. Q.e.d. (Claim 1)
Proof of Claim 2: Trivial. Q.e.d. (Claim 2)
Proof of Claim 3: As $x^{\mathbb{B}}$ does not occur in B, we get $B \vdash \forall x^{\mathbb{B}}.\ A_i$. The rest is trivial. Q.e.d. (Claim 3)

References

[Ackermann, 1925] Wilhelm Ackermann, 1925. Begründung des „tertium non datur“ mittels der Hilbertschen Theorie der Widerspruchsfreiheit. *Mathematische Annalen*, 93:1–36. Received March 30, 1924. Inauguraldissertation, 1924, Göttingen.

[Ackermann, 1938] Wilhelm Ackermann, 1938. Mengentheoretische Begründung der Logik. *Mathematische Annalen*, 115:1–22. Received April 23, 1937.

[Ackermann, 1940] Wilhelm Ackermann, 1940. Zur Widerspruchsfreiheit der Zahlentheorie. *Mathematische Annalen*, 117:163–194. Received Aug. 15, 1939.

[Anon, 1899] Anon, editor, 1899. *Festschrift zur Feier der Enthüllung des* Gauss-Weber-*Denkmals in Göttingen, herausgegeben von dem Fest-Comitee.* Verlag von B. G. Teubner, Leipzig.

[Asser, 1957] Günter Asser, 1957. Theorie der logischen Auswahlfunktionen. *Zeitschrift für math. Logik und Grundlagen der Mathematik*, 3:30–68.

[Bell, 1993a] John Lane Bell, 1993a. Hilbert's ε-Operator and Classical Logic. *J. Philosophical Logic*, 22:1–18.

[Bell, 1993b] John Lane Bell, 1993b. Hilbert's ε-Operator in Intuitionistic Type Theories. *Math. Logic Quart. (until 1993: Zeitschrift für math. Logik und Grundlagen der Mathematik)*, 39:323–337.

[Bell &al., 2001] John Lane Bell, David DeVidi, and Graham Solomon, 2001. *Logical Options: An Introduction to Classical and Alternative Logics.* Broadview Press.

[Berka & Kreiser, 1973] Karel Berka and Lothar Kreiser, editors, 1973. *Logik-Texte – Kommentierte Auswahl zur Geschichte der modernen Logik.* Akademie Verlag GmbH, Berlin. 2nd rev. edn. (1st edn. 1971; 4th rev. rev. edn. 1986).

[Bibel & Schmitt, 1998] Wolfgang Bibel and Peter H. Schmitt, editors, 1998. *Automated Deduction — A Basis for Applications.* Number 8–10 in Applied Logic Series. Kluwer (Springer Science+Business Media). Vol. I: Foundations — Calculi and Methods. Vol. II: Systems and Implementation Techniques. Vol. III: Applications.

[Blass & Gurevich, 2000] Andreas Blass and Yuri Gurevich, 2000. The logic of choice. *J. Symbolic Logic*, 65:1264–1310.

[Bourbaki, 1939ff.] Nicolas Bourbaki, 1939ff.. *Éléments des Mathématique — Livres 1–9.* Actualités Scientifiques et Industrielles. Hermann, Paris.

[Caferra & Salzer, 2000] Ricardo Caferra and Gernot Salzer, editors, 2000. *Automated Deduction in Classical and Non-Classical Logics.* Number 1761 in Lecture Notes in Artificial Intelligence. Springer.

[Clocksin & Mellish, 2003] William F. Clocksin and Christopher S. Mellish, 2003. *Programming in* Prolog. Springer. 5th edn. (1st edn. 1981).

[DeVidi, 1995] David DeVidi, 1995. Intuitionistic ε- and τ-calculi. *Math. Logic Quart. (until 1993: Zeitschrift für math. Logik und Grundlagen der Mathematik)*, 41:523–546.

[DIAMOND, 1976] Cora Diamond, editor, 1976. WITTGENSTEIN*'s Lectures on the Foundations of Mathematics, Cambridge, 1939.* Cornell Univ., Ithaca (NY). From the notes of R. G. BOSANQUET, NORMAN MALCOLM, RUSH RHEES, and YORICK SMYTHIES. German translation is [DIAMOND, 1978].

[DIAMOND, 1978] Cora Diamond, editor, 1978. WITTGENSTEIN*s Vorlesungen über die Grundlagen der Mathematik, Cambridge, 1939.* Suhrkamp Verlag, Frankfurt am Main. German translation of [DIAMOND, 1976] by JOACHIM SCHULTE. 1st edn. as "LUDWIG WITTGENSTEIN, Schriften 7" (ISBN 3518072471). Note that this volume was excluded from the later Suhrkamp edn. "LUDWIG WITTGENSTEIN, Werkausgabe".

[EGLY & FERMÜLLER, 2002] Uwe Egly and Christian G. Fermüller, editors, 2002. *11th Int. Conf. on Tableaux and Related Methods, København, 2002*, number 2381 in Lecture Notes in Artificial Intelligence. Springer.

[EWALD, 1996] William Ewald, editor, 1996. *From* KANT *to* HILBERT *— A source book in the foundations of mathematics.* Oxford Univ. Press.

[FITTING, 1975] Melvin Fitting, 1975. A modal logic ε-calculus. *Notre Dame J. of Formal Logic*, XVI:1–16.

[FITTING, 1990] Melvin Fitting, 1990. *First-order logic and automated theorem proving.* Springer. 1st edn. (2nd rev. edn. is [FITTING, 1996]).

[FITTING, 1996] Melvin Fitting, 1996. *First-order logic and automated theorem proving.* Springer. 2nd rev. edn. (1st edn. is [FITTING, 1990]).

[FITTING, 1999] Melvin Fitting, 1999. On quantified modal logic. *Fundamenta Informaticae*, 39:105–121.

[FREGE, 1893/1903] Gottlob Frege, 1893/1903. *Grundgesetze der Arithmetik — Begriffsschriftlich abgeleitet.* Verlag von Hermann Pohle, Jena. Vol. I/II. As facsimile with corrigenda by CHRISTIAN THIEL: Georg Olms Verlag, Hildesheim (Germany), 1998. English translations: [FREGE, 1964], [FREGE, 2013].

[FREGE, 1964] Gottlob Frege, 1964. *The Basic Laws of Arithmetic — Exposition of the System.* Univ. of California Press. English translation of [FREGE, 1893/1903, Vol I], with an introduction, by MONTGOMERY FURTH.

[FREGE, 2013] Gottlob Frege, 2013. *Basic Laws of Arithmetic.* Oxford Univ. Press. English translation of [FREGE, 1893/1903] by PHILIP EBERT and MARCUS ROSSBERG, with an Introduction by CRISPIN WRIGHT and an appendix by ROY T. COOK.

[FRIED, 2012] Johannes Fried, 2012. *Canossa – Entlarvung einer Legende. Eine Streitschrift.* Akademie Verlag GmbH, Berlin.

[GABBAY, 1981] Dov Gabbay, 1981. *Semantical Investigations in* HEYTING*'s Intuitionistic Logic.* Kluwer (Springer Science+Business Media).

[GABBAY & WOODS, 2004ff.] Dov Gabbay and John Woods, editors, 2004ff.. *Handbook of the History of Logic.* North-Holland (Elsevier).

[GABBAY & PITTS, 2002] Murdoch J. Gabbay and Andrew M. Pitts, 2002. A new approach to abstract syntax with variable binding. *Formal Asp. Comput.*, 13:341–363.

[Gentzen, 1935] Gerhard Gentzen, 1935. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210,405–431. Also in [Berka & Kreiser, 1973, pp. 192–253]. English translation in [Gentzen, 1969].

[Gentzen, 1969] Gerhard Gentzen, 1969. *The Collected Papers of* Gerhard Gentzen. North-Holland (Elsevier). Ed. by Manfred E. Szabo.

[Giese & Ahrendt, 1999] Martin Giese and Wolfgang Ahrendt, 1999. Hilbert's ε-terms in automated theorem proving. In [Murray, 1999, pp. 171–185].

[Gillman, 1987] Leonard Gillman, 1987. *Writing Mathematics Well.* The Mathematical Association of America.

[Heijenoort, 1971] Jean van Heijenoort, 1971. *From* Frege *to* Gödel*: A Source Book in Mathematical Logic, 1879–1931.* Harvard Univ. Press. 2nd rev. edn. (1st edn. 1967).

[Hermes, 1965] Hans Hermes, 1965. *Eine Termlogik mit Auswahloperator.* Number 6 in LNM. Springer.

[Heusinger, 1997] Klaus von Heusinger, 1997. *Salienz und Referenz — Der Epsilonoperator in der Semantik der Nominalphrase und anaphorischer Pronomen.* Number 43 in Studia grammatica. Akademie Verlag GmbH, Berlin.

[Hilbert, 1899] David Hilbert, 1899. Grundlagen der Geometrie. In [Anon, 1899, pp. 1–92]. 1st edn. without appendixes. Reprinted in [Hilbert, 2004, pp. 436–525]. *(Last edition of "Grundlagen der Geometrie" by* Hilbert *is [*Hilbert*, 1930b], which is also most complete regarding the appendixes. Last three editions by* Paul Bernays *are [*Hilbert*, 1962; 1968; 1972], which are also most complete regarding supplements and figures. Its first appearance as a separate book was the French translation [*Hilbert*, 1900b]. Two substantially different English translations are [*Hilbert*, 1902] and [*Hilbert*, 1971]).*

[Hilbert, 1900a] David Hilbert, 1900a. Über den Zahlbegriff. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 8:180–184. Received Dec. 1899. Reprinted as Appendix VI of [Hilbert, 1909; 1913; 1922; 1923b; 1930b].

[Hilbert, 1900b] David Hilbert, 1900b. Les principes fondamentaux de la géométrie. *Annales Scientifiques de l'École Normale Supérieure*, Série 3, 17:103–209. French translation by Léonce Laugel of special version of [Hilbert, 1899], revised and authorized by Hilbert. Also in published as a separate book by the same publisher (Gauthier-Villars, Paris).

[Hilbert, 1902] David Hilbert, 1902. *The Foundations of Geometry.* Open Court, Chicago. English translation by E. J. Townsend of special version of [Hilbert, 1899], revised and authorized by Hilbert, `http://www.gutenberg.org/etext/17384`.

[Hilbert, 1903] David Hilbert, 1903. *Grundlagen der Geometrie. — Zweite, durch Zusätze vermehrte und mit fünf Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Druck und Verlag von B. G. Teubner, Leipzig. 2nd rev. extd. edn. of [Hilbert, 1899], rev. and extd. with five appendixes, newly added figures, and an index of notion names.

[Hilbert, 1905] David Hilbert, 1905. Über die Grundlagen der Logik und der Arithmetik. In [Krazer, 1905, pp. 174–185]. Reprinted as Appendix VII of [Hilbert, 1909; 1913; 1922; 1923b; 1930b]. English translation *On the foundations of logic and arithmetic* by Beverly Woodward with an introduction by Jean van Heijenoort in [Heijenoort, 1971, pp. 129–138].

[Hilbert, 1909] David Hilbert, 1909. *Grundlagen der Geometrie. — Dritte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Number VII in Wissenschaft und Hypothese. Druck und Verlag von B. G. Teubner, Leipzig, Berlin. 3rd rev. extd. edn. of [Hilbert, 1899], rev. edn. of [Hilbert, 1903], extd. with a bibliography and two additional appendixes (now seven in total) (Appendix VI: [Hilbert, 1900a]) (Appendix VII: [Hilbert, 1905]).

[Hilbert, 1913] David Hilbert, 1913. *Grundlagen der Geometrie. — Vierte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Druck und Verlag von B. G. Teubner, Leipzig, Berlin. 4th rev. extd. edn. of [Hilbert, 1899], rev. edn. of [Hilbert, 1909].

[Hilbert, 1922] David Hilbert, 1922. *Grundlagen der Geometrie. — Fünfte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Verlag und Druck von B. G. Teubner, Leipzig, Berlin. 5th extd. edn. of [Hilbert, 1899]. Contrary to what the sub-title may suggest, this is an anastatic reprint of [Hilbert, 1913], extended by a very short preface on the changes w.r.t. [Hilbert, 1913], and with augmentations to Appendix II, Appendix III, and Chapter IV, § 21.

[Hilbert, 1923a] David Hilbert, 1923a. Die logischen Grundlagen der Mathematik. *Mathematische Annalen*, 88:151–165. Received Sept. 29, 1922. Talk given at the Deutsche Naturforschergesellschaft in Leipzig, Sept. 1922. English translation in [Ewald, 1996, pp. 1134–1148].

[Hilbert, 1923b] David Hilbert, 1923b. *Grundlagen der Geometrie. — Sechste unveränderte Auflage. Anastatischer Nachdruck. Mit zahlreichen in den Text gedruckten Figuren.* Verlag und Druck von B. G. Teubner, Leipzig, Berlin. 6th rev. extd. edn. of [Hilbert, 1899], anastatic reprint of [Hilbert, 1922].

[Hilbert, 1926] David Hilbert, 1926. Über das Unendliche — Vortrag, gehalten am 4. Juni 1925 gelegentlich einer zur Ehrung des Andenkens an Weierstrass von der Westfälischen Math. Ges. veranstalteten Mathematiker-Zusammenkunft in Münster i. W. *Mathematische Annalen*, 95:161–190. Received June 24, 1925. Reprinted as Appendix VIII of [Hilbert, 1930b]. English translation *On the infinite* by Stefan Bauer-Mengelberg with an introduction by Jean van Heijenoort in [Heijenoort, 1971, pp. 367–392].

[Hilbert, 1928] David Hilbert, 1928. Die Grundlagen der Mathematik — Vortrag, gehalten auf Einladung des Mathematischen Seminars im Juli 1927 in Hamburg. *Abhandlungen aus dem mathematischen Seminar der Univ. Hamburg*, 6:65–85. Reprinted as Appendix IX of [Hilbert, 1930b]. English translation *The foundations of mathematics* by Stefan Bauer-Mengelberg and Dagfinn Føllesdal with a short introduction by Jean van Heijenoort in [Heijenoort, 1971, pp. 464–479].

[Hilbert, 1930a] David Hilbert, 1930a. Probleme der Grundlegung der Mathematik. *Mathematische Annalen*, 102:1–9. Vortrag gehalten auf dem Internationalen Mathematiker-Kongreß in Bologna, Sept. 3, 1928. Received March 25, 1929. Reprinted as Appendix X of [Hilbert, 1930b]. Short version in *Atti del congresso internationale dei matematici, Bologna, 3–10 settembre 1928*, Vol. 1, pp. 135–141, Bologna, 1929.

[Hilbert, 1930b] David Hilbert, 1930b. *Grundlagen der Geometrie. — Siebente umgearbeitete und vermehrte Auflage. Mit 100 in den Text gedruckten Figuren.* Verlag und Druck von B. G. Teubner, Leipzig, Berlin. 7th rev. extd. edn. of [Hilbert, 1899], thoroughly revised edition of [Hilbert, 1923b], extd. with three new appendixes (now ten in total) (Appendix VIII: [Hilbert, 1926]) (Appendix IX: [Hilbert, 1928]) (Appendix X: [Hilbert, 1930a]).

[Hilbert, 1956] David Hilbert, 1956. *Grundlagen der Geometrie. — Achte Auflage, mit Revisionen und Ergänzungen von Dr.* Paul Bernays*. Mit 124 Abbildungen.* B. G. Teubner Verlagsgesellschaft, Stuttgart. 8th rev. extd. edn. of [Hilbert, 1899], rev. edn. of [Hilbert, 1930b], omitting appendixes VI–X, extd. by Paul Bernays, now with 24 additional figures and 3 additional supplements.

[Hilbert, 1962] David Hilbert, 1962. *Grundlagen der Geometrie. — Neunte Auflage, revidiert und ergänzt von Dr.* Paul Bernays*. Mit 129 Abbildungen.* B. G. Teubner Verlagsgesellschaft, Stuttgart. 9th rev. extd. edn. of [Hilbert, 1899], rev. edn. of [Hilbert, 1956], extd. by Paul Bernays, now with 129 figures, 5 appendixes, and 8 supplements (I 1, I 2, II, III, IV 1, IV 2, V 1, V 2).

[Hilbert, 1968] David Hilbert, 1968. *Grundlagen der Geometrie. — Zehnte Auflage, revidiert und ergänzt von Dr.* Paul Bernays*. Mit 124 Abbildungen.* B. G. Teubner Verlagsgesellschaft, Stuttgart. 10th rev. extd. edn. of [Hilbert, 1899], rev. edn. of [Hilbert, 1962] by Paul Bernays.

[Hilbert, 1971] David Hilbert, 1971. *The Foundations of Geometry.* Open Court, Chicago and La Salle (IL). Newly translated and fundamentally different 2nd edn. of [Hilbert, 1902], actually an English translation of [Hilbert, 1968] by Leo Unger.

[Hilbert, 1972] David Hilbert, 1972. *Grundlagen der Geometrie. — 11. Auflage. Mit Supplementen von Dr.* Paul Bernays. B. G. Teubner Verlagsgesellschaft, Stuttgart. 11th rev. extd. edn. of [Hilbert, 1899], rev. edn. of [Hilbert, 1968] by Paul Bernays.

[Hilbert, 2004] David Hilbert, 2004. David Hilbert*'s Lectures on the Foundations of Geometry, 1891–1902.* Springer. Ed. by Michael Hallett and Ulrich Majer.

[Hilbert & Ackermann, 1928] David Hilbert and Wilhelm Ackermann, 1928. *Grundzüge der theoretischen Logik.* Number XXVII in Grundlehren der mathematischen Wissenschaften. Springer. 1st edn., the final version in a serious of three thorough revisions is [Hilbert & Ackermann, 1959].

[Hilbert & Ackermann, 1938] David Hilbert and Wilhelm Ackermann, 1938. *Grundzüge der theoretischen Logik.* Number XXVII in Grundlehren der mathematischen Wissenschaften. Springer. 2nd edn., most thoroughly revised edition of [Hilbert & Ackermann, 1928]. English translation is [Hilbert & Ackermann, 1950].

[Hilbert & Ackermann, 1949] David Hilbert and Wilhelm Ackermann, 1949. *Grundzüge der theoretischen Logik.* Number 27 in Grundlehren der mathematischen Wissenschaften. Springer. 3rd edn. of [Hilbert & Ackermann, 1928], thoroughly revised edition of [Hilbert & Ackermann, 1938].

[Hilbert & Ackermann, 1950] David Hilbert and Wilhelm Ackermann, 1950. *Principles of Mathematical Logic.* Chelsea, New York. English translation of [Hilbert & Ackermann, 1938] by Lewis M. Hammond, George G. Leckie, and F. Steinhardt, ed. and annotated by Robert E. Luce. Reprinted by American Math. Soc. 1999.

[Hilbert & Ackermann, 1959] David Hilbert and Wilhelm Ackermann, 1959. *Grundzüge der theoretischen Logik.* Number 27 in Grundlehren der mathematischen Wissenschaften. Springer. 4th edn. of [Hilbert & Ackermann, 1928], most thoroughly revised and extd. edition of [Hilbert & Ackermann, 1949].

[Hilbert & Bernays, 1934] David Hilbert and Paul Bernays, 1934. *Grundlagen der Mathematik — Erster Band.* Number XL in Grundlehren der mathematischen Wissenschaften. Springer.

1st edn. (2nd edn. is [Hilbert & Bernays, 1968]). Reprinted by J. W. Edwards Publ., Ann Arbor (MI), 1944. English translation is [Hilbert & Bernays, 2017a; 2017b].

[Hilbert & Bernays, 1939] David Hilbert and Paul Bernays, 1939. *Grundlagen der Mathematik — Zweiter Band.* Number L in Grundlehren der mathematischen Wissenschaften. Springer. 1st edn. (2nd edn. is [Hilbert & Bernays, 1970]). Reprinted by J. W. Edwards Publ., Ann Arbor (MI), 1944.

[Hilbert & Bernays, 1968] David Hilbert and Paul Bernays, 1968. *Grundlagen der Mathematik I.* Number 40 in Grundlehren der mathematischen Wissenschaften. Springer. 2nd rev. edn. of [Hilbert & Bernays, 1934]. English translation is [Hilbert & Bernays, 2017a; 2017b].

[Hilbert & Bernays, 1970] David Hilbert and Paul Bernays, 1970. *Grundlagen der Mathematik II.* Number 50 in Grundlehren der mathematischen Wissenschaften. Springer. 2nd rev. extd. edn. of [Hilbert & Bernays, 1939].

[Hilbert & Bernays, 2017a] David Hilbert and Paul Bernays, 2017a. *Grundlagen der Mathematik I — Foundations of Mathematics I, Part A: Title Pages, Prefaces, and §§ 1–2.* Springer. First English translation and bilingual facsimile edn. of the 2nd German edn. [Hilbert & Bernays, 1968], incl. the annotation and translation of all differences of the 1st German edn. [Hilbert & Bernays, 1934]. Ed. by Claus-Peter Wirth, Jörg Siekmann, Volker Peckhaus, Michael Gabbay, Dov Gabbay. Translated and commented by Claus-Peter Wirth &al. Thoroughly rev. 3rd edn. (1st edn. College Publications, London, 2011; 2nd edn. `http://wirth.bplaced.net/p/hilbertbernays`, 2013).

[Hilbert & Bernays, 2017b] David Hilbert and Paul Bernays, 2017b. *Grundlagen der Mathematik I — Foundations of Mathematics I, Part B: §§ 3–5 and Deleted Part I of the 1st Edn..* Springer. First English translation and bilingual facsimile edn. of the 2nd German edn. [Hilbert & Bernays, 1968], incl. the annotation and translation of all deleted texts of the 1st German edn. [Hilbert & Bernays, 1934]. Ed. by Claus-Peter Wirth, Jörg Siekmann, Volker Peckhaus, Michael Gabbay, Dov Gabbay. Translated and commented by Claus-Peter Wirth &al. Thoroughly rev. 3rd edn. (1st edn. College Publications, London, 2012; 2nd edn. `http://wirth.bplaced.net/p/hilbertbernays`, 2013).

[Hintikka, 1974] K. Jaakko J. Hintikka, 1974. Quantifiers vs. quantification theory. *Linguistic Inquiry*, V(2):153–177.

[Hintikka, 1996] K. Jaakko J. Hintikka, 1996. *The Principles of Mathematics Revisited.* Cambridge Univ. Press.

[Howard & Rubin, 1998] Paul Howard and Jean E. Rubin, 1998. *Consequences of the Axiom of Choice.* American Math. Soc.. `http://consequences.emich.edu/conseq.htm`.

[Kanger, 1963] Stig Kanger, 1963. A simplified proof method for elementary logic. In [Siekmann & Wrightson, 1983, Vol. 1, pp. 364–371].

[Kennedy, 1973] Hubert C. Kennedy, 1973. *Selected works of* Guiseppe Peano. George Allen & Unwin, London.

[Kohlhase, 1998] Michael Kohlhase, 1998. Higher-order automated theorem proving. In [Bibel & Schmitt, 1998, Vol. 1, pp. 431–462].

[Kowalski, 1974] Robert A. Kowalski, 1974. Predicate logic as a programming language. In [Rosenfeld, 1974, pp. 569–574].

[Krazer, 1905] A. Krazer, editor, 1905. *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses, Heidelberg, Aug. 8–13, 1904*. Verlag von B. G. Teubner, Leipzig.

[Lambert, 1764] Johann Heinrich Lambert, 1764. *Neues Organon oder Gedanken über die Erforschung und Bezeichnung des Wahren und dessen Unterscheidung von Irrthum und Schein.* Johann Wendler, Leipzig. Vol. I (Dianoiologie oder die Lehre von den Gesetzen des Denkens, Alethiologie oder Lehre von der Wahrheit) (http://books.google.de/books/about/Neues_Organon_oder_Gedanken_Uber_die_Erf.html?id=ViS3XCuJEw8C) & Vol. II (Semiotik oder Lehre von der Bezeichnung der Gedanken und Dinge, Phänomenologie oder Lehre von dem Schein) (http://books.google.de/books/about/Neues_Organon_oder_Gedanken_%C3%BCber_die_Er.html?id=X8UAAAAAcAAj). Facsimile reprint by Georg Olms Verlag, Hildesheim (Germany), 1965, with a German introduction by Hans Werner Arndt.

[Leisenring, 1969] Albert C. Leisenring, 1969. *Mathematical Logic and* Hilbert*'s ε-Symbol.* Gordon and Breach, New York.

[Mattick & Wirth, 1999] Volker Mattick and Claus-Peter Wirth, 1999. An algebraic Dexter-based hypertext reference model. Research Report (green/grey series) 719/1999, FB Informatik, Univ. Dortmund. http://wirth.bplaced.net/p/gr719, http://arxiv.org/abs/0902.3648.

[Meyer-Viol, 1995] Wilfried P. M. Meyer-Viol, 1995. *Instantial Logic — An Investigation into Reasoning with Instances.* PhD thesis, Univ. Utrecht. ILLC dissertation series 1995–11.

[Miller, 1992] Dale A. Miller, 1992. Unification under a mixed prefix. *J. Symbolic Computation*, 14:321–358.

[Murray, 1999] Neil V. Murray, editor, 1999. *8th Int. Conf. on Tableaux and Related Methods, Saratoga Springs (NY), 1999*, number 1617 in Lecture Notes in Artificial Intelligence. Springer.

[Peano, 1884] Guiseppe Peano, editor, 1884. *Angelo Genocchi — Calcolo differenziale e principii di calcolo integrale.* Fratelli Bocca, Torino (i.e. Turin, Italy). German translation: [Peano, 1899a].

[Peano, 1890] Guiseppe Peano, 1890. Démonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, 37:182–228. Facsimile also in [Peano, 1990, pp. 76–122].

[Peano, 1896f.] Guiseppe Peano, 1896f.. Studii di logica matematica. *Atti della Reale Accademia delle Scienze di Torino (i.e. Turin, Italy) — Classe di Scienze Morali, Storiche e Filologiche e Classe di Scienze Fisiche, Matematiche e Naturali*, 32:565–583. Also in Atti della Reale Accademia delle Scienze di Torino (i.e. Turin, Italy) — Classe di Scienze Fisiche, Matematiche e Naturali **32**, pp. 361–397. English translation *Studies in Mathematical Logic* in [Kennedy, 1973, pp. 190–205]. German translation: [Peano, 1899b].

[Peano, 1899a] Guiseppe Peano, editor, 1899a. *Angelo Genocchi — Differentialrechnung und Grundzüge der Integralrechnung.* B. G. Teubner Verlagsgesellschaft, Leipzig. German translation of [Peano, 1884].

[Peano, 1899b] Guiseppe Peano, 1899b. Über mathematische Logik. German translation of [Peano, 1896f.]. In [Peano, 1899a, Appendix 1]. Facsimile also in [Peano, 1990, pp. 10–26].

[Peano, 1990] Guiseppe Peano, 1990. Guiseppe Peano — *Arbeiten zur Analysis und zur mathematischen Logik.* Number 13 in Teubner-Archiv zur Mathematik. B. G. Teubner Verlagsgesellschaft. With an essay by Günter Asser (ed.).

[PINKAL &AL., 2001] Manfred Pinkal, Jörg Siekmann, and Christoph Benzmüller, 2001. Teilprojekt MI 3: DIALOG: Tutorieller Dialog mit einem Mathematik-Assistenten. In SFB 378 Resource-Adaptive Cognitive Processes, Proposal Jan. 2002 - Dez. 2004, Saarland Univ..

[PRAWITZ, 1960] Dag Prawitz, 1960. An improved proof procedure. *Theoria: A Swedish Journal of Philosophy*, 26:102–139. Also in [SIEKMANN & WRIGHTSON, 1983, Vol. 1, pp. 159–199].

[QUINE, 1981] Willard Van O. Quine, 1981. *Mathematical Logic.* Harvard Univ. Press. 4th rev. edn. (1st edn. 1940).

[ROSENFELD, 1974] Jack L. Rosenfeld, editor, 1974. *Proc. of the Congress of the Int. Federation for Information Processing (IFIP), Stockholm (Sweden), Aug. 5–10, 1974.* North-Holland (Elsevier).

[RUBIN & RUBIN, 1985] Herman Rubin and Jean E. Rubin, 1985. *Equivalents of the Axiom of Choice.* North-Holland (Elsevier). 2nd rev. edn. (1st edn. 1963).

[RUSSELL, 1905] Bertrand Russell, 1905. On Denoting. *Mind*, 14:479–493.

[RUSSELL, 1919] Bertrand Russell, 1919. *Introduction to Mathematical Philosophy.* George Allen & Unwin, London.

[SCHMIDT-SAMOA, 2006] Tobias Schmidt-Samoa, 2006. Flexible heuristics for simplification with conditional lemmas by marking formulas as forbidden, mandatory, obligatory, and generous. *J. Applied Non-Classical Logics*, 16:209–239. `http://dx.doi.org/10.3166/jancl.16.208-239`.

[SIEKMANN & WRIGHTSON, 1983] Jörg Siekmann and Graham Wrightson, editors, 1983. *Automation of Reasoning.* Springer.

[SLATER, 1994] B. Hartley Slater, 1994. *Intensional Logic — an essay in analytical metaphysics.* Avebury Series in Philosophy. Ashgate Publ. Ltd (Taylor & Francis), Aldershot (England).

[SLATER, 2002] B. Hartley Slater, 2002. *Logic Reformed.* Peter Lang AG, Bern, Switzerland.

[SLATER, 2007a] B. Hartley Slater, 2007a. Completing RUSSELL's logic. *Russell: the Journal of Bertrand Russell Studies, McMaster Univ., Hamilton (Ontario), Canada*, 27:144–158. Also in [SLATER, 2011, Chapter 3 (pp. 15–27)].

[SLATER, 2007b] B. Hartley Slater, 2007b. *The De-Mathematisation of Logic.* Polimetrica, Monza, Italy. Open access publication, `wirth.bplaced.net/op/fullpaper/hilbertbernays/Slater_2007_De-Mathematisation.pdf`.

[SLATER, 2009] B. Hartley Slater, 2009. HILBERT's epsilon calculus and its successors. In [GABBAY & WOODS, 2004ff., Vol. 5: Logic from RUSSELL to CHURCH, pp. 365–448].

[SLATER, 2011] B. Hartley Slater, 2011. *Logic is Not Mathematical.* College Publications, London.

[SMULLYAN, 1968] Raymond M. Smullyan, 1968. *First-Order Logic.* Springer. 1st edn., 2nd extd. edn. is [SMULLYAN, 1995].

[SMULLYAN, 1995] Raymond M. Smullyan, 1995. *First-Order Logic.* Dover Publications, New York. 2nd extd. edn. of [SMULLYAN, 1968], with a new preface and some corrections of the author.

[STRAWSON, 1950] Peter F. Strawson, 1950. On Referring. *Mind*, 59:320–344.

[URBAN &AL., 2004] Christian Urban, Murdoch J. Gabbay, and Andrew M. Pitts, 2004. Nominal unification. *Theoretical Computer Sci.*, 323:473–497.

[Wallen, 1990] Lincoln A. Wallen, 1990. *Automated Proof Search in Non-Classical Logics — efficient matrix proof methods for modal and intuitionistic logics.* MIT Press. Phd thesis.

[Whitehead & Russell, 1910–1913] Alfred North Whitehead and Bertrand Russell, 1910–1913. *Principia Mathematica.* Cambridge Univ. Press. 1st edn..

[Wirth, 1998] Claus-Peter Wirth, 1998. Full first-order sequent and tableau calculi with preservation of solutions and the liberalized δ-rule but without Skolemization. Research Report (green/grey series) 698/1998, FB Informatik, Univ. Dortmund. http://arxiv.org/abs/0902.3730. Short version in Gernot Salzer, Ricardo Caferra (eds.). Proc. 2nd Int. Workshop on First-Order Theorem Proving (FTP'98), pp. 244–255, Tech. Univ. Vienna, 1998. Short version also in [Caferra & Salzer, 2000, pp. 283–298].

[Wirth, 2002] Claus-Peter Wirth, 2002. A new indefinite semantics for Hilbert's epsilon. In [Egly & Fermüller, 2002, pp. 298–314]. http://wirth.bplaced.net/p/epsi.

[Wirth, 2004] Claus-Peter Wirth, 2004. Descente Infinie + Deduction. *Logic J. of the IGPL*, 12:1–96. http://wirth.bplaced.net/p/d.

[Wirth, 2006] Claus-Peter Wirth, 2006. *lim+, δ^+, and Non-Permutability of β-Steps.* SEKI-Report SR–2005–01 (ISSN 1437–4447). SEKI Publications, Saarland Univ. Rev. edn. July 2006 (1st edn. 2005), ii+36 pp., http://arxiv.org/abs/0902.3635. Thoroughly improved version is [Wirth, 2012b].

[Wirth, 2008] Claus-Peter Wirth, 2008. Hilbert's epsilon as an operator of indefinite committed choice. *J. Applied Logic*, 6:287–317. http://dx.doi.org/10.1016/j.jal.2007.07.009.

[Wirth, 2012a] Claus-Peter Wirth, 2012a. Herbrand's Fundamental Theorem in the eyes of Jean van Heijenoort. *Logica Universalis*, 6:485–520. Received Jan. 12, 2012. Published online June 22, 2012, http://dx.doi.org/10.1007/s11787-012-0056-7.

[Wirth, 2012b] Claus-Peter Wirth, 2012b. lim +, δ^+, and Non-Permutability of β-Steps. *J. Symbolic Computation*, 47:1109–1135. Received Jan. 18, 2011. Published online July 15, 2011, http://dx.doi.org/10.1016/j.jsc.2011.12.035.

[Wirth, 2012c] Claus-Peter Wirth, 2012c. Hilbert*'s epsilon as an Operator of Indefinite Committed Choice.* SEKI-Report SR–2006–02 (ISSN 1437–4447). SEKI Publications, Saarland Univ. Rev. edn. Jan. 2012, ii+73 pp., http://arxiv.org/abs/0902.3749.

[Wirth, 2014] Claus-Peter Wirth, 2014. Herbrand*'s Fundamental Theorem: The Historical Facts and their Streamlining.* SEKI-Report SR–2014–01 (ISSN 1437–4447). SEKI Publications. ii+47 pp., http://arxiv.org/abs/1405.6317.

[Wirth, 2016] Claus-Peter Wirth, 2016. *The Explicit Definition of Quantifiers via* Hilbert*'s ε is Confluent and Terminating.* SEKI-Report SR–2015–02 (ISSN 1437–4447). SEKI Publications. ii+20 pp., https://arxiv.org/abs/1611.06389. To appear also in: IfCoLog Journal of Logics and their Applications.

[Wirth &al., 2009] Claus-Peter Wirth, Jörg Siekmann, Christoph Benzmüller, and Serge Autexier, 2009. Jacques Herbrand: Life, logic, and automated deduction. In [Gabbay & Woods, 2004ff., Vol. 5: Logic from Russell to Church, pp. 195–254].

[Wirth &al., 2014] Claus-Peter Wirth, Jörg Siekmann, Christoph Benzmüller, and Serge Autexier, 2014. *Lectures on* Jacques Herbrand *as a Logician.* SEKI-Report SR–2009–01 (ISSN 1437–4447). SEKI Publications. Rev. edn. May 2014, ii+82 pp., http://arxiv.org/abs/0902.4682.

Index