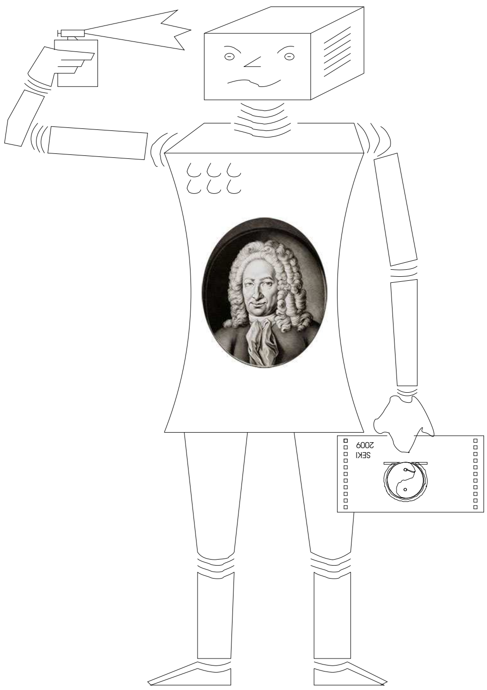

SEKI Report

http://www.dfki.de/seki
ISSN 1437-4447

A Simplified and Improved Free-Variable Framework for Hilbert's epsilon as an Operator of Indefinite Committed Choice and for Fermat's *Descente Infinie*

Claus-Peter Wirth
Dept. of Computer Sci., Saarland Univ.,
D–66123 Saarbrücken, Germany
wirth@logic.at

SEKI Report SR–2011–01

SEKI is published by the following institutions:

German Research Center for Artificial Intelligence (DFKI GmbH), Germany

- Robert Hooke Str. 5, D–28359 Bremen
- Trippstadter Str. 122, D–67663 Kaiserslautern
- Campus D 3 2, D–66123 Saarbrücken

Jacobs University Bremen, School of Engineering & Science, Campus Ring 1, D–28759 Bremen, Germany

Universität des Saarlandes, FR 6.2 Informatik, Campus, D–66123 Saarbrücken, Germany

SEKI Editor:

Claus-Peter Wirth
E-mail: wirth@logic.at
WWW: http://www.ags.uni-sb.de/~cp

Please send surface mail exclusively to:

DFKI Bremen GmbH
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A Simplified and Improved Free-Variable Framework for HILBERT's epsilon as an Operator of Indefinite Committed Choice and for FERMAT's *Descente Infinie*

CLAUS-PETER WIRTH
Dept. of Computer Sci., Saarland Univ., D–66123 Saarbrücken, Germany
wirth@logic.at

Submitted March 02, 2011

Abstract

Free variables occur in many different calculi and reasoning contexts with ad hoc and altering semantics. We present the most recent version of our free-variable framework for two-valued logics with properly improved functionality, but only two kinds of free variables left (instead of three): implicitly universally and implicitly existentially quantified ones, now simply called "free atoms" and "free variables", respectively. The quantificational expressibility and the problem-solving facilities of our framework exceed standard first-order and even higher-order modal logics, and directly support FERMAT's *descente infinie*. With the improved version of our framework, we can now model HENKIN quantification as well. We propose a new semantics for HILBERT's ε as a choice operator with the following features: We avoid overspecification (such as right-uniqueness), but admit indefinite choice, committed choice, and classical logics. Moreover, our semantics for the ε supports reductive proof search optimally and is natural in the sense that it mirrors some cases of referential interpretation of indefinite articles in natural language.

Keywords: Logical Foundations; Theories of Truth and Validity; Formalized Mathematics; Human-Oriented Interactive Theorem Proving; Automated Theorem Proving; HILBERT's ε-Operator; HENKIN Quantification; FERMAT's *Descente Infinie*

Contents

1 Overview

1.1 What is new?

Driven by a weakness in representing Henkin quantification (described in [Wirth, 2010, § 6.4.1]) and inspired by nominal terms (cf. e.g. [Urban &al., 2004]), we have improved our semantical free-variable framework for two-valued logics as presented in this paper significantly:

1. We have replaced the two-layered construction of free δ^+-variables on top of free γ-variables over free δ^--variables of [Wirth, 2004; 2008; 2010] with a one-layered construction of *free variables* over *free atoms*: Free variables with empty choice-condition now play the former rôle of the γ-variables. Free variables with non-empty choice-condition now play the former rôle of the δ^+-variables. Free atoms now play the former rôle of the δ^--variables.

2. As a consequence the proofs of the lemmas and theorems have shortened by more than a factor of 2. Therefore, we now present all the proofs in this paper and make it self-contained in this aspect; whereas in [Wirth, 2008; 2010], we had to point to [Wirth, 2004] for most of the proofs.

3. Compared to [Wirth, 2004], besides shortening the proofs, we have made the meta-level presuppositions more explicit in this paper; cf. § 4.7.

4. The difference between free variables and atoms and their names are now more standard and more clear than those of the different free variables before.

5. Last but not least, we can now treat Henkin quantification in a direct way; cf. § 4.10.

Taking all these points together, the version of our free-variable framework presented in this paper, is the version that we recommend for further reference, application, and development, because it is much easier to handle than its predecessors. And so we found it appropriate, to present the material from [Wirth, 2008; 2010] anew in this paper (omitting only the discussions on the history of an extended semantics for Hilbert's ε, on Leisenring's axiom (E2), on the tailoring of operators similar to our ε, and on the analysis of natural-language semantics). The material on mathematical induction in the style of Fermat's *descente infinie* in our framework is to be reorganized accordingly in a later publication.

1.2 Organization

This paper is organized as follows: After introductions to our free variables and atoms and their relation to reductive quantificational inference rules (§ 2) and to Hilbert's ε (§ 3), we explain and formalize our novel approach to the semantics of our free variables and atoms and the ε (§ 4), and summarize and discuss it (§ 5). We conclude in § 6.

2 Introduction to Free Variables and Atoms

2.1 Introduction to Free Variables and Atoms

Free variables or free atoms frequently occur in mathematical practice. The logical function of these free symbols varies locally; it is typically determined by the context and the obviously intended semantics in an *implicit* way. In this paper, however, we will make this function *explicit* by using disjoint sets of symbols for these different logical functions, namely $\mathbb{V}$ (the set of free *variables*), $\mathbb{A}$ (the set of *free atoms*), $\mathbb{B}$ (the set of *bound*[1] *atoms*).

An *atom* typically stands for an arbitrary object in a proof attempt or in a discourse, of which nothing is known and of which we will never want to know anything but whether it is an atom, and, if yes, whether it is a free or a bound one and whether it is identical to another atom or not. The name "atom" for such an object has a tradition in set theories with atoms. (In German, beside "Atom", also "Urelement", but with a slightly stronger semantical emphasis on origin of creation.)

A *variable*, however, in the sense as we will use the word in this paper, is a place-holder in a proof attempt or in a discourse, which gathers and stores information and which may be replaced with a definition or a description during the discourse or proof attempt. The name "free variable" for such a place-holder has a tradition in free-variable semantic tableaus; cf. [FITTING, 1996].

Both variables and atoms may be instantiated with terms. Only variables, however, may refer to free variables or atoms, or may depend on them; and only variables suffer from their instantiation in the following three aspects:

1. If a variable is instantiated, then this affects *all* of its occurrences in the whole state of the proof attempt (i.e. it is *rigid* in the terminology of semantic tableaus). Thus, if the instantiation is executed eagerly, the variable must be replaced *globally* in all terms of the whole state of the proof attempt.

2. If a variable is instantiated, it can be eliminated completely from the current state of the proof attempt without any effect on the chance to complete it into a successful proof.

3. The instantiation may be relevant for the outcome of the successful proof because the global replacement may affect the input proposition.

In contrast, atoms cannot refer to any other symbols, nor depend on them in any form. Moreover, free atoms never suffer from their instantiation in any of these aspects: They may be instantiated both locally and repeatedly in the application of lemmas or induction hypotheses, provided that the instantiation is admissible. Although the instantiation of atoms may be relevant for bookkeeping or for a replay mechanism, it can never influence the outcome of a proof.

2.2 Semantics of Free Variables and Atoms

The classification as a free variable or a free or bound atom will be indicated by adjoining a "$\mathbb{V}$", an "$\mathbb{A}$", or a "$\mathbb{B}$", as a label to the upper right of the meta-variable for the symbol, respectively. If a meta-variable stands for a symbol of the union of some of these sets, we will indicate this by listing all possible sets; e.g. "$x^{\mathbb{V}\mathbb{A}}$" is a meta-variable for a symbol that may be either a free variable or a free atom.

Meta-variables whose labels are disjoint will always denote different symbols; e.g. "$x^{\mathbb{V}}$" and "$x^{\mathbb{A}}$" will always denote different symbols; whereas "$x^{\mathbb{V}\mathbb{A}}$" may denote the same symbol as "$x^{\mathbb{A}}$". Moreover, in concrete examples, we will implicitly assume that different meta-variables denote different symbols; whereas in formal discussions, "$x^{\mathbb{A}}$" and "$y^{\mathbb{A}}$" may denote the same symbol.

As already noted in [Russell, 1919, p.155], free symbols of a (quasi-) formula often have an obviously universal intention in mathematical practice, such as the free symbols $m^{\mathbb{V}\mathbb{A}}$, $p^{\mathbb{V}\mathbb{A}}$, and $q^{\mathbb{V}\mathbb{A}}$ of the (quasi-) formula Since this is informal, it should not be tagged.

$$(m^{\mathbb{V}\mathbb{A}})^{(p^{\mathbb{V}\mathbb{A}}+q^{\mathbb{V}\mathbb{A}})} = (m^{\mathbb{V}\mathbb{A}})^{(p^{\mathbb{V}\mathbb{A}})} * (m^{\mathbb{V}\mathbb{A}})^{(q^{\mathbb{V}\mathbb{A}})}.$$

Moreover, the (quasi-) formula itself is not meant to denote a propositional function, but actually stands for the explicitly universally quantified, closed formula

$$\forall m^{\mathbb{B}}, p^{\mathbb{B}}, q^{\mathbb{B}}.\ \Big(\ (m^{\mathbb{B}})^{(p^{\mathbb{B}}+q^{\mathbb{B}})} = (m^{\mathbb{B}})^{(p^{\mathbb{B}})} * (m^{\mathbb{B}})^{(q^{\mathbb{B}})}\ \Big).$$

In this paper, however, we indicate by

$$(m^{\mathbb{A}})^{(p^{\mathbb{A}}+q^{\mathbb{A}})} = (m^{\mathbb{A}})^{(p^{\mathbb{A}})} * (m^{\mathbb{A}})^{(q^{\mathbb{A}})},$$

a proper formula with free atoms, which — independently of its context — is logically equivalent to the explicitly universally quantified formula, but which also admits the reference to the free atoms, which is required for mathematical induction in the style of Fermat's *descente infinie*, and which may also be beneficial for solving reference problems in the analysis of natural language. So the third version of these formulas combines the practical advantages of the first version with the semantical clarity of the second version.

Changing from universal to existential intention, it is somehow clear that the linear system of the (quasi-) formula

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{V}\mathbb{A}} \\ y^{\mathbb{V}\mathbb{A}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

asks us to find solutions for $x^{\mathbb{V}\mathbb{A}}$ and $y^{\mathbb{V}\mathbb{A}}$. The mere existence of such solutions is expressed by the explicitly existentially quantified, closed formula

$$\exists x^{\mathbb{B}}, y^{\mathbb{B}}.\ \left(\ \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{B}} \\ y^{\mathbb{B}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}\ \right).$$

In this paper, however, we indicate by

$$\begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix} \begin{pmatrix} x^{\mathbb{V}} \\ y^{\mathbb{V}} \end{pmatrix} = \begin{pmatrix} 11 \\ 13 \end{pmatrix}$$

a proper formula with free variables, which — independently of its context — is logically equivalent to the explicitly existentially quantified formula, but which also admits the reference to the free variables, which is required for retrieving solutions for $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$ as instantiations for $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$ chosen in a formal proof. So the third version of these formulas again combines the practical advantages of the first version with the semantical clarity of the second version.

2.3 γ- and δ-Rules

RAYMOND M. SMULLYAN has classified reductive inference rules into α-, β-, γ-, and δ-rules, and invented a uniform notation for them; cf. [SMULLYAN, 1968].

Suppose we want to prove an existential proposition $\exists y^{\mathbb{B}}.\ A$. Then the γ-rules of old-fashioned inference systems (such as [GENTZEN, 1935] or [SMULLYAN, 1968]) require us to choose a *fixed* witnessing term t as a substitute for the bound variable *immediately* when eliminating the quantifier.

Let A be a formula. We assume that all binders have minimal scope; e.g. $\forall x^{\mathbb{B}}, y^{\mathbb{B}}.\ A \wedge B$ reads $(\forall x^{\mathbb{B}}.\ \forall y^{\mathbb{B}}.\ A) \wedge B$. Let Γ and Π be *sequents*, i.e. disjunctive lists of formulas.

γ-rules: Let t be any term:

$$\frac{\qquad \Gamma \quad \exists y^{\mathbb{B}}.\ A \quad \Pi}{A\{y^{\mathbb{B}}\mapsto t\} \quad \Gamma \quad \exists y^{\mathbb{B}}.\ A \quad \Pi} \qquad\qquad \frac{\qquad \Gamma \quad \neg\forall y^{\mathbb{B}}.\ A \quad \Pi}{\neg A\{y^{\mathbb{B}}\mapsto t\} \quad \Gamma \quad \neg\forall y^{\mathbb{B}}.\ A \quad \Pi}$$

free B-variables?

Note that in the good old days when trees grew upwards, GERHARD GENTZEN (1909–1945) would have inverted the inference rules such that passing the line means consequence. In our case, passing the line means reduction, and trees grow downwards.

More modern inference systems (such as the ones in [FITTING, 1996]), however, enable us to delay the crucial choice of the term t until the state of the proof attempt may provide more information to make a successful decision. This delay is achieved by introducing a special kind of variable, called "dummy" in [PRAWITZ, 1960] and [KANGER, 1963], "free variable" in [FITTING, 1996] and in Footnote 11 of [PRAWITZ, 1960], "meta variable" in the field of planning and constraint solving, and "free γ-variable" in [WIRTH, 2004; 2008; 2010].

In this paper, we call these variables simply "free variables" and write them like "$y^{\mathbb{V}}$". When these additional variables are available, we can reduce $\exists y^{\mathbb{B}}.\ A$ first to $A\{y^{\mathbb{B}} \mapsto y^{\mathbb{V}}\}$ and then sometime later in the proof we may globally replace $y^{\mathbb{V}}$ with an appropriate term.

The addition of these free variables changes the notion of a term, but not the notation of the γ-rules, whereas it becomes visible in the δ-rules. A δ-rule may introduce either a free atom (δ^{-}*-rule*) or an ε-constrained free variable (δ^{+}*-rule*).

Forward ref

Forward ref

δ^--rules: Let $x^{\mathbb{A}}$ be a new free atom:

$$\frac{\Gamma \quad \forall x^{\mathbb{B}}.\ A \quad \Pi}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \quad \Gamma \quad \Pi} \qquad \mathbb{V}(\Gamma \ \ \forall x^{\mathbb{B}}.\ A \ \ \Pi) \times \{x^{\mathbb{A}}\}$$

$$\frac{\Gamma \quad \neg\exists x^{\mathbb{B}}.\ A \quad \Pi}{\neg A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \quad \Gamma \quad \Pi} \qquad \mathbb{V}(\Gamma \ \ \neg\exists x^{\mathbb{B}}.\ A \ \ \Pi) \times \{x^{\mathbb{A}}\}$$

Note that $\mathbb{V}(\Gamma \ \ \forall x^{\mathbb{B}}.\ A \ \ \Pi)$ stands for the set of all symbols from $\mathbb{V}$ (in this case the free variables) that occur in the sequent $\Gamma \ \ \forall x^{\mathbb{B}}.\ A \ \ \Pi$.

The free atom $x^{\mathbb{A}}$ introduced by the δ^--rules is sometimes also called "parameter", "eigenvariable", or "free δ-variable", or even also "free variable" in HILBERT-calculi, cf. [HILBERT & BERNAYS, 1968/70, Vol. I, p.102, Schema (α)]. A free atom typically stands for an arbitrary object in a discourse of which nothing is known.

The occurrence of the free atom $x^{\mathbb{A}}$ of the δ^--rules must be disallowed in the terms that may replace those free variables which have already been in use when $x^{\mathbb{A}}$ was introduced by application of the δ^--rule, i.e. the free variables of the upper sequent to which the δ^--rule was applied. The reason for this restriction of instantiation of free variables is that the dependence (or scoping) of the quantifiers must be somehow reflected in a dependence of the free variables and the free atoms. In our framework, this dependence is to be captured in binary relations on the free variables and the free atoms, called *variable-conditions*. Indeed, it is sometimes unsound to instantiate a free variable $x^{\mathbb{V}}$ with a term containing a free atom $y^{\mathbb{A}}$ that was introduced later than $x^{\mathbb{V}}$:

Permissions

Example 2.1 The formula $\exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})$
is not generally valid. We can start a proof attempt as follows:

γ-step: $\forall x^{\mathbb{B}}.\ (y^{\mathbb{V}} = x^{\mathbb{B}}),\quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})$

δ^--step: $(y^{\mathbb{V}} = x^{\mathbb{A}}),\quad \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})$

Now, if the free variable $y^{\mathbb{V}}$ could be replaced with the free atom $x^{\mathbb{A}}$, then we would get the tautology $(x^{\mathbb{A}} = x^{\mathbb{A}})$, i.e. we would have proved an invalid formula. To prevent this, the δ^--step has to record $\mathbb{V}(\forall x^{\mathbb{B}}.\ (y^{\mathbb{V}} = x^{\mathbb{B}}),\ \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (y^{\mathbb{B}} = x^{\mathbb{B}})) \times \{x^{\mathbb{A}}\} = \{(y^{\mathbb{V}}, x^{\mathbb{A}})\}$ in a variable-condition, where $(y^{\mathbb{V}}, x^{\mathbb{A}})$ means that $y^{\mathbb{V}}$ is somehow "necessarily older" than $x^{\mathbb{A}}$, so that we will never instantiate the free variable $y^{\mathbb{V}}$ with a term containing the free atom $x^{\mathbb{A}}$. Starting with empty variable-conditions, we extend the variable-conditions during proof attempts by δ-steps and by global instantiations of free variables. Roughly speaking, this kind of global instantiation of these *rigid* free variables is consistent if the resulting variable-condition (seen as a directed graph) has no cycle after adding, for each free variable $y^{\mathbb{V}}$ instantiated with a term t and for each free variable or atom $x^{\mathbb{VA}}$ occurring in t, the pair $(x^{\mathbb{VA}}, y^{\mathbb{V}})$.

Number this comment?

To make things more complicated, there are basically two different versions of the δ-rules: standard δ^--rules (also simply called "δ-rules") and δ^+-rules (also called "*liberalized* δ-rules"). They differ in the kind of symbol they introduce and — crucially — in the way they enlarge the variable-condition, depicted to the lower right of the bar:

δ^+-rules: Let $x^{\mathbb{V}}$ be a new free variable:

$$\frac{\varGamma \quad \forall x^{\mathbb{B}}.\ A \quad \varPi}{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\} \quad \varGamma \quad \varPi} \qquad \begin{array}{c}(\ x^{\mathbb{V}},\ \ \varepsilon x^{\mathbb{B}}.\ \neg A\ )\\ \mathbb{VA}(\forall x^{\mathbb{B}}.\ A) \times \{x^{\mathbb{V}}\}\end{array}$$

$$\frac{\varGamma \quad \neg\exists x^{\mathbb{B}}.\ A \quad \varPi}{\neg A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\} \quad \varGamma \quad \varPi} \qquad \begin{array}{c}(\ x^{\mathbb{V}},\ \ \varepsilon x^{\mathbb{B}}.\ A\ )\\ \mathbb{VA}(\neg\exists x^{\mathbb{B}}.\ A) \times \{x^{\mathbb{V}}\}\end{array}$$

Em, typo?

Note that, in the first δ^--rule, $\mathbb{V}(\varGamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \varPi)$ denotes the set of the free variables occurring in the whole upper sequent, whereas in the first δ^+-rule, $\mathbb{VA}(\forall x^{\mathbb{B}}.\ A)$ denotes the set of all free variables and all free atoms, but only the ones occurring in the *principal*[2] *formula* $\forall x^{\mathbb{B}}.\ A$. The smaller variable-conditions generated by the δ^+-rules mean more proofs. Indeed, the δ^+-rules enable additional proofs on the same level of γ-*multiplicity* (i.e. the maximal number of repeated γ-steps applied to the identical principal formula); cf. e.g. [WIRTH, 2004, Example 2.8, p. 21]. For certain classes of theorems, some of these proofs are exponentially and even non-elementarily shorter than the shortest proofs which apply only δ^--rules; for a survey cf. [WIRTH, 2004, § 2.1.5]. Moreover, the δ^+-rules provide additional proofs that are not only shorter but also more natural and easier to find, both automatically and for human beings; see the discussion on design goals for inference systems in [WIRTH, 2004, § 1.2.1], and the proof of the limit theorem for $+$ in [WIRTH, 2006]. All in all, the name "liberalized" for the δ^+-rules is indeed justified: They provide more freedom to the prover.[3] But, you can still prove the same things? Say so.

Moreover, note that the pairs indicated to the upper right of the bar of the δ^+-rules are to augment another global binary relation beside the variable-condition, namely a function called the *choice-condition*. This will be explained in § 3.8f. Make hyperlink.

There is a popular alternative to variable-conditions, namely Skolemization, where the δ^-- and δ^+-rules introduce functions (i.e. the order of the replacements for the bound variables is incremented) which are given the free variables of $\mathbb{V}(\varGamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \varPi)$ and $\mathbb{V}(\forall x^{\mathbb{B}}.\ A)$ as initial arguments, respectively. Then, the occur-check of unification implements the restrictions on the instantiation of free variables. In some inference systems, however, Skolemization is unsound (e.g. for higher-order systems such as the one in [KOHLHASE, 1998] or the system in [WIRTH, 2004] for *descente infinie*) or inappropriate (e.g. in the matrix systems of [WALLEN, 1990]). We prefer inference systems with variable-conditions as this is a simpler, more general, and not less efficient approach compared to Skolemizing inference systems. Note that variable-conditions do not add unnecessary complexity here:

- We will need the variable-conditions anyway for our choice-conditions, which again are needed to formalize our approach to HILBERT's ε-operator.
- If variable-conditions are superfluous, however, then we can work with empty variable-conditions as if there would be no variable-conditions at all.

3 Introduction to HILBERT's ε

3.1 Motivation, requirements specification, and overview

HILBERT's ε-symbol is a binder that forms terms; just like PEANO's ι-symbol, which is sometimes written as $\bar{\iota}$ or as an inverted ι. Roughly speaking, the term $\varepsilon x^{\mathbb{B}}.\ A$, formed from a bound variable $x^{\mathbb{B}}$ and a formula A, denotes *just some* object that is *chosen* such that — if possible — A holds for it (seen as a predicate on $x^{\mathbb{B}}$).

For ACKERMANN, BERNAYS, and HILBERT, the ε was an intermediate tool in proof theory, to be eliminated in the end. Instead of giving a model-theoretic semantics for the ε, they just specified those axioms which were essential in their proof transformations. These axioms did not provide a complete definition, but left the ε underspecified.

Descriptive terms such as $\varepsilon x^{\mathbb{B}}.\ A$ and $\iota x^{\mathbb{B}}.\ A$ are of universal interest and applicability. We suppose that our novel treatment will turn out to be useful in many areas where logic is designed or applied as a tool for description and reasoning.

For the usefulness of such descriptive terms we consider the requirements listed below to be the most important ones. Our new indefinite ε-operator satisfies these requirements and — as it is defined by novel semantical techniques — may serve as the paradigm for the design of similar operators satisfying these requirements.

Requirement I (Syntax): The syntax must clearly express where exactly a *commitment* to a choice of a special object is required, and where — to the contrary — different objects corresponding with the description may be chosen for different occurrences of the same descriptive term.

Requirement II (Reasoning): In a reductive proof step, it must be possible to replace a descriptive term with a term that corresponds with its description. The soundness of such a replacement must be expressible and should be verifiable in the original calculus.

Requirement III (Semantics): The semantics should be simple, straightforward, natural, formal, and model-based. Overspecification should be carefully avoided. Furthermore, the semantics should be modular and abstract in the sense that it adds the operator to a variety of logics, independently of the details of a concrete logic.

In [WIRTH, 2008], we have reviewed the literature on extended semantics given to HILBERT's ε-operator in the 2^{nd} half of the 20^{th} century. In this paper, we introduce to the ι and the ε (§ 3.2), to the ε's proof-theoretic origin (§ 3.3), and to our contrasting semantical objective (§ 3.4) with its emphasis on *indefinite* and *committed choice* (§ 3.5).

3.2 From the ι to the ε

The first occurrence of a descriptive ι-operator seems to be in [FREGE, 1893/1903, Vol. I], where a boldface backslash is written instead of the ι. In [PEANO, 1896f.], '$\bar{\iota}$' is written instead of 'ι'. In [PEANO, 1899b], we find an alternative notation besides '$\bar{\iota}$', namely a ι-symbol upside-down. Both notations were meant to denote the inverse of PEANO's ι-function, which constructs the singleton set of its argument. Today, we write '$\{y\}$' for PEANO's 'ιy', the upside-down ι is not easily available in typesetting, and we write a simple non-inverted ι for the descriptive ι-operator.

All the slightly differing definitions of semantics for the ι-operator agree on the following: If there is a unique x such that the formula A holds (seen as a predicate on $x^{\mathbb{B}}$), then the ι-term $\iota x^{\mathbb{B}}.\ A$ denotes this unique object.

Example 3.1 (ι-binder) For an informal introduction to the ι-binder, consider Father to be a predicate for which $\mathsf{Father}(\mathsf{Heinrich\,III}, \mathsf{Heinrich\,IV})$ holds, i.e. "Heinrich III is father of Heinrich IV". Now, "*the* father of Heinrich IV" can be denoted by $\iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV})$, and because this is nobody but Heinrich III, i.e. $\mathsf{Heinrich\,III} = \iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV})$, we know that $\mathsf{Father}(\iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV}), \mathsf{Heinrich\,IV})$. Similarly,

$$\mathsf{Father}(\iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Adam}), \mathsf{Adam}), \tag{3.1.1}$$

and thus $\exists y^{\mathbb{B}}.\ \mathsf{Father}(y^{\mathbb{B}}, \mathsf{Adam})$, but, oops! Adam and Eve do not have any fathers.
If you do not agree, you would probably appreciate the following problem that occurs when somebody has God as an additional father.

$$\mathsf{Father}(\mathsf{Holy\,Ghost}, \mathsf{Jesus}) \quad \wedge \quad \mathsf{Father}(\mathsf{Joseph}, \mathsf{Jesus}). \tag{3.1.2}$$

Then the Holy Ghost is *the* father of Jesus and Joseph is *the* father of Jesus:

$$\mathsf{Holy\,Ghost} = \iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus}) \ \wedge\ \mathsf{Joseph} = \iota x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus}) \tag{3.1.3}$$

This implies something *the* Pope may not accept, namely $\mathsf{Holy\,Ghost} = \mathsf{Joseph}$, and he anathematized Heinrich IV in the year 1076:

$$\mathsf{Anathematized}(\iota x^{\mathbb{B}}.\ \mathsf{Pope}(x^{\mathbb{B}}), \mathsf{Heinrich\,IV}, 1076). \tag{3.1.4}$$

From FREGE [1893/1903] to QUINE [1981], we find a multitude of ι-operators that are arbitrarily overspecified for the sake of completeness and syntactic eliminability. There are basically three ways of giving a semantics to the ι-terms without overspecification:

RUSSELL's ι-operator: In [WHITEHEAD & RUSSELL, 1910–1913], the ι-terms do not refer to an object but make sense only in the context of a sentence. This was nicely described already in [RUSSELL, 1905], without using any symbol for the ι, however.

HILBERT's ι-operator: To overcome the complex difficulties of that non-referential definition, in [HILBERT & BERNAYS, 1968/70, Vol. I, p. 392ff.], a completed proof of $\exists! x^{\mathbb{B}}.\ A$ was required to precede each formation of the term $\iota x^{\mathbb{B}}.\ A$, which otherwise could not be considered a well-formed term at all.

PEANO's ι-operator: Since the inflexible treatment of HILBERT's ι-operator makes the ι quite impractical and the formal syntax of logic undecidable in general, in Vol. II of the same book, the ε, however, is already given a more flexible treatment. There, the simple idea is to leave the ε-terms uninterpreted, as will be described below. In this paper, we present this more flexible view also for the ι. Moreover, this view is already PEANO's original one, cf. [PEANO, 1896f.].

At least in non-modal classical logics, it is a well justified standard that *each term denotes.* More precisely — in each model or structure $\mathcal{S}$ under consideration — each occurrence of a proper term must denote an object in the universe of $\mathcal{S}$. Following that standard, to be able to write down $\iota x^{\mathbb{B}}.\ A$ without further consideration, we have to treat $\iota x^{\mathbb{B}}.\ A$ as an uninterpreted term about which we only know

$$\exists! x^{\mathbb{B}}.\ A \quad\Rightarrow\quad A\{x^{\mathbb{B}} \mapsto \iota x^{\mathbb{B}}.\ A\} \tag{ι_0}$$

or in different notation

$$(\exists! x^{\mathbb{B}}.\ (A(x^{\mathbb{B}}))) \quad\Rightarrow\quad A(\iota x^{\mathbb{B}}.\ (A(x^{\mathbb{B}}))),$$

where, for some new $y^{\mathbb{B}}$, we can define $\exists! x^{\mathbb{B}}.\ A \ := \ \exists y^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ (x^{\mathbb{B}}{=}y^{\mathbb{B}} \ \Leftrightarrow \ A)$.

With (ι_0) as the only axiom for the ι, the term $\iota x^{\mathbb{B}}.\ A$ has to satisfy A (seen as a predicate on $x^{\mathbb{B}}$) only if there exists a unique object such that A holds for it. Moreover, the problems presented in Example 3.1 do not appear because (3.1.1) and (3.1.3) are not valid. Indeed, the description of (3.1.1) lacks existence and the descriptions of (3.1.3) and (3.1.4) lack uniqueness. The price we have to pay here is that — roughly speaking — the term $\iota x^{\mathbb{B}}.\ A$ is of no use unless the unique existence $\exists! x^{\mathbb{B}}.\ A$ can be derived.

3.3 On the ε's proof-theoretic origin

Compared to ι, the ε is more useful because — instead of (ι_0) — it comes with the stronger axiom

$$\exists x^{\mathbb{B}}.\ A \ \Rightarrow\ A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \tag{ε_0}$$

More precisely, as the formula $\exists x^{\mathbb{B}}.\ A$ (which has to be true to guarantee a meaningful interpretation of the ε-term $\varepsilon x^{\mathbb{B}}.\ A$) is weaker than the corresponding formula $\exists! x^{\mathbb{B}}.\ A$ (for the resp. ι-term), the area of useful application is wider for the ε- than for the ι-operator. Moreover, in case of $\exists! x^{\mathbb{B}}.\ A$, the ε-operator picks the same element as the ι-operator, i.e. $\exists! x^{\mathbb{B}}.\ A \ \Rightarrow\ \big(\ \varepsilon x^{\mathbb{B}}.\ A \ = \ \iota x^{\mathbb{B}}.\ A\ \big)$.

As the basic methodology of Hilbert's programme is to treat all symbols as meaningless, he does not give us any semantics but only the axiom (ε_0). Although no meaning is required, it furthers the understanding. And therefore, in [Hilbert & Bernays, 1968/70], the fundamental work which summarizes the foundational contributions of David Hilbert and his group, Paul Bernays writes:

> $\varepsilon x^{\mathbb{B}}.\ A$ ... "ist ein Ding des Individuenbereichs, und zwar ist dieses Ding gemäß der inhaltlichen Übersetzung der Formel (ε_0) *ein solches, auf das jenes Prädikat A zutrifft, vorausgesetzt, dass es überhaupt auf ein Ding des Individuenbereichs zutrifft.*"
>
> [Hilbert & Bernays, 1968/70, Vol. II, p.12, modernized orthography]
>
> $\varepsilon x^{\mathbb{B}}.\ A$ ... "is a thing of the domain of individuals for which — according to the contentual translation of the formula (ε_0) — *the predicate A holds, provided that A holds for any thing of the domain of individuals at all.*"

Example 3.2 (ε instead of ι, part I) *(continuing Example 3.1)*

Just as for the ι, for the ε we have $\mathsf{Heinrich\,III} \ = \ \varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV})$ and

$$\mathsf{Father}(\varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Heinrich\,IV}), \mathsf{Heinrich\,IV}).$$

But, from the contrapositive of (ε_0) and $\neg\mathsf{Father}(\varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Adam}), \mathsf{Adam})$, we now conclude that $\neg\exists y^{\mathbb{B}}.\ \mathsf{Father}(y^{\mathbb{B}}, \mathsf{Adam})$.

Hilbert did not need any semantics or precise intention for the ε-symbol because it was introduced merely as a formal syntactical device to facilitate proof-theoretic investigations, motivated by the possibility to get rid of the existential and universal quantifiers via

$$\exists x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \qquad (\varepsilon_1)$$
$$\forall x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg A\} \qquad (\varepsilon_2)$$

When we remove all quantifiers in a derivation of the (Hilbert-style) predicate calculus of [Hilbert & Bernays, 1968/70] along (ε_1) and (ε_2), the following transformations occur: Tautologies are turned into tautologies. The axioms

$$A\{x^{\mathbb{B}}\mapsto t\} \quad\Rightarrow\quad \exists x^{\mathbb{B}}.\ A \qquad\text{and}\qquad \forall x^{\mathbb{B}}.\ A \quad\Rightarrow\quad A\{x^{\mathbb{B}}\mapsto t\}$$

are turned into

$$A\{x^{\mathbb{B}}\mapsto t\} \quad\Rightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \qquad (\varepsilon\text{-}formula)$$

and — roughly speaking w.r.t. two-valued logics — its contrapositive, respectively. The inference steps are turned into inference steps: modus ponens into modus ponens; instantiation of free variables as well as quantifier introduction into instantiation including ε-terms. Finally, the ε-formula is taken as a new axiom scheme instead of (ε_0) because it has the advantage of being free of quantifiers.

This argumentation is actually the start of the proof transformation which constructively proves the first of Bernays' two theorems on ε-elimination in first-order logic, the so-called *1st ε-theorem*. In its extended form, this theorem may be stated as follows:

Theorem 3.3 (Extd. 1st ε-Thm., [Hilbert & Bernays, 1968/70, Vol. II, p.79f.])
From a derivation of $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.\ A$ *(containing no bound variables besides the ones bound by the prefix* $\exists x_1^{\mathbb{B}}.\ \ldots \exists x_r^{\mathbb{B}}.$*) from the formulas* $P_1, \ldots, P_k$ *(containing no bound variables) in the predicate calculus (incl., as axiom schemes, ε-formula and (to specify equality) reflexivity and substitutability), we can construct a (finite) disjunction of the form* $\bigvee_{i=0}^{s} A\{x_1^{\mathbb{B}}, \ldots, x_r^{\mathbb{B}} \mapsto t_{i,1}, \ldots, t_{i,r}\}$ *and a derivation of it, in which bound variables do not occur at all, from* $P_1, \ldots, P_k$ *in the elementary calculus (i.e. tautologies plus the inference schema (of modus ponens) and substitution of free variables).*

Note that r, s range over natural numbers including 0, and that A, $t_{i,j}$, and P_i are ε-free because otherwise they would have to include (additional) bound variables.

Moreover, the *2nd ε-Theorem* in [Hilbert & Bernays, 1968/70, Vol. II], states that the ε (just as the ι, cf. [Hilbert & Bernays, 1968/70, Vol. I]) is a conservative extension of the predicate calculus in the sense that each formal proof of an ε-free formula can be transformed into a formal proof that does not use the ε at all. Generally, however, it is not a conservative extension when we add the ε either with (ε_0), with (ε_1), or with the ε-formula to other first-order logics — may they be weaker such as intuitionistic logic, or stronger such as set theories with axiom schemes over arbitrary terms including the ε; cf. [Wirth, 2008, § 3.1.3]. Moreover, even in standard first-order logic there is no translation from the formulas containing the ε to formulas not containing it.

What is the more general case?
Axioms?

3.4 Our objective

While the historiographical and technical research on the ε-theorems is still going on and the methods of ε-elimination and ε-substitution did not die with HILBERT's Programme, this is not our subject here. We are less interested in HILBERT's formal programme and the consistency of mathematics than in the powerful use of logic in creative processes. And, instead of the tedious syntactical proof transformations, which easily lose their usefulness and elegance within their technical complexity and which — more importantly — can only refer to an already existing logic, we look for *semantical* means for finding new logics and new applications. And the question that still has to be answered in this field is: *What would be a proper semantics for* HILBERT*'s ε?*

3.5 Indefinite and committed choice

Just as the ι-symbol is usually taken to be the referential interpretation of the *definite* articles in natural languages, it is our opinion that the ε-symbol should be that of the *indefinite* determiners (articles and pronouns) such as "a(n)" or "some".

Example 3.4 (ε instead of ι, part II) *(continuing Example 3.1)*
It may well be the case that

$$\mathsf{Holy\,Ghost} = \varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus}) \quad\wedge\quad \mathsf{Joseph} = \varepsilon x^{\mathbb{B}}.\ \mathsf{Father}(x^{\mathbb{B}}, \mathsf{Jesus})$$

i.e. that "The Holy Ghost is $\underline{a}$ father of Jesus and Joseph is $\underline{a}$ father of Jesus." But this does not bring us into trouble with the Pope because we do not know whether all fathers of Jesus are equal. This will become clearer when we reconsider this in Example 3.12.

Closely connected to indefinite choice (also called "indeterminism" or "don't care nondeterminism") is the notion of *committed choice*. For example, when we have a new telephone, we typically *don't care* which number we get, but once a number has been chosen for our telephone, we will insist on a *commitment to this choice*, so that our phone number is not changed between two incoming calls.

Example 3.5 (Committed choice)
Suppose we want to prove $\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$
According to (ε_1) from § 3.3 this reduces to $\varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}}) \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$
Since there is no solution to $x^{\mathbb{B}} \neq x^{\mathbb{B}}$ we can replace $\varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$ with anything. Thus, the above reduces to $\mathsf{0} \neq \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$
and then, by exactly the same argumentation, to $\mathsf{0} \neq \mathsf{1}$
which is true in the natural numbers.
Thus, we have proved our original formula $\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$, which, however, is false.
What went wrong? Of course, we have to commit to our choice for all occurrences of the ε-term introduced when eliminating the existential quantifier: If we choose $\mathsf{0}$ on the left-hand side, we have to commit to the choice of $\mathsf{0}$ on the right-hand side as well.

3.6 Quantifier Elimination and Subordinate ε-terms

Before we can introduce to our treatment of the ε, we also have to get more acquainted with the ε in general.

The elimination of $\forall$- and $\exists$-quantifiers with the help of ε-terms (cf. § 3.3) may be more difficult than expected when some ε-terms become "subordinate" to others.

Definition 3.6 (Subordinate) An ε-term $\varepsilon v^{\mathbb{B}}.\ B$ (or, more generally, a binder on $v^{\mathbb{B}}$ together with its scope B) is *superordinate* to an (occurrence of an) ε-term $\varepsilon x^{\mathbb{B}}.\ A$ if

1. $\varepsilon x^{\mathbb{B}}.\ A$ is a subterm of B and
2. an occurrence of the variable $v^{\mathbb{B}}$ in $\varepsilon x^{\mathbb{B}}.\ A$ is free in B
   (i.e. the binder on $v^{\mathbb{B}}$ binds an occurrence of $v^{\mathbb{B}}$ in $\varepsilon x^{\mathbb{B}}.\ A$).

An (occurrence of an) ε-term $\varepsilon x^{\mathbb{B}}.\ A$ is *subordinate* to an ε-term $\varepsilon v^{\mathbb{B}}.\ B$ (or, more generally, to a binder on $v^{\mathbb{B}}$ together with its scope B) if $\varepsilon v^{\mathbb{B}}.\ B$ is superordinate to $\varepsilon x^{\mathbb{B}}.\ A$.

In [Hilbert & Bernays, 1968/70, Vol. II, p. 24], these subordinate ε-terms, which are responsible for the difficulty to prove the ε-theorems constructively, are called "untergeordnete ε-Ausdrücke". Note that we will not use a special name for ε-terms with free occurrences of variables or atoms here — such as "ε-Ausdrücke" ("quasi ε-terms") instead of "ε-Terme" ("ε-terms") — but simply call them "ε-terms", too.

Example 3.7 (Quantifier Elimination and Subordinate ε-Terms)
Let us repeat the formulas (ε_1) and (ε_2) from § 3.3 here:

$$\exists x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ A\} \qquad (\varepsilon_1)$$

$$\forall x^{\mathbb{B}}.\ A \quad\Leftrightarrow\quad A\{x^{\mathbb{B}} \mapsto \varepsilon x^{\mathbb{B}}.\ \neg A\} \qquad (\varepsilon_2)$$

Let us consider the formula

$$\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$$

and apply (ε_1) and (ε_2) to remove the three quantifiers completely.

We introduce the following abbreviations:

$$\begin{array}{lcl} z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y^{\mathbb{B}}) & = & \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z_a^{\mathbb{B}}) \\ y_a(w^{\mathbb{B}})(x^{\mathbb{B}}) & = & \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y_a^{\mathbb{B}})) \\ x_a(w^{\mathbb{B}}) & = & \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}(w^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a(w^{\mathbb{B}})(x_a^{\mathbb{B}}), z_a(w^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a(w^{\mathbb{B}})(x_a^{\mathbb{B}}))), \\ w_a & = & \varepsilon w_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_a(w_a^{\mathbb{B}})(x_a(w_a^{\mathbb{B}})), z_a(w_a^{\mathbb{B}})(x_a(w_a^{\mathbb{B}}))(y_a(w_a^{\mathbb{B}})(x_a(w_a^{\mathbb{B}})))), \end{array}$$

Hmm, forgot what B is.

In [Wirth, 2008; 2010], we have shown that the outside-in elimination leads to the same result as the inside-out elimination, but is not linear in the number of steps. Thus, we eliminate inside-out, i.e. we start with the elimination of $\forall z^{\mathbb{B}}$. The transformation is:

$$\begin{array}{ll} \exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}}), \\ \exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y^{\mathbb{B}})), \\ \exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y_a(w^{\mathbb{B}})(x^{\mathbb{B}}), z_a(w^{\mathbb{B}})(x^{\mathbb{B}})(y_a(w^{\mathbb{B}})(x^{\mathbb{B}}))), \\ \exists w^{\mathbb{B}}. & \mathsf{P}(w^{\mathbb{B}}, x_a(w^{\mathbb{B}}), y_a(w^{\mathbb{B}})(x_a(w^{\mathbb{B}})), z_a(w^{\mathbb{B}})(x_a(w^{\mathbb{B}}))(y_a(w^{\mathbb{B}})(x_a(w^{\mathbb{B}})))), \\ & \mathsf{P}(w_a, x_a(w_a), y_a(w_a)(x_a(w_a)), z_a(w_a)(x_a(w_a))(y_a(w_a)(x_a(w_a)))). \end{array}$$

Note that the resulting formula is quite deep and has more than one thousand occurrences of the ε-binder. Indeed, in general n nested quantifiers result in an ε-nesting depth of $2^n - 1$.

To understand this a bit, let us have a closer look a the resulting formula. Let us write it as

$$\mathsf{P}(w_a, x_b, y_d, z_h) \tag{3.7.1}$$

then (after renaming some bound atoms) we have

$$
\begin{array}{rcll}
z_h & = & \varepsilon z_h^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b, y_d, z_h^{\mathbb{B}}), & (3.7.2)\\
y_d & = & \varepsilon y_d^{\mathbb{B}}.\ \mathsf{P}(w_a, x_b, y_d^{\mathbb{B}}, z_g(y_d^{\mathbb{B}})) & (3.7.3)\\
 & & \text{with } z_g(y_d^{\mathbb{B}}) = \varepsilon z_g^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b, y_d^{\mathbb{B}}, z_g^{\mathbb{B}}), & (3.7.4)\\
x_b & = & \varepsilon x_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c(x_b^{\mathbb{B}}), z_f(x_b^{\mathbb{B}})) & (3.7.5)\\
 & & \text{with } z_f(x_b^{\mathbb{B}}) = \varepsilon z_f^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c(x_b^{\mathbb{B}}), z_f^{\mathbb{B}}) & (3.7.6)\\
 & & \text{and } \ y_c(x_b^{\mathbb{B}}) = \varepsilon y_c^{\mathbb{B}}.\ \mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e(x_b^{\mathbb{B}})(y_c^{\mathbb{B}})) & (3.7.7)\\
 & & \qquad\qquad \text{with } z_e(x_b^{\mathbb{B}})(y_c^{\mathbb{B}}) = \varepsilon z_e^{\mathbb{B}}.\ \neg\mathsf{P}(w_a, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e^{\mathbb{B}}), & (3.7.8)\\
w_a & = & \varepsilon w_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b(w_a^{\mathbb{B}}), z_d(w_a^{\mathbb{B}})) & (3.7.9)\\
 & & \text{with } z_d(w_a^{\mathbb{B}}) = \varepsilon z_d^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b(w_a^{\mathbb{B}}), z_d^{\mathbb{B}}) & (3.7.10)\\
 & & \text{and } \ y_b(w_a^{\mathbb{B}}) = \varepsilon y_b^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c(w_a^{\mathbb{B}})(y_b^{\mathbb{B}})) & (3.7.11)\\
 & & \qquad\qquad \text{with } z_c(w_a^{\mathbb{B}})(y_b^{\mathbb{B}}) = \varepsilon z_c^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c^{\mathbb{B}}), & (3.7.12)\\
 & & \quad x_a(w_a^{\mathbb{B}}) = \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})) & (3.7.13)\\
 & & \qquad\qquad \text{with } z_b(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}) = \varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b^{\mathbb{B}}) & (3.7.14)\\
 & & \qquad\qquad \text{and } \ y_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}) = \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}})) & (3.7.15)\\
 & & \qquad\qquad\qquad \text{with } z_a(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}}) = \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}}). & (3.7.16)
\end{array}
$$

Firstly, note that the free-occurring bound atoms

$$
\begin{array}{c}
z_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_b^{\mathbb{B}}, x_a^{\mathbb{B}},\\
z_c^{\mathbb{B}}, y_b^{\mathbb{B}}, z_d^{\mathbb{B}}, w_a^{\mathbb{B}},\\
z_e^{\mathbb{B}}, y_c^{\mathbb{B}}, z_f^{\mathbb{B}}, x_b^{\mathbb{B}},\\
z_g^{\mathbb{B}}, y_d^{\mathbb{B}}, z_h^{\mathbb{B}},
\end{array}
$$

in the indented ε-terms are actually bound by the next ε to the left, to which the respective ε-terms thus become subordinate. For example, the ε-term $z_g(y_d^{\mathbb{B}})$ is subordinate to the ε-term y_d. Secondly, the ε-terms of these equations are exactly those that require a commitment to their choice. This means that each of z_a, z_b, z_c, z_d z_e, z_f, z_g, z_h, each of y_a, y_b, y_c, y_d, and each of x_a, x_b may be chosen differently without affecting soundness of the equivalence transformation. Note that the variables are strictly nested into each other; so we must choose in the order of z_a, y_a, z_b, x_a, z_c, y_b, z_d, w_a, z_e, y_c, z_f, x_b, z_g, y_d, z_h. Moreover, in case of all ε-terms except w_a, x_b, y_d, z_h, we actually have to choose a function instead of a simple object. In HILBERT's view, however, there are neither functions nor objects at all, but only terms (and quasi-terms (i.e. with free occurrences of bound atoms)), where $x_a(w_a^{\mathbb{B}})$ reads

$$
\varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}\left(
\begin{array}{l}
w_a^{\mathbb{B}},\\
x_a^{\mathbb{B}},\\
\varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}\big(\ w_a^{\mathbb{B}},\ \ x_a^{\mathbb{B}},\ \ y_a^{\mathbb{B}},\ \ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}})\ \big),\\
\varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}\Big(\ w_a^{\mathbb{B}},\ \ x_a^{\mathbb{B}},\ \ \varepsilon y_a^{\mathbb{B}}.\ \mathsf{P}\big(\ w_a^{\mathbb{B}},\ \ x_a^{\mathbb{B}},\ \ y_a^{\mathbb{B}},\ \ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}})\ \big),\ \ z_b^{\mathbb{B}}\ \Big)
\end{array}
\right),
$$

$y_b(w_a^{\mathbb{B}})$ reads

$$\varepsilon y^{\mathbb{B}}_{b}.\ \neg\mathsf{P}\left(\begin{array}{l}
w^{\mathbb{B}}_{a},\\
\varepsilon x^{\mathbb{B}}_{a}.\ \ \neg\mathsf{P}\left(\begin{array}{l}
w^{\mathbb{B}}_{a},\\
x^{\mathbb{B}}_{a},\\
\varepsilon y^{\mathbb{B}}_{a}.\ \mathsf{P}(\ w^{\mathbb{B}}_{a},\ \ x^{\mathbb{B}}_{a},\ \ y^{\mathbb{B}}_{a},\ \ \varepsilon z^{\mathbb{B}}_{a}.\ \neg\mathsf{P}(w^{\mathbb{B}}_{a},x^{\mathbb{B}}_{a},y^{\mathbb{B}}_{a},z^{\mathbb{B}}_{a})\ \ ),\\
\varepsilon z^{\mathbb{B}}_{b}.\ \neg\mathsf{P}\Big(\ w^{\mathbb{B}}_{a},\ \ x^{\mathbb{B}}_{a},\ \ \varepsilon y^{\mathbb{B}}_{a}.\ \mathsf{P}(\ w^{\mathbb{B}}_{a},\ \ x^{\mathbb{B}}_{a},\ \ y^{\mathbb{B}}_{a},\ \ \varepsilon z^{\mathbb{B}}_{a}.\ \neg\mathsf{P}(w^{\mathbb{B}}_{a},x^{\mathbb{B}}_{a},y^{\mathbb{B}}_{a},z^{\mathbb{B}}_{a})\ \ ),\ \ z^{\mathbb{B}}_{b}\ \Big)
\end{array}\right),\\
y^{\mathbb{B}}_{b},\\
\varepsilon z^{\mathbb{B}}_{c}.\ \neg\mathsf{P}\left(\begin{array}{l}
w^{\mathbb{B}}_{a},\\
\varepsilon x^{\mathbb{B}}_{a}.\ \ \neg\mathsf{P}\left(\begin{array}{l}
w^{\mathbb{B}}_{a},\\
x^{\mathbb{B}}_{a},\\
\varepsilon y^{\mathbb{B}}_{a}.\ \mathsf{P}(\ w^{\mathbb{B}}_{a},\ \ x^{\mathbb{B}}_{a},\ \ y^{\mathbb{B}}_{a},\ \ \varepsilon z^{\mathbb{B}}_{a}.\ \neg\mathsf{P}(w^{\mathbb{B}}_{a},x^{\mathbb{B}}_{a},y^{\mathbb{B}}_{a},z^{\mathbb{B}}_{a})\ \ ),\\
\varepsilon z^{\mathbb{B}}_{b}.\ \neg\mathsf{P}\Big(\ w^{\mathbb{B}}_{a},\ \ x^{\mathbb{B}}_{a},\ \ \varepsilon y^{\mathbb{B}}_{a}.\ \mathsf{P}(\ w^{\mathbb{B}}_{a},\ \ x^{\mathbb{B}}_{a},\ \ y^{\mathbb{B}}_{a},\ \ \varepsilon z^{\mathbb{B}}_{a}.\ \neg\mathsf{P}(w^{\mathbb{B}}_{a},x^{\mathbb{B}}_{a},y^{\mathbb{B}}_{a},z^{\mathbb{B}}_{a})\ \ ),\ \ z^{\mathbb{B}}_{b}\ \Big)
\end{array}\right),\\
y^{\mathbb{B}}_{b},\\
z^{\mathbb{B}}_{c}
\end{array}\right)
\end{array}\right).$$

	ε-nesting depth	number of ε-binders	Ackermann rank	Ackermann degree
$z_a(w^{\mathbb{B}}_a)(x^{\mathbb{B}}_a)(y^{\mathbb{B}}_a)$	1	1	1	undefined
$y_a(w^{\mathbb{B}}_a)(x^{\mathbb{B}}_a)$	2	2	2	undefined
$z_b(w^{\mathbb{B}}_a)(x^{\mathbb{B}}_a)$	3	3	1	undefined
$x_a(w^{\mathbb{B}}_a)$	4	6	3	undefined
$z_c(w^{\mathbb{B}}_a)(y^{\mathbb{B}}_b)$	5	7	1	undefined
$y_b(w^{\mathbb{B}}_a)$	6	14	2	undefined
$z_d(w^{\mathbb{B}}_a)$	7	21	1	undefined
w_a	8	42	4	1
$z_e(y^{\mathbb{B}}_c)(w^{\mathbb{B}}_a)$	9	43	1	undefined
$y_c(x^{\mathbb{B}}_b)$	10	86	2	undefined
$z_f(x^{\mathbb{B}}_b)$	11	129	1	undefined
x_b	12	258	3	2
$z_g(y^{\mathbb{B}}_d)$	13	301	1	undefined
y_d	14	602	2	3
z_h	15	903	1	4
$\mathsf{P}(w_a, x_b, y_d, z_h)$	15	1805	undefined	undefined

For $\forall w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \forall y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$ instead of $\exists w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$, we get the same exponential growth of nesting depth as in Example 3.7 above, when we completely eliminate the quantifiers using (ε_2). The only difference is that we get additional occurrences of '$\neg$'. But when we have quantifiers of the same kind like '$\exists$' or '$\forall$', we had better choose them in parallel; e.g., for $\forall w^{\mathbb{B}}.\ \forall x^{\mathbb{B}}.\ \forall y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{B}}, x^{\mathbb{B}}, y^{\mathbb{B}}, z^{\mathbb{B}})$, we choose

$$v_a := \varepsilon v^{\mathbb{B}}.\ \neg\mathsf{P}(1^{\text{st}}(v^{\mathbb{B}}), 2^{\text{nd}}(v^{\mathbb{B}}), 3^{\text{rd}}(v^{\mathbb{B}}), 4^{\text{th}}(v^{\mathbb{B}})),$$

and then take $\mathsf{P}(1^{\text{st}}(v_a), 2^{\text{nd}}(v_a), 3^{\text{rd}}(v_a), 4^{\text{th}}(v_a))$ as result of the elimination.

Roughly speaking, in today's theorem proving, cf. e.g. [Fitting, 1996], [Wirth, 2004], the exponential explosion of term depth of Example 3.7 is avoided by an outside-in removal of δ-quantifiers *without removing the quantifiers below ε-binders* and by a replacement of γ-quantified variables with free variables without choice-conditions. For the formula of Example 3.7, this yields $\mathsf{P}(w^{\mathbb{V}}, x_e, y^{\mathbb{V}}, z_e)$ with $x_e = \varepsilon x^{\mathbb{B}}_e.\ \neg\exists y^{\mathbb{B}}.\ \forall z^{\mathbb{B}}.\ \mathsf{P}(w^{\mathbb{V}}, x^{\mathbb{B}}_e, y^{\mathbb{B}}, z^{\mathbb{B}})$ and $z_e = \varepsilon z^{\mathbb{B}}_e.\ \neg\mathsf{P}(w^{\mathbb{V}}, x_e, y^{\mathbb{V}}, z^{\mathbb{B}}_e)$. Thus, in general, the nesting of binders for the complete elimination of a prenex of n quantifiers does not become deeper than $\frac{1}{4}(n{+}1)^2$.

Moreover, if we are only interested in reduction and not in equivalence transformation of a formula, we can abstract Skolem terms from the ε-terms and just reduce to the formula $\mathsf{P}(w^{\mathbb{V}}, x^{\mathbb{A}}(w^{\mathbb{V}}), y^{\mathbb{V}}, z^{\mathbb{A}}(w^{\mathbb{V}})(y^{\mathbb{V}}))$. In non-Skolemizing inference systems with variable-conditions we get $\mathsf{P}(w^{\mathbb{V}}, x^{\mathbb{A}}, y^{\mathbb{V}}, z^{\mathbb{A}})$ instead, with $\{(w^{\mathbb{V}}, x^{\mathbb{A}}),\ (w^{\mathbb{V}}, z^{\mathbb{A}}),\ (y^{\mathbb{V}}, z^{\mathbb{A}})\}$ as an extension to the variable-condition. Note that with Skolemization or variable-conditions we have no growth of nesting depth at all, and the same will be the case for our approach to ε-terms.

3.7 Do not be afraid of Indefiniteness!

From the discussion in § 3.5, one could get the impression that an indefinite logical treatment of the ε is not easy to find. Indeed, on the first sight, there is the problem that some standard axiom schemes cannot be taken for granted, such as substitutability

$$s = t \qquad \Rightarrow \qquad f(s) = f(t)$$

and reflexivity

$$t = t$$

Note that substitutability is similar to ALBERT C. LEISENRING's

$$\forall x^{\mathbb{B}}.\ (A_0 \Leftrightarrow A_1) \qquad \Rightarrow \qquad \varepsilon x^{\mathbb{B}}.\ A_0 \ = \ \varepsilon x^{\mathbb{B}}.\ A_1 \tag{E2}$$

(cf. [LEISENRING, 1969]) when we take logical equivalence as equality. Moreover, note that

$$\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \ = \ \varepsilon x^{\mathbb{B}}.\ \mathsf{true} \tag{REFLEX}$$

is an instance of reflexivity.

This means that it is not definitely okay to replace a subterm with an equal term and that even syntactically equal terms may not be definitely equal.

It may be interesting to see that — in computer programs — we are quite used to committed choice and to an indefinite behavior of choosing, and that the violation of substitutability and even reflexivity is no problem there:

Example 3.8 (Violation of Substitutability and Reflexivity in Programs)

In the implementation of the specification of the web-based hypertext system of [MATTICK & WIRTH, 1999], we needed a function that chooses an element from a set implemented as a list. Its ML code is:

```
fun choose s = case s of Set (i :: _) => i | _ => raise Empty;
```

And, of course, it simply returns the first element of the list. For another set that is equal — but where the list may have another order — the result may be different. Thus, the behavior of the function `choose` is indefinite for a given set, but any time it is called for an implemented set, it chooses a special element and *commits to this choice*, i.e.: when called again, it returns the same value. In this case we have `choose s` = `choose s`, but `s` = `t` does not imply `choose s` = `choose t`. In an implementation where some parallel reordering of lists may take place, even `choose s` = `choose s` may be wrong.

From this example we may learn that the question of `choose s` = `choose s` may be indefinite until the choice steps have actually been performed. *This is exactly how we will treat our ε.* The steps that are performed in logic are related to proving: Reductive inference steps that make proof trees grow toward the leaves, and choice steps that instantiate variables and atoms for various purposes.

Thus, on the one hand, when we want to prove

$$\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \ = \ \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$$

we can choose $\mathsf{0}$ for both occurrences of $\varepsilon x^{\mathbb{B}}.\ \mathsf{true}$, get $\mathsf{0} = \mathsf{0}$, and the proof is successful.

On the other hand, when we want to prove

$$\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \ \neq\ \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$$

we can choose $\mathsf{0}$ for one occurrence and $\mathsf{1}$ for the other, get $\mathsf{0} \neq \mathsf{1}$, and the proof is successful again. This procedure may seem wondrous again, but is very similar to something quite common with free variables with empty choice-conditions (cf. § 2.1):

On the one hand, when we want to prove

$$x^{\mathbb{V}} = y^{\mathbb{V}}$$

we can choose $\mathsf{0}$ to replace both $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$, get $\mathsf{0} = \mathsf{0}$, and the proof is successful.

On the other hand, when we want to prove

$$x^{\mathbb{V}} \neq y^{\mathbb{V}}$$

we can choose $\mathsf{0}$ to replace $x^{\mathbb{V}}$ and $\mathsf{1}$ to replace $y^{\mathbb{V}}$, get $\mathsf{0} \neq \mathsf{1}$, and the proof is successful again.

3.8 Replacing ε-terms with Free Variables

There is an important difference between the inequations $\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \neq \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$ and $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ at the end of the previous § 3.7: The latter does not violate the reflexivity axiom! And we are going to cure the violation of the former immediately with the help of our free variables, but now with non-empty choice-conditions. Instead of $\varepsilon x^{\mathbb{B}}.\ \mathsf{true} \neq \varepsilon x^{\mathbb{B}}.\ \mathsf{true}$ we write $x^{\mathbb{V}} \neq y^{\mathbb{V}}$ and remember what these free variables stand for by storing this into a function C, called a *choice-condition*:

$$\begin{aligned} C(x^{\mathbb{V}}) &:= \varepsilon x^{\mathbb{B}}.\ \mathsf{true}, \\ C(y^{\mathbb{V}}) &:= \varepsilon x^{\mathbb{B}}.\ \mathsf{true}. \end{aligned}$$

For a first step, suppose that our ε-terms are not subordinate to any outside binder, cf. Definition 3.6. Then, we can replace an ε-term $\varepsilon z^{\mathbb{B}}.\ A$ with a new free variable $z^{\mathbb{V}}$ and extend the partial function C by

$$C(z^{\mathbb{V}}) \ := \ \varepsilon z^{\mathbb{B}}.\ A.$$

By this procedure we can eliminate all ε-terms without loosing any syntactical information.

As a first consequence of this elimination, the substitutability and reflexivity axioms are immediately regained, and the problems discussed in § 3.7 disappear.

A second reason for replacing the ε-terms with free variables is that the latter can solve the question whether a committed choice is required: We can express (on the one hand) a committed choice by using the same free variable and (on the other hand) a choice without commitment by using a fresh variable with the same choice-condition.

Indeed, this also solves our problems with committed choice of Example 3.5 of § 3.5: Now, again using (ε_1), $\exists x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$ reduces to $x^{\mathbb{V}} \neq x^{\mathbb{V}}$ with

$$C(x^{\mathbb{V}}) \ := \ \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} \neq x^{\mathbb{B}})$$

and the proof attempt immediately fails because of the now regained reflexivity axiom.

As the second step, we still have to explain what to do with subordinate ε-terms. If the ε-term $\varepsilon v_l^{\mathbb{B}}.\ A$ contains free occurrences of exactly the distinct bound atoms $v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}$, then we have to replace this ε-term with the application term $z^{\mathbb{V}}(v_0^{\mathbb{B}})\cdots(v_{l-1}^{\mathbb{B}})$ of the same type as $v_l^{\mathbb{B}}$ (for a new free variable $z^{\mathbb{V}}$) and to extend the choice-condition C by

$$C(z^{\mathbb{V}}) \ := \ \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ A.$$

Example 3.9 (Higher-Order Choice-Condition) *(continuing Example 3.7 of § 3.6)*

In our framework, the complete elimination of ε-terms in (3.7.1) of Example 3.7 results in

$$\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}}) \qquad \text{(cf. (3.7.1)!)}$$

with the following higher-order choice-condition:

$$\begin{array}{lcrl}
C(z_h^{\mathbb{V}}) & := & \varepsilon z_h^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{B}}) & \text{(cf. (3.7.2)!)} \\
C(y_d^{\mathbb{V}}) & := & \varepsilon y_d^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{B}}, z_c^{\mathbb{V}}(y_d^{\mathbb{B}})) & \text{(cf. (3.7.3)!)} \\
C(z_g^{\mathbb{V}}) & := & \lambda y_d^{\mathbb{B}}.\ \varepsilon z_g^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{B}}, z_g^{\mathbb{B}}) & \text{(cf. (3.7.4)!)} \\
C(x_b^{\mathbb{V}}) & := & \varepsilon x_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{V}}(x_b^{\mathbb{B}}), z_f^{\mathbb{V}}(x_b^{\mathbb{B}})) & \text{(cf. (3.7.5)!)} \\
C(z_f^{\mathbb{V}}) & := & \lambda x_b^{\mathbb{B}}.\ \varepsilon z_f^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{V}}(x_b^{\mathbb{B}}), z_f^{\mathbb{B}}) & \text{(cf. (3.7.6)!)} \\
C(y_c^{\mathbb{V}}) & := & \lambda x_b^{\mathbb{B}}.\ \varepsilon y_c^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e^{\mathbb{V}}(x_b^{\mathbb{B}})(y_c^{\mathbb{B}})) & \text{(cf. (3.7.7)!)} \\
C(z_e^{\mathbb{V}}) & := & \lambda x_b^{\mathbb{B}}.\ \lambda y_c^{\mathbb{B}}.\ \varepsilon z_e^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_c^{\mathbb{B}}, z_e^{\mathbb{B}}) & \text{(cf. (3.7.8)!)} \\
C(w_a^{\mathbb{V}}) & := & \varepsilon w_a^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{V}}(w_a^{\mathbb{B}}), z_d^{\mathbb{V}}(w_a^{\mathbb{B}})) & \text{(cf. (3.7.9)!)} \\
C(z_d^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \varepsilon z_d^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{V}}(w_a^{\mathbb{B}}), z_d^{\mathbb{B}}) & \text{(cf. (3.7.10)!)} \\
C(y_b^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \varepsilon y_b^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c^{\mathbb{V}}(w_a^{\mathbb{B}})(y_b^{\mathbb{B}})) & \text{(cf. (3.7.11)!)} \\
C(z_c^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda y_b^{\mathbb{B}}.\ \varepsilon z_c^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{V}}(w_a^{\mathbb{B}}), y_b^{\mathbb{B}}, z_c^{\mathbb{B}}) & \text{(cf. (3.7.12)!)} \\
C(x_a^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \varepsilon x_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})) & \text{(cf. (3.7.13)!)} \\
C(z_b^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda x_a^{\mathbb{B}}.\ \varepsilon z_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}}), z_b^{\mathbb{B}}) & \text{(cf. (3.7.14)!)} \\
C(y_a^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda x_a^{\mathbb{B}}.\ \varepsilon y_a^{\mathbb{B}}.\ \ \mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{V}}(w_a^{\mathbb{B}})(x_a^{\mathbb{B}})(y_a^{\mathbb{B}})) & \text{(cf. (3.7.15)!)} \\
C(z_a^{\mathbb{V}}) & := & \lambda w_a^{\mathbb{B}}.\ \lambda x_a^{\mathbb{B}}.\ \lambda y_a^{\mathbb{B}}.\ \varepsilon z_a^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{B}}, x_a^{\mathbb{B}}, y_a^{\mathbb{B}}, z_a^{\mathbb{B}}) & \text{(cf. (3.7.16)!)}
\end{array}$$

Note that this representation of (3.7.1) is smaller and easier to understand than all previous ones. Indeed, by combination of λ-abstraction and term sharing via free variables, in our framework the ε becomes practically feasible.

3.9 Instantiating Free Variables (“ε-Substitution”)

Having realized Requirement I (Syntax) of § 3.1 in the previous § 3.8, in this § 3.9 we are now going to explain how to satisfy Requirement II (Reasoning). To this end, we have to explain how to replace free variables with terms that satisfy their choice-conditions.

The first thing to know about free variables with choice-conditions is: Just like the the free variables without choice-conditions (introduced by γ-rules e.g.) and contrary to free atoms, the free variables with choice-conditions (introduced by δ^{+}-rules e.g.) are *rigid* in the sense that the only way to replace a free variable is to do it *globally*, i.e. in all formulas and all choice-conditions in an atomic transaction.

In *reductive* theorem proving such as in sequent, tableau, or matrix calculi we are in the following situation: While a free variable without choice-condition can be replaced with nearly everything, the replacement of a free variable with a choice-condition requires some proof work, and a free atom cannot be instantiated at all.

Contrariwise, when formulas are used as tools instead of tasks, free atoms can indeed be replaced — and this even locally (i.e. non-rigidly). This is the case not only for purely *generative* calculi, (such as resolution and paramodulation calculi) and HILBERT-style calculi (such as the predicate calculus of [HILBERT & BERNAYS, 1968/70]), but also for the lemma and induction hypothesis application in the otherwise reductive calculi of [WIRTH, 2004], cf. [WIRTH, 2004, § 2.5.2].

More precisely — again considering *reductive* theorem proving, where formulas are proof tasks — a free variable without choice-condition may be instantiated with any term (of appropriate type) that does not violate the current variable-condition, cf. § 4.6 for details. The instantiation of a free variable with choice-condition additionally requires some proof work depending on the current choice-condition, cf. Definition 4.13 for the formal details. In general, if a substitution σ replaces — possibly among other free variables — the free variable $y^{\mathbb{V}}$ in the domain of the choice-condition C, then — to know that the global instantiation of the whole proof forest with σ is consistent — we have to prove $(Q_C(y^{\mathbb{V}}))\sigma$, where Q_C is given as follows:

Definition 3.10 (Q_C)
Q_C is the function that maps every $y^{\mathbb{V}} \in \mathrm{dom}(C)$ with

$$C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$$

(for some bound atoms $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}$ and some formula B) to the single-formula sequent

$$\forall v_0^{\mathbb{B}}.\ \ldots \forall v_{l-1}^{\mathbb{B}}.\ \big(\ \exists v_l^{\mathbb{B}}.\ B \quad \Rightarrow \quad B\{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}\ \big),$$

and is otherwise undefined.

Note that $Q_C(y^{\mathbb{V}})$ is nothing but a formulation of axiom (ε_0) from § 3.3 in our framework, and Lemma 4.19 states its validity.

Example 3.11 (Predecessor Function)

Suppose that our domain is natural numbers and that $y^{\mathbb{V}}$ has the choice-condition

$$C(y^{\mathbb{V}}) \quad = \quad \lambda v_0^{\mathbb{B}}.\ \varepsilon v_1^{\mathbb{B}}.\ \big(\ v_0^{\mathbb{B}} = v_1^{\mathbb{B}} + 1\ \big).$$

Then, before we may instantiate $y^{\mathbb{V}}$ with the symbol p for the predecessor function specified by

$$\forall x^{\mathbb{B}}.\ \big(\ \mathsf{p}(x^{\mathbb{B}}{+}1) = x^{\mathbb{B}}\ \big),$$

we have to prove the single-formula sequent $(Q(y^{\mathbb{V}}))\{y^{\mathbb{V}} \mapsto \mathsf{p}\}$, which reads

$$\forall v_0^{\mathbb{B}}.\ \Big(\ \exists v_1^{\mathbb{B}}.\ \big(\ v_0^{\mathbb{B}} = v_1^{\mathbb{B}} + 1\ \big) \Rightarrow \big(\ v_0^{\mathbb{B}} = \mathsf{p}(v_0^{\mathbb{B}}) + 1\ \big)\ \Big),$$

Moreover, the single formula of this sequent immediately follows from the specification of p.

Example 3.12 (Canossa 1077) *(continuing Example 3.4)*

The situation of Example 3.4 now reads

$$\mathsf{Holy\ Ghost} \ = \ z_0^{\mathbb{V}} \quad \wedge \quad \mathsf{Joseph} \ = \ z_1^{\mathbb{V}} \tag{3.12.1}$$

with $\quad C(z_0^{\mathbb{V}}) = \varepsilon z_0^{\mathbb{B}}.\ \mathsf{Father}(z_0^{\mathbb{B}}, \mathsf{Jesus}),$
and $\quad C(z_1^{\mathbb{V}}) = \varepsilon z_1^{\mathbb{B}}.\ \mathsf{Father}(z_1^{\mathbb{B}}, \mathsf{Jesus}).$

This does not bring us into the old trouble with the Pope because nobody knows whether $z_0^{\mathbb{V}} = z_1^{\mathbb{V}}$ holds or not.

On the one hand, knowing (3.1.2) from Example 3.1 of § 3.2, we can prove (3.12.1) as follows: Let us replace $z_0^{\mathbb{V}}$ with $\mathsf{Holy\ Ghost}$ because, for $\sigma_0 := \{z_0^{\mathbb{V}} \mapsto \mathsf{Holy\ Ghost}\}$, from $\mathsf{Father}(\mathsf{Holy\ Ghost}, \mathsf{Jesus})$ we obtain

$$\exists z_0^{\mathbb{B}}.\ \mathsf{Father}(z_0^{\mathbb{B}}, \mathsf{Jesus}) \quad \Rightarrow \quad \mathsf{Father}(\mathsf{Holy\ Ghost}, \mathsf{Jesus}),$$

which is nothing but the required $(Q_C(z_0^{\mathbb{V}}))\sigma_0$.

Analogously, we replace $z_1^{\mathbb{V}}$ with Joseph because, for $\sigma_1 := \{z_1^{\mathbb{V}} \mapsto \mathsf{Joseph}\}$, from (3.1.2) we also obtain the required $(Q_C(z_1^{\mathbb{V}}))\sigma_1$. After these replacements, (3.12.1) becomes the tautology

$$\mathsf{Holy\ Ghost} \ = \ \mathsf{Holy\ Ghost} \quad \wedge \quad \mathsf{Joseph} \ = \ \mathsf{Joseph}$$

On the other hand, if we want to have trouble, we can apply the substitution

$$\sigma' \ = \ \{z_0^{\mathbb{V}} \mapsto \mathsf{Joseph},\ z_1^{\mathbb{V}} \mapsto \mathsf{Joseph}\}$$

to (3.12.1) because of $(Q_C(z_0^{\mathbb{V}}))\sigma' = (Q_C(z_1^{\mathbb{V}}))\sigma_1 = (Q_C(z_1^{\mathbb{V}}))\sigma'$.
Then our task is to show

$$\mathsf{Holy\ Ghost} \ = \ \mathsf{Joseph} \quad \wedge \quad \mathsf{Joseph} \ = \ \mathsf{Joseph}$$

Note that this course of action is stupid already under the aspect of theorem proving alone.

4 Formal Presentation of Our Indefinite Semantics

To satisfy Requirement III (Semantics) of § 3.1, in this § 4 we present our novel semantics for HILBERT's ε formally. This is required for precision and consistency. As consistency of our new semantics is not trivial at all, technical rigor cannot be avoided. From §§ 2 and 3, the reader should have a good intuition of our intended representation and semantics of HILBERT's ε, free variables, atoms, and choice-conditions in our framework.

4.1 Organization of § 4

After some preliminary subsections (§§ 4.1–4.3), we formalize variable-conditions and their consistency (§ 4.4). The following discussion of alternatives to the design decisions in the formalization of variable-conditions may be skipped (§ 4.5).

Moreover, we explain how to deal with free variables syntactically (§ 4.6) and semantically (§§ 4.7 and 4.8).

Furthermore, after formalizing choice-conditions and their compatibility (§ 4.9), we define our notion of $(C,(R,N))$-validity and discuss some examples (§ 4.10). One of these examples is especially interesting because we show that — with our new more careful treatment of negative information in our *positive/negative* variable-conditions — we now can model HENKIN quantification directly.

Our interest goes beyond soundness in that we want "*preservation of solutions*". By this we mean the following: All *closing substitutions* for the free variables — i.e. all solutions that transform a proof attempt (to which a proposition has been reduced) into a closed proof — are also solutions of the original proposition. This is similar to a proof in PROLOG, computing answers to a query proposition that contains free variables. Therefore, we discuss this solution-preserving notion of *reduction* (§ 4.13), especially under the aspects of extensions of variable-conditions and choice-conditions (§ 4.11) and of global instantiation of free variables with choice-conditions ("ε-substitution") (§ 4.12).

Finally, in § 4.14, we show soundness, safeness, and solution-preservation for our γ-, δ^-, and δ^+-rules of § 2.3.

All in all, in this § 4, we extend and simplify the presentation of [WIRTH, 2008], which is extended with additional linguistic applications in [WIRTH, 2010] and which again simplifies and extends the presentation of [WIRTH, 2004], which, however, additionally contains some comparative discussions and compatible extensions for *descente infinie*.

4.2 Basic Notions and Notation

'**N**' denotes the set of natural numbers and '$\prec$' the ordering on **N**. Let $\mathbf{N}_{+} := \{\, n \in \mathbf{N} \mid 0 \neq n \,\}$. We use '$\uplus$' for the union of disjoint classes and 'id' for the identity function. For classes R, A, and B we define:

$\mathrm{dom}(R) := \{\, a \mid \exists b.\ (a,b) \in R \,\}$	*domain*	
${}_{A}{\upharpoonleft}R := \{\, (a,b) \in R \mid a \in A \,\}$	*restriction to* A	Just write "image_R(A)".
$\langle A \rangle R := \{\, b \mid \exists a \in A.\ (a,b) \in R \,\}$	*image of* A, i.e. $\langle A \rangle R = \mathrm{ran}({}_{A}{\upharpoonleft}R)$	

And the dual ones:

$\mathrm{ran}(R) := \{\, b \mid \exists a.\ (a,b) \in R \,\}$	*range*
$R{\restriction}_{B} := \{\, (a,b) \in R \mid b \in B \,\}$	*range-restriction to* B
$R\langle B \rangle := \{\, a \mid \exists b \in B.\ (a,b) \in R \,\}$	*reverse-image of* B, i.e. $R\langle B \rangle = \mathrm{dom}(R{\restriction}_{B})$

Furthermore, we use '$\emptyset$' to denote the empty set as well as the empty function. Functions are (right-) unique relations and the meaning of '$f{\circ}g$' is extensionally given by $(f{\circ}g)(x) = g(f(x))$. The *class of total functions from* A *to* B is denoted as $A \rightarrow B$. The *class of (possibly) partial functions from* A *to* B is denoted as $A \rightsquigarrow B$. Both $\rightarrow$ and $\rightsquigarrow$ associate to the right, i.e. $A \rightsquigarrow B \rightarrow C$ reads $A \rightsquigarrow (B \rightarrow C)$.

Let R be a binary relation. R is said to be a relation *on* A if $\mathrm{dom}(R) \cup \mathrm{ran}(R) \subseteq A$. R is *irreflexive* if $\mathrm{id} \cap R = \emptyset$. It is A*-reflexive* if ${}_{A}{\upharpoonleft}\mathrm{id} \subseteq R$. Speaking of a *reflexive* relation we refer to the largest A that is appropriate in the local context, and referring to this A we write R^0 to ambiguously denote ${}_{A}{\upharpoonleft}\mathrm{id}$. With $R^1 := R$, and $R^{n+1} := R^n{\circ}R$ for $n \in \mathbf{N}_{+}$, R^m denotes the m-step relation for R. The *transitive closure* of R is $R^{+} := \bigcup_{n \in \mathbf{N}_{+}} R^n$. The *reflexive & transitive closure* of R is $R^{*} := \bigcup_{n \in \mathbf{N}} R^n$. A relation R (on A) is *well-founded* if any non-empty class B ($\subseteq A$) has an R-minimal element, i.e. $\exists a \in B.\ \neg\exists a' \in B.\ a' R\, a$.

To be useful in context with HILBERT's ε, the notion of a "choice function" must be generalized here: We need a *total* function on the power set of any universe. Thus, a value must be supplied even at the empty set:

Definition 4.1 (Generalized Choice Function)
f is a *generalized choice function* if f is function with
$f : \mathrm{dom}(f) \rightarrow \bigcup(\mathrm{dom}(f))$ and $\forall x \in \mathrm{dom}(f).\ (x = \emptyset \ \vee\ f(x) \in x)$.

Corollary 4.2
The empty function $\emptyset$ *is both a choice function and a generalized choice function.*
If $\mathrm{dom}(f) = \{\emptyset\}$, *then* f *is neither a choice function nor a generalized choice function.*
If $\emptyset \notin \mathrm{dom}(f)$, *then* f *is a generalized choice function iff* f *is a choice function.*
If $\emptyset \in \mathrm{dom}(f)$, *then* f *is a generalized choice function iff there is a choice function* f' *and an* $x \in \bigcup(\mathrm{dom}(f'))$ *such that* $f = f' \cup \{(\emptyset, x)\}$.

4.3 Variables, Atoms, Constants, and Substitutions

We assume the following sets of symbols to be disjoint:

$\mathbb{V}$ (free) (rigid) *variables*, serving as unknowns or the free variables of [FITTING, 1996]

$\mathbb{A}$ *(free) atoms*, which serve as parameters and must not be bound

$\mathbb{B}$ *bound atoms*, which may be bound May or must? Why do we need this?

Σ *constants*, i.e. the function and predicate symbols from the signature

We define:

$$\mathbb{VA} := \mathbb{V} \uplus \mathbb{A}$$

$$\mathbb{VAB} := \mathbb{V} \uplus \mathbb{A} \uplus \mathbb{B}$$

By slight abuse of notation, for $S \in \{\mathbb{V}, \mathbb{A}, \mathbb{VA}, \mathbb{VAB}\}$, we write "$S(\Gamma)$" to denote the set of symbols from S that have free occurrences in Γ.

Let σ be a substitution.

σ is a *substitution on V* if $\operatorname{dom}(\sigma) \subseteq V$.

The following indented statement (as simple as it is) will require some discussion.

> We denote with "$\Gamma\sigma$" the result of replacing each (free) occurrence of a symbol $x \in \operatorname{dom}(\sigma)$ in Γ with $\sigma(x)$; possibly after renaming in Γ some symbols that are bound in Γ, especially because a capture of their free occurrences in $\sigma(x)$ must be avoided.

Note that such a renaming will hardly be required for the following reason: We will bind only symbols from the set $\mathbb{B}$ of bound atoms. And — unless explicitly stated otherwise — we tacitly assume that all occurrences of bound atoms from $\mathbb{B}$ in a term or formula or in the range of a substitution are *bound occurrences* (i.e. that a bound atom $x^{\mathbb{B}} \in \mathbb{B}$ occurs only in the scope of a binder on $x^{\mathbb{B}}$). Thus, in standard situations, even without renaming, no additional occurrences can become bound (i.e. captured) when applying a substitution.

Actually, however, we may still have to rename some of the bound atoms in Γ when we want to exclude the binding of a bound atom within the scope of another binding of the same bound atom. For example, for Γ being the formula $\forall x^{\mathbb{B}}.\ (x^{\mathbb{B}} = y^{\mathbb{V}})$ and σ being the substitution $\{y^{\mathbb{V}} \mapsto \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = x^{\mathbb{B}})\}$, we may want the result of $\Gamma\sigma$ to be something like $\forall z^{\mathbb{B}}.\ (z^{\mathbb{B}} = \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = x^{\mathbb{B}}))$ instead of $\forall x^{\mathbb{B}}.\ (x^{\mathbb{B}} = \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = x^{\mathbb{B}}))$.

Moreover — unless explicitly stated otherwise — in this paper we will use only substitutions on $\mathbb{V}$. Thus, also the occurrence of "(free)" in the statement indented above is hardly of any relevance here, because we never bind elements of $\mathbb{V}$ anyway.

4.4 Consistent Positive/Negative Variable-Conditions

Variable-conditions are binary relations on free variables and free atoms. They put conditions on the possible instantiation of free variables, and on the dependence of their valuations. In this paper, for clarity of presentation, a variable-condition is formalized as a pair (R, N) of binary relations, which we will call "positive/negative variable-conditions":

What are R and N ranging over? Be explicit.

- The first component (R) of such a pair is a binary relation that is meant to express a *positive* dependence. It comes with the intention of transitivity, although it will typically not be closed up to transitivity for reasons of presentation and efficiency.

  The overall idea is that the occurrence of a pair $(x^{\mathbb{VA}}, y^{\mathbb{V}})$ in this positive relation means something like

  $$\text{“the value of } y^{\mathbb{V}} \text{ may well depend on } x^{\mathbb{VA}}\text{”}$$

  or

  $$\text{“the value of } y^{\mathbb{V}} \text{ is described in terms of } x^{\mathbb{VA}}\text{”.}$$

  A relation exactly like this positive relation (R) was the only component of a variable-condition as defined and used identically throughout [Wirth, 2002; 2004; 2008; 2010]. Note, however, that, in these publications, we had to admit this single positive relation to be a subset of $\mathbb{VA}\times\mathbb{VA}$ (instead of the restriction to $\mathbb{VA}\times\mathbb{V}$ of Definition 4.3 in this paper), because it had to simulate the negative relation (N) in addition; thereby losing some expressive power as compared to our positive/negative variable-conditions here (cf. Example 4.20).

- The second component (N), however, is meant to capture a *negative* dependence.

  The overall idea is that the occurrence of a pair $(x^{\mathbb{V}}, y^{\mathbb{A}})$ in this negative relation means something like

  $$\text{“the value of } x^{\mathbb{V}} \text{ is necessarily older than } y^{\mathbb{A}}\text{”}$$

  or

  $$\text{“the value of } x^{\mathbb{V}} \text{ must not depend on } y^{\mathbb{A}}\text{”}$$

  or

  $$\text{“} y^{\mathbb{A}} \text{ is fresh for } x^{\mathbb{V}}\text{”.}$$

  Relations similar to this negative relation (N) occurred as the only component of a variable-condition already in [Wirth, 1998], and later — with a completely different motivation — also as "*freshness* conditions" in [Gabbay & Pitts, 2002].

Definition 4.3 (Positive/Negative Variable-Condition)
A *positive/negative variable-condition* is a pair (R, N) with

$$R \subseteq \mathbb{VA} \times \mathbb{V}$$

and

$$N \subseteq \mathbb{V} \times \mathbb{A}.$$

Note that, in a positive/negative variable-condition (R, N), the relations R and N are always disjoint because their ranges are always subsets of the disjoint sets $\mathbb{V}$ and $\mathbb{A}$, respectively.

In the following definition, the well-foundedness guarantees that all dependences can be traced back to independent symbols and that no variable may transitively depend on itself, whereas the irreflexivity makes sure that no contradictious dependences can occur.

Definition 4.4 (Consistency)
A pair (R, N) is *consistent* if

$$R \text{ is well-founded}$$

and

$$R^{+} \circ N \text{ is irreflexive.}$$

Let (R, N) be positive/negative variable-condition. Let us think of our (binary) relations R and N as edges of a directed graph whose vertices are the (atom and variable) symbols currently in use. Then, $R^{+}\circ N$ is irreflexive iff there is no cycle in $R \cup N$ that contains exactly one edge from N. Moreover, in practice, a positive/negative variable-condition (R, N) can always be chosen to be finite in both components. In this case, R is well-founded iff R is acyclic. Thus we get:

Corollary 4.5 *If (R, N) is a positive/negative variable-condition with $|R|, |N| \in \mathbf{N}$, then (R, N) is consistent iff each cycle in the directed graph of $R \uplus N$ contains more than one edge from N. The latter can be effectively tested with time complexity of $|R| + |N|$.*

Note that, in the finite case, the test of Corollary 4.5 seems to be both the most efficient and the most human-oriented way to represent the question of consistency of positive/negative variable-conditions.

4.5 Further Discussion of our Formalization of Variable-Conditions

Let us recall that the two relations R and N of a positive/negative variable-condition (R, N) are always disjoint because their ranges must be disjoint according to Definition 4.3. Thus, from a technical point of view, we could merge R and N into a single relation, but we prefer to have two relations for the two different functions (positive and negative) of the variable-conditions in this paper, instead of the one relation for one function of [WIRTH, 2002; 2004; 2008; 2010], which realized the negative function only with a significant loss of relevant information.

Moreover, in Definition 4.3, we have excluded the possibility that two atoms $a^{\mathbb{A}}, b^{\mathbb{A}} \in \mathbb{A}$ may be related to each other in any of the components of a positive/negative variable-condition (R, N):

- $y^{\mathbb{VA}}\ R\ a^{\mathbb{A}}$ is indeed excluded for intentional reasons: An atom $a^{\mathbb{A}}$ cannot depend on any other symbol $y^{\mathbb{VA}}$. In this sense an atom is indeed atomic and can be seen as a black box.

- $b^{\mathbb{A}}\ N\ a^{\mathbb{A}}$, however, is excluded for technical reasons only.

  Two atoms $a^{\mathbb{A}}, b^{\mathbb{A}}$ in nominal terms [URBAN &AL., 2004] are indeed always fresh for each other: $a^{\mathbb{A}} \# b^{\mathbb{A}}$. In our free-variable framework, this would read: $b^{\mathbb{A}}\ N\ a^{\mathbb{A}}$.

  The reason why we did not include $\mathbb{A}\times\mathbb{A}$ into the negative component N is simply that we want to be close to the data structures of a both efficient and human-oriented graph implementation.

  Furthermore, consistency of a positive/negative variable-condition (R, N) is equivalent to consistency of $(\ R,\ \ N \uplus (\mathbb{A}\times\mathbb{A})\ )$.

  Indeed, if we added $\mathbb{A}\times\mathbb{A}$ to N, the result of the acyclicity test of Corollary 4.5 would not be changed: If there were a cycle with a single edge from $\mathbb{A}\times\mathbb{A}$, then its previous edge would have to be one of the original edges of N, and so this cycle would have more than one edge from N and thus would not count as a counterexample to consistency.

Moreover, we could remove the set $\mathbb{B}$ of bound atoms from our sets of symbols and consider its elements to be elements of the set $\mathbb{A}$ of (free) atoms. Beside some additional care on free occurrences of atoms in § 4.3, an additional price we would have to pay for this removal is that we would have to take $\mathbb{V}\times\mathbb{B}$ to be a part of the second component (N) of all our positive/negative variable-conditions (R, N). The reason for this is that we must guarantee that a bound atom $b^{\mathbb{B}}$ cannot be read by any variable $x^{\mathbb{V}}$, especially not after an elimination of binders; indeed, in case of $b^{\mathbb{B}}\ R^{+}\ x^{\mathbb{V}}$, we would then get a cycle $b^{\mathbb{B}}\ R^{+}\ x^{\mathbb{V}}\ N\ b^{\mathbb{B}}$ with only one edge from N. Although, in practical contexts, we can always get along with a finite subset of $\mathbb{V}\times\mathbb{B}$, the essential pairs of this subset will still be quite many and most confusing. For instance, already for the higher-order choice-condition of Example 3.9, three and a half dozens of pairs from $\mathbb{V}\times\mathbb{B}$ are essential, compared to 14 pairs of useful information in R (cf. Example 4.14(a)).

4.6 Extensions, σ-Updates, and (R,N)-Substitutions

Within a reasoning process, positive/negative variable-conditions may be subject to only one kind of transformation, which we will simply call "extension".

Motivate this.

Definition 4.6 ([Weak] Extension)
(R',N') is an [*weak*] *extension of* (R,N) if
(R',N') is a positive/negative variable-condition, $R \subseteq R'$ [or at least $R \subseteq (R')^{+}$], and $N \subseteq N'$.

As an immediate corollary of Definitions 4.6 and 4.4 we get:

Corollary 4.7
If (R',N') is a consistent positive/negative variable-condition and a [weak] extension of (R,N), then (R,N) is a consistent positive/negative variable-condition as well.

A σ-update is a special form of an extension:

Definition 4.8 (σ-Update, Dependence)
Let (R,N) be a positive/negative variable-condition and σ be a substitution on $\mathbb{V}$.
The *dependence of* σ is

$$D \quad := \quad \{\ (z^{\mathbb{V}\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathrm{dom}(\sigma) \wedge z^{\mathbb{V}\mathbb{A}} \in \mathbb{V}\mathbb{A}(\sigma(x^{\mathbb{V}}))\ \}.$$

The *σ-update of* (R,N) is $(R \cup D, N)$.

Definition 4.9 ((R,N)-Substitution)
Let (R,N) be a positive/negative variable-condition.
σ is an *(R,N)-substitution* if
σ is a substitution on $\mathbb{V}$ and the σ-update of (R,N) is consistent.

Syntactically, $(x^{\mathbb{V}}, a^{\mathbb{A}}) \in N$ is to express that an (R,N)-substitution σ must not replace $x^{\mathbb{V}}$ with a term in which $a^{\mathbb{A}}$ could ever occur; i.e. that $a^{\mathbb{A}}$ is fresh for $x^{\mathbb{V}}$: $a^{\mathbb{A}} \# x^{\mathbb{V}}$. This is indeed guaranteed if any σ-update (R',N') of (R,N) is again required to be consistent, and so on. We can see this as follows: For $z^{\mathbb{V}} \in \mathbb{V}(\sigma(x^{\mathbb{V}}))$, we get

$$z^{\mathbb{V}} \;\; R' \;\; x^{\mathbb{V}} \;\; N' \;\; a^{\mathbb{A}}.$$

If we now try to apply a second substitution σ' with $a^{\mathbb{A}} \in \mathbb{A}(\sigma'(z^{\mathbb{V}}))$ (so that $a^{\mathbb{A}}$ occurs in $(x^{\mathbb{V}}\sigma)\sigma'$, contrary to what we initially expressed as our freshness intention), then σ' is not an (R',N')-substitution because, for the σ'-update (R'',N'') of (R',N'), we have

$$a^{\mathbb{A}} \;\; R'' \;\; z^{\mathbb{V}} \;\; R'' \;\; x^{\mathbb{V}} \;\; N'' \;\; a^{\mathbb{A}}.$$

so $(R'')^{+} \circ N''$ is not irreflexive. All in all, the positive/negative variable-condition

- (R',N') blocks any instantiation of $(x^{\mathbb{V}}\sigma)$ resulting in a term containing $a^{\mathbb{A}}$, just as
- (R,N) blocked $x^{\mathbb{V}}$ before the application of σ.

4.7 Semantical Presuppositions

Instead of defining truth from scratch, we require some abstract properties typically holding in two-valued model semantics.

Truth is given relative to some Σ-structure $\mathcal{S}$, assigning a non-empty universe (or "carrier") to each type. More precisely, we assume that, for every type v, every Σ-structure $\mathcal{S}$ maps "$\forall v$" (i.e. the string consisting of the symbol "$\forall$" and the sort v) to a set $\mathcal{S}(\forall v)$ with

$$\mathcal{S}(\varepsilon v) \;\in\; \mathcal{S}(\forall v),$$

i.e. $(\mathcal{S}(\varepsilon v))_v$ serves as a choice function for the family of universes $(\mathcal{S}(\forall v))_v$.

For $\mathrm{X} \subseteq \mathbb{VAB}$, we denote the set of total $\mathcal{S}$-valuations of X (i.e. functions mapping atoms and variables to objects of the universe of $\mathcal{S}$ (respecting types)) with

$$\mathrm{X} \to \mathcal{S}$$

and the set of (possibly) partial $\mathcal{S}$-valuations of X with

$$\mathrm{X} \leadsto \mathcal{S}$$

For $\delta : \mathrm{X} \to \mathcal{S}$, we denote with "$\mathcal{S} \uplus \delta$" the extension of $\mathcal{S}$ to X. More precisely, we assume some evaluation function "eval" such that $\mathrm{eval}(\mathcal{S} \uplus \delta)$ maps every term whose free-occurring symbols are from $\Sigma \uplus \mathrm{X}$ into the universe of $\mathcal{S}$ (respecting types). Moreover, $\mathrm{eval}(\mathcal{S} \uplus \delta)$ maps every formula B whose free-occurring symbols are from $\Sigma \uplus \mathrm{X}$ to TRUE or FALSE, such that:

$$B \text{ is true in } \mathcal{S} \uplus \delta \quad \text{iff} \quad \mathrm{eval}(\mathcal{S} \uplus \delta)(B) = \mathsf{TRUE}.$$

We leave open what our formulas and what our Σ-structures exactly are. The latter can range from first-order Σ-structures to higher-order modal Σ-models; provided that the following three properties — which (explicitly or implicitly) belong to the standard of most logic textbooks — hold for every term or formula B, every Σ-structure $\mathcal{S}$, and every $\mathcal{S}$-valuation $\delta : \mathbb{VAB} \leadsto \mathcal{S}$.

Explicitness Lemma

The value of the evaluation of B depends only on the valuation of those variables and atoms that actually have free occurrences in B; i.e., for $\mathrm{X} := \mathbb{VAB}(B)$, if $\mathrm{X} \subseteq \mathrm{dom}(\delta)$, then:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(B) \quad = \quad \mathrm{eval}(\mathcal{S} \;\uplus\; {}_{\mathrm{X}}{\upharpoonleft}\delta)(B).$$

Substitution [Value] Lemma

Let σ be a substitution on $\mathbb{VAB}$. If $\mathbb{VAB}(B\sigma) \subseteq \mathrm{dom}(\delta)$, then:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) \quad = \quad \mathrm{eval}\Big(\; \mathcal{S} \;\;\uplus\;\; \big(\; (\; \sigma \;\uplus\; {}_{\mathbb{VAB}\setminus\mathrm{dom}(\sigma)}{\upharpoonleft}\mathrm{id} \;) \;\circ\; \mathrm{eval}(\mathcal{S} \uplus \delta) \;\big) \;\Big)\Big(\; B \;\Big).$$

Valuation Lemma

The evaluation function treats application terms from $\mathbb{VAB}$ straightforwardly in the sense that for every $v_0^{\mathbb{VAB}}, \ldots, v_{l-1}^{\mathbb{VAB}}, y^{\mathbb{VAB}} \in \mathrm{dom}(\delta)$ with $v_0^{\mathbb{VAB}} : \alpha_0, \;\ldots,\; v_{l-1}^{\mathbb{VAB}} : \alpha_{l-1}$, $y^{\mathbb{VAB}} : \alpha_0 \to \cdots \to \alpha_{l-1} \to \alpha_l$ for some types $\alpha_0, \ldots, \alpha_{l-1}, \alpha_l$, we have:

$$\mathrm{eval}(\mathcal{S} \uplus \delta)(y^{\mathbb{VAB}}(v_0^{\mathbb{VAB}}) \cdots (v_{l-1}^{\mathbb{VAB}})) \;=\; \delta(y^{\mathbb{VAB}})(\delta(v_0^{\mathbb{VAB}})) \cdots (\delta(v_{l-1}^{\mathbb{VAB}})).$$

Note that we need the case where $y^{\mathbb{VAB}}$ is a higher-order symbol (i.e. the case of $l \succ 0$) only in the rare case that higher-order choice-conditions are required. Beside this, the basic language of the general reasoning framework, however, may well be first-order and does not have to include function application.

Moreover, for the few cases where we refer to quantifiers and implication, such as rules of § 2.3 or our version Q_C of the axiom (ε_0) (cf. Definition 3.10), and the lemmas and theorems that refer to these (namely Lemmas 4.19 and 4.25, Theorem 4.27(6), and Theorem 4.28),[4] we have to know that the quantifiers and the implication show the standard semantical behavior of classical logic:

∀-Lemma
Assume $\mathbb{VAB}(\forall x^{\mathbb{B}}.\ A) \subseteq \mathrm{dom}(\delta)$. The following two are logically equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(\forall x^{\mathbb{B}}.\ A) = \mathsf{TRUE}$
- $\mathrm{eval}(\mathcal{S} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}\!\upharpoonright\!\delta \uplus \chi)(A) = \mathsf{TRUE}$ for every $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$

∃-Lemma
Assume $\mathbb{VAB}(\exists x^{\mathbb{B}}.\ A) \subseteq \mathrm{dom}(\delta)$. The following two are logically equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(\exists x^{\mathbb{B}}.\ A) = \mathsf{TRUE}$,
- $\mathrm{eval}(\mathcal{S} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}\!\upharpoonright\!\delta \uplus \chi)(A) = \mathsf{TRUE}$ for some $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$

⇒-Lemma
Assume $\mathbb{VAB}(A \Rightarrow B) \subseteq \mathrm{dom}(\delta)$. The following two are logically equivalent:

- $\mathrm{eval}(\mathcal{S} \uplus \delta)(A \Rightarrow B) = \mathsf{TRUE}$
- $\mathrm{eval}(\mathcal{S} \uplus \delta)(A) = \mathsf{FALSE}$ or $\mathrm{eval}(\mathcal{S} \uplus \delta)(B) = \mathsf{TRUE}$

4.8 Semantical Relations and $\mathcal{S}$-Semantical Valuations

We now come to some technical definitions required for our (model-) semantical counterparts of our syntactical (R, N)-substitutions.

Let $\mathcal{S}$ be a Σ-structure. An "$\mathcal{S}$-semantical valuation" π plays the rôle of a *raising function* (a dual of a SKOLEM function as defined in [MILLER, 1992]). This means that π does not simply map each variable directly to an object of $\mathcal{S}$ (of the same type), but may additionally read the values of some atoms under an $\mathcal{S}$-valuation $\tau : \mathbb{A} \to \mathcal{S}$. More precisely, we assume that π takes some restriction of τ as a second argument, say $\tau' : \mathbb{A} \leadsto \mathcal{S}$ with $\tau' \subseteq \tau$. In short:

$$\pi : \mathbb{V} \to (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}.$$

Moreover, for each variable $x^{\mathbb{V}}$, we require that the set $\mathrm{dom}(\tau')$ of atoms read by $\pi(x^{\mathbb{V}})$ is identical for all τ. This identical set will be denoted with $S_\pi\langle\{x^{\mathbb{V}}\}\rangle$ below. Technically, we require that there is some "semantical relation" $S_\pi \subseteq \mathbb{A}\times\mathbb{V}$ such that for all $x^{\mathbb{V}} \in \mathbb{V}$:

$$\pi(x^{\mathbb{V}}) \ : \ (S_\pi\langle\{x^{\mathbb{V}}\}\rangle \to \mathcal{S}) \to \mathcal{S}.$$

This means that $\pi(x^{\mathbb{V}})$ can read the τ-value of $y^{\mathbb{A}}$ if and only if $(y^{\mathbb{A}}, x^{\mathbb{V}}) \in S_\pi$. Note that, for each $\pi : \mathbb{V} \to (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}$, at most one such semantical relation exists, namely the one of the following definition.

Definition 4.10 (Semantical Relation (S_π))

The *semantical relation for* π is

$$S_\pi \ := \ \{\ (y^{\mathbb{A}}, x^{\mathbb{V}}) \mid x^{\mathbb{V}} \in \mathbb{V} \wedge y^{\mathbb{A}} \in \mathrm{dom}(\bigcup(\mathrm{dom}(\pi(x^{\mathbb{V}}))))\ \}.$$

Definition 4.11 ($\mathcal{S}$-Semantical Valuation)

Let $\mathcal{S}$ be a Σ-structure.

π is an *$\mathcal{S}$-semantical valuation* if

$$\pi : \mathbb{V} \to (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}$$

and, for all $x^{\mathbb{V}} \in \mathrm{dom}(\pi)$:

$$\pi(x^{\mathbb{V}}) : (S_\pi\langle\{x^{\mathbb{V}}\}\rangle \to \mathcal{S}) \to \mathcal{S}.$$

Finally, we need the technical means to turn an $\mathcal{S}$-semantical valuation π together with an $\mathcal{S}$-valuation τ of the atoms into an $\mathcal{S}$-valuation $\epsilon(\pi)(\tau)$ of the variables:

Definition 4.12 (ϵ)

We define the function $\epsilon : \ (\mathbb{V} \to (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}) \ \to \ (\mathbb{A} \to \mathcal{S}) \ \to \ \mathbb{V} \ \leadsto \ \mathcal{S}$

for $\pi : \mathbb{V} \to (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}, \quad \tau : \mathbb{A} \to \mathcal{S}, \quad x^{\mathbb{V}} \in \mathbb{V}$

by $\epsilon(\pi)(\tau)(x^{\mathbb{V}}) := \pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}{\upharpoonright}\tau).$

So this epsilon isn't hilbert's epsilon?

4.9 Choice-Conditions and Compatibility

Choice-conditions are completely syntactical objects according to the following Definition 4.13; how they influence our semantics will be described in Definition 4.15.

Definition 4.13 (Choice-Condition, Return Type)

C is an (R,N)-*choice-condition* if

- (R,N) is a consistent positive/negative variable-condition and
- C is a partial function from $\mathbb{V}$ into the set of higher-order ε-terms

notation before used

such that, for every $y^{\mathbb{V}} \in \mathrm{dom}(C)$, the following items hold for some types $\alpha_0, \ldots, \alpha_l$:

1. The value $C(y^{\mathbb{V}})$ is of the form

$$\lambda v_0^{\mathbb{B}}.\ \ldots\ \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$$

   for some formula B and for some mutually distinct bound atoms $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}} \in \mathbb{B}$ with $v_0^{\mathbb{B}} : \alpha_0, \ \ldots, \ v_l^{\mathbb{B}} : \alpha_l$, and with $\mathbb{B}(B) \subseteq \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\}$.

2. $y^{\mathbb{V}} : \alpha_0 \rightarrow \cdots \rightarrow \alpha_{l-1} \rightarrow \alpha_l$.

3. $z^{\mathbb{VA}}\ R^+\ y^{\mathbb{V}}$ for all $z^{\mathbb{VA}} \in \mathbb{VA}(C(y^{\mathbb{V}}))$.

In the situation described, α_l is *the return type of* $C(y^{\mathbb{V}})$.
β is *a return type of* C if there is a $z^{\mathbb{V}} \in \mathrm{dom}(C)$ such that β is the return type of $C(z^{\mathbb{V}})$.

Example 4.14 (Choice-Condition) *(continuing Example 3.9)*

(a) If (R,N) is a consistent positive/negative variable-condition that satisfies

$$z_a^{\mathbb{V}}\ R\ y_a^{\mathbb{V}}\ R\ z_b^{\mathbb{V}}\ R\ x_a^{\mathbb{V}}\ R\ z_c^{\mathbb{V}}\ R\ y_b^{\mathbb{V}}\ R\ z_d^{\mathbb{V}}\ R\ w_a^{\mathbb{V}}\ R\ z_e^{\mathbb{V}}\ R\ y_c^{\mathbb{V}}\ R\ z_f^{\mathbb{V}}\ R\ x_b^{\mathbb{V}}\ R\ z_g^{\mathbb{V}}\ R\ y_d^{\mathbb{V}}\ R\ z_h^{\mathbb{V}},$$

then the C of Example 3.9 is an (R,N)-choice-condition, indeed.

(b) If some clever person tried to do the whole quantifier elimination of Example 3.9 by

$$\begin{array}{lcl} C'(z_h^{\mathbb{V}}) & := & \varepsilon z_h^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{B}}) \\ C'(y_d^{\mathbb{V}}) & := & \varepsilon y_d^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{B}}, z_h^{\mathbb{V}}) \\ C'(x_b^{\mathbb{V}}) & := & \varepsilon x_b^{\mathbb{B}}.\ \neg\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{B}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}}) \\ C'(w_a^{\mathbb{V}}) & := & \varepsilon w_a^{\mathbb{B}}.\ \mathsf{P}(w_a^{\mathbb{B}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}}) \end{array}$$

then he would — among other constraints — have to satisfy $z_h^{\mathbb{V}}\ R^+\ y_d^{\mathbb{V}}\ R^+\ z_h^{\mathbb{V}}$, because of Item 3 of Definition 4.13 and the values of C' at $y_d^{\mathbb{V}}$ and $z_h^{\mathbb{V}}$. This would make R non-well-founded. Thus, this C' cannot be an (R,N)-choice-condition for any (R,N), because the consistency of (R,N) is required in Definition 4.13. Note that the choices required by C' for $y_d^{\mathbb{V}}$ and $z_h^{\mathbb{V}}$ are in an unsolvable conflict, indeed.

(c) For a more elementary example, take

$$C''(x^{\mathbb{V}}) \ := \ \varepsilon x^{\mathbb{B}}.\ (x^{\mathbb{B}} = y^{\mathbb{V}}) \qquad\qquad C''(y^{\mathbb{V}}) \ := \ \varepsilon y^{\mathbb{B}}.\ (x^{\mathbb{V}} \neq y^{\mathbb{B}})$$

Then $x^{\mathbb{V}}$ and $y^{\mathbb{V}}$ form a vicious circle of conflicting choices for which no valuation can be found that is compatible with C''. But C'' is no choice-condition at all because there is no (consistent(!)) positive/negative variable-condition (R,N) such that C'' is an (R,N)-choice-condition.

Definition 4.15 (Compatibility)

Let C be an (R,N)-choice-condition. Let $\mathcal{S}$ be a Σ-structure.

π is *$\mathcal{S}$-compatible with* $(C,(R,N))$ if the following items hold:

1. π is an $\mathcal{S}$-semantical valuation (cf. Definition 4.11) and $(R \cup S_\pi, N)$ is consistent (cf. Definitions 4.4 and 4.10).
2. For every $y^{\mathbb{V}} \in \mathrm{dom}(C)$ with $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots\ \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ for some formula B, and for every $\tau : \mathbb{A} \to \mathcal{S}$, and for every $\chi : \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\} \to \mathcal{S}$:

   If B is true in $\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$,
   then $B\{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$ is true in $\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$ as well.

   (For ϵ, cf. Definition 4.12.)

To understand Item 2 of Definition 4.15, consider an (R,N)-choice-condition

$$C := \{(y^{\mathbb{V}},\ \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B)\},$$

which restricts the value of $y^{\mathbb{V}}$ with the higher-order ε-term $\lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$. Then, roughly speaking, this choice-condition C requires that whenever there is a χ-value of $v_l^{\mathbb{B}}$ such that B is true in $\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$, the π-value of $y^{\mathbb{V}}$ is chosen such that $B\{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$ becomes true in $\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$ as well. Note that, because free variables can never read any bound atoms, the free variables of the latter term cannot read the χ-value of any of the bound atoms $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}$.

Moreover, Item 2 of Definition 4.15 is closely related to HILBERT's ε-operator in the sense that — roughly speaking — $y^{\mathbb{V}}$ must be given one of the values admissible for

$$\lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B.$$

As the choice for $y^{\mathbb{V}}$ depends on the symbols that have a free occurrence in that higher-order ε-term, we included this dependence into the positive relation R of the consistent positive/negative variable-condition (R,N) in Item 3 of Definition 4.13. This inclusion excludes conflicts as in Example 4.14(c).

Let (R,N) be a consistent positive/negative variable-condition. Then the empty function $\emptyset$ is an (R,N)-choice-condition. Moreover, each $\pi : \mathbb{V} \to \{\emptyset\} \to \mathcal{S}$ is $\mathcal{S}$-compatible with $(\emptyset,(R,N))$ because of $S_\pi = \emptyset$. In fact, as shown by the following Lemma 4.16, assuming an adequate principle of choice on the meta level, a compatible π always exists according to the following Lemma 4.16. This existence relies on Item 3 of Definition 4.13 and on the well-foundedness of R.

Lemma 4.16

Let $\mathcal{S}$ be a Σ-structure. Let C be an (R,N)-choice-condition.
Assume that for every return type α of C (cf. Definition 4.13),
there is a generalized choice function on the power-set of the universe of $\mathcal{S}$ for the type α.
[Let ρ be an $\mathcal{S}$-semantical valuation with $S_\rho \subseteq R^+$.]

Then there is an $\mathcal{S}$-semantical valuation π

- *that is $\mathcal{S}$-compatible with $(C,(R,N))$, and*
- *that satisfies $S_\pi = {}_{\mathbb{A}}{\upharpoonleft}(R^+)$ [and ${}_{\mathbb{V}\setminus\mathrm{dom}(C)}{\upharpoonleft}\pi = {}_{\mathbb{V}\setminus\mathrm{dom}(C)}{\upharpoonleft}\rho$].*

Proof of Lemma 4.16

Under the given assumptions, set $\lhd := R^+$ and $S_\pi := {}_{\mathbb{A}}\!\upharpoonright\!\lhd$.

Claim A: $\lhd = R^+ = (R \cup S_\pi)^+$ is a well-founded ordering.

Claim B: $(R \cup S_\pi, N)$ is a consistent positive/negative variable-condition.

Claim C: $S_\rho \subseteq {}_{\mathbb{A}}\!\upharpoonright\!\lhd = S_\pi \subseteq \lhd$.

Claim D: $S_\pi \circ \lhd \subseteq S_\pi$.

Proof of Claims A, B, C, and D: (R, N) is consistent because C is an (R, N)-choice-condition. Thus, R is well-founded and $\lhd = R^+ = (R \cup S_\pi)^+$ is a well-founded ordering. Moreover, we have $S_\rho, S_\pi, R \subseteq \lhd$. Thus, (R, N) is a weak extension of $(R \cup S_\pi, N)$. Thus, by Corollary 4.7, $(R \cup S_\pi, N)$ is a consistent positive/negative variable-condition. Finally, $S_\pi \circ \lhd = {}_{\mathbb{A}}\!\upharpoonright\!\lhd \circ \lhd \subseteq {}_{\mathbb{A}}\!\upharpoonright\!\lhd = S_\pi$. Q.e.d. (Claims A, B, C, and D)

By recursion on $y^{\mathbb{V}} \in \mathbb{V}$ in $\lhd$ we can define $\pi(y^{\mathbb{V}}) : (S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle \to \mathcal{S}) \to \mathcal{S}$ as follows.

Let $\tau' : S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle \to \mathcal{S}$ be arbitrary.

$y^{\mathbb{V}} \in \mathbb{V}\backslash\mathrm{dom}(C)$: If an $\mathcal{S}$-semantical valuation ρ is given, then we set

$$\pi(y^{\mathbb{V}})(\tau') := \rho(y^{\mathbb{V}})({}_{S_\rho\langle\!\{y^{\mathbb{V}}\}\!\rangle}\!\upharpoonright\!\tau');$$

which is well-defined according to Claim C. Otherwise, we choose an arbitrary value for $\pi(y^{\mathbb{V}})(\tau')$ from the universe of $\mathcal{S}$ (of the appropriate type). Note that universes $\mathcal{S}(\forall v)$ of $\mathcal{S}$ are assumed to be non-empty and $\mathcal{S}$ is assumed to provide some choice function $\mathcal{S}(\varepsilon v)$, cf. § 4.7.

$y^{\mathbb{V}} \in \mathrm{dom}(C)$: In this case, we have the following situation: $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ for some formula B and some $v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}} \in \mathbb{B}$ with $v_0^{\mathbb{B}} : \alpha_0, \ \ldots, \ v_l^{\mathbb{B}} : \alpha_l$, $y^{\mathbb{V}} : \alpha_0 \to \ldots \to \alpha_{l-1} \to \alpha_l$, and $z^{\mathbb{VA}} \lhd y^{\mathbb{V}}$ for all $z^{\mathbb{VA}} \in \mathbb{VA}(B)$, because C is an (R, N)-choice-condition. In particular, by Claim A, $y^{\mathbb{V}} \notin \mathbb{V}(B)$.

In this case, with the help of the assumed generalized choice function on the power-set of the universe of $\mathcal{S}$ of the sort α_l, we let $\pi(y^{\mathbb{V}})(\tau')$ be the function f that for $\chi : \{v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}\} \to \mathcal{S}$ chooses a value from the universe of $\mathcal{S}$ of type α_l for $f(\chi(v_0^{\mathbb{B}}))\cdots(\chi(v_{l-1}^{\mathbb{B}}))$, such that,

$$\text{if possible, } B \text{ is true in } \mathcal{S} \uplus \delta' \uplus \chi',$$

for $\delta' := \epsilon(\pi)(\tau' \uplus \tau'') \uplus \tau' \uplus \tau'' \uplus \chi$ for an arbitrary $\tau'' : (\mathbb{A}\backslash\mathrm{dom}(\tau')) \to \mathcal{S}$, and for $\chi' := \{v_l^{\mathbb{B}} \mapsto f(\chi(v_0^{\mathbb{B}}))\cdots(\chi(v_{l-1}^{\mathbb{B}}))\}$.

Note that the point-wise definition of f is correct: by the Explicitness Lemma and because of $y^{\mathbb{V}} \notin \mathbb{V}(B)$, the definition of the value of $f(\chi(v_0^{\mathbb{B}}))\cdots(\chi(v_{l-1}^{\mathbb{B}}))$ does not depend on the values of $f(\chi''(v_0^{\mathbb{B}}))\cdots(\chi''(v_{l-1}^{\mathbb{B}}))$ for a different $\chi'' : \{v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}\} \to \mathcal{S}$. Therefore, the function f is well-defined, because it also does not depend on τ'' according to the Explicitness Lemma and Claim 1 below. Finally, π is well-defined by induction on $\lhd$ according to Claim 2 below.

Claim 1: For $z^{\mathbb{VA}} \lhd y^{\mathbb{V}}$, the application term $(\delta' \uplus \chi')(z^{\mathbb{VA}})$ has the the value $\tau'(z^{\mathbb{VA}})$ in case of $z^{\mathbb{VA}} \in \mathbb{A}$, and the value $\pi(z^{\mathbb{VA}})({}_{S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle}\!\upharpoonright\!\tau')$ in case of $z^{\mathbb{VA}} \in \mathbb{V}$.

Claim 2: The definition of $\pi(y^{\mathbb{V}})(\tau')$ depends only on such values of $\pi(v^{\mathbb{V}})$ with $v^{\mathbb{V}} \lhd y^{\mathbb{V}}$, and does not depend on τ'' at all.

<u>Proof of Claim 1:</u> For $z^{\mathbb{VA}} \in \mathbb{A}$ the application term has the value $\tau'(z^{\mathbb{VA}})$ because of $z^{\mathbb{VA}} \in S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle$. Moreover, for $z^{\mathbb{VA}} \in \mathbb{V}$, we have $S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle \subseteq S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle$ by Claim D, and therefore the applicative term has the value $\pi(z^{\mathbb{VA}})({}_{S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle}{\upharpoonleft}(\tau' \uplus \tau'')) = \pi(z^{\mathbb{VA}})({}_{S_\pi\langle\!\{z^{\mathbb{VA}}\}\!\rangle}{\upharpoonleft}\tau')$. <u>Q.e.d. (Claim 1)</u>

<u>Proof of Claim 2:</u> In case of $y^{\mathbb{V}} \notin \mathrm{dom}(C)$, the definition of $\pi(y^{\mathbb{V}})(\tau')$ is immediate and independent. Otherwise, we have $z^{\mathbb{VA}} \lhd y^{\mathbb{V}}$ for all $z^{\mathbb{VA}} \in \mathbb{VA}(C(y^{\mathbb{V}}))$. Thus, Claim 2 follows from the EXPLICITNESS LEMMA and Claim 1. <u>Q.e.d. (Claim 2)</u>

Moreover, $\pi : \mathbb{V} \to (\mathbb{A} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$ is obviously an $\mathcal{S}$-semantical valuation. Thus, Item 1 of Definition 4.15 is satisfied for π by Claim B.

To show that also Item 2 of Definition 4.15 is satisfied, let us assume $y^{\mathbb{V}} \in \mathrm{dom}(C)$ and $\tau : \mathbb{A} \to \mathcal{S}$ to be arbitrary with $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}. \ \ldots \lambda v_{l-1}^{\mathbb{B}}. \ \varepsilon v_l^{\mathbb{B}}. \ B$, and let us then assume to the contrary of Item 2 that, for some $\chi : \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\} \to \mathcal{S}$ and for $\delta := \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$ and $\sigma := \{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$, we have $\mathrm{eval}(\mathcal{S} \uplus \delta)(B) = \mathsf{TRUE}$ and $\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) = \mathsf{FALSE}$.

Set $\tau' := {}_{S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonleft}\tau$ and $\tau'' := {}_{\mathbb{A}\setminus\mathrm{dom}(\tau')}{\upharpoonleft}\tau$.

Set $\delta' := {}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\delta$ and $f := \pi(y^{\mathbb{V}})(\tau')$.

Set $\chi' := \{v_l^{\mathbb{B}} \mapsto f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))\}$.

Then $\delta' = \epsilon(\pi)(\tau' \uplus \tau'') \uplus \tau' \uplus \tau'' \uplus \chi$. Moreover, by the EXPLICITNESS LEMMA, we have $\delta' = {}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id} \circ \mathrm{eval}(\mathcal{S} \uplus \delta)$.

By the VALUATION LEMMA we have

$$\begin{array}{rl} & \mathrm{eval}(\mathcal{S} \uplus \delta)(y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})) \\ = & \delta(y^{\mathbb{V}})(\delta(v_0^{\mathbb{B}})) \cdots (\delta(v_{l-1}^{\mathbb{B}})) \\ = & \epsilon(\pi)(\tau)(y^{\mathbb{V}})(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}})) \\ = & \pi(y^{\mathbb{V}})(\tau')(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}})) \\ = & f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}})). \end{array}$$

Thus, $\chi' = \sigma \circ \mathrm{eval}(\mathcal{S} \uplus \delta)$.

Thus, $\delta' \uplus \chi' = ({}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id} \uplus \sigma) \circ \mathrm{eval}(\mathcal{S} \uplus \delta)$.

Thus, we have, on the one hand,

$$\begin{array}{rl} & \mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi')(B) \\ = & \mathrm{eval}(\mathcal{S} \uplus (({}_{\mathbb{VAB}\setminus\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\mathrm{id} \uplus \sigma) \circ \mathrm{eval}(\mathcal{S} \uplus \delta)))(B) \\ = & \mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) \\ = & \mathsf{FALSE}, \end{array}$$

where the second equation holds by the SUBSTITUTION [VALUE] LEMMA.

Moreover, on the other hand, we have

$$\begin{array}{rl} & \mathrm{eval}(\mathcal{S} \uplus \delta' \uplus {}_{\{v_l^{\mathbb{B}}\}}{\upharpoonleft}\chi)(B) \\ = & \mathrm{eval}(\mathcal{S} \uplus \delta)(B) \\ = & \mathsf{TRUE}. \end{array}$$

This means that a value (such as $\chi(v_l^{\mathbb{B}})$) could have been chosen for $f(\chi(v_0^{\mathbb{B}})) \cdots (\chi(v_{l-1}^{\mathbb{B}}))$ to make B true in $\mathcal{S} \uplus \delta' \uplus \chi'$, but it was not. This contradicts the definition of f.

Q.e.d. (Lemma 4.16)

4.10 $(C,(R,N))$-Validity

Motivation.

Definition 4.17 (**$(C,(R,N))$-Validity,** K)

Let C be an (R,N)-choice-condition. Let G be a set of sequents.
Let $\mathcal{S}$ be a Σ-structure. Let $\delta : \mathbb{VA} \rightsquigarrow \mathcal{S}$ be an $\mathcal{S}$-valuation.

G is *$(C,(R,N))$-valid in $\mathcal{S}$* if
G is $(\pi,\mathcal{S})$-valid for some π that is $\mathcal{S}$-compatible with $(C,(R,N))$.

G is *$(\pi,\mathcal{S})$-valid* if G is true in $\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau$ for every $\tau : \mathbb{A} \to \mathcal{S}$.

G is *true in $\mathcal{S}{\uplus}\delta$* if Γ is true in $\mathcal{S}{\uplus}\delta$ for all $\Gamma \in G$.

A sequent Γ is *true in $\mathcal{S}{\uplus}\delta$* if there is some formula listed in Γ that is true in $\mathcal{S}{\uplus}\delta$.

Validity in a class of Σ-structures is understood as validity in each of the Σ-structures of that class. If we omit the reference to a special Σ-structure we mean validity in some fixed class K of Σ-structures, such as the class of all Σ-structures or the class of HERBRAND Σ-structures.

Example 4.18 (**$(\emptyset,(R,N))$-Validity**)

For $x^{\mathbb{V}} \in \mathbb{V}$, $y^{\mathbb{A}} \in \mathbb{A}$, the single-formula sequent $x^{\mathbb{V}} = y^{\mathbb{A}}$ is $(\emptyset,(\emptyset,\emptyset))$-valid in any $\mathcal{S}$ because we can choose $S_\pi := \mathbb{A}{\times}\mathbb{V}$ and $\pi(x^{\mathbb{V}})(\tau) := \tau(y^{\mathbb{A}})$ for $\tau : \mathbb{A} \to \mathcal{S}$, resulting in

$$\epsilon(\pi)(\tau)(x^{\mathbb{V}}) = \pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) = \pi(x^{\mathbb{V}})({}_{\mathbb{A}}{\upharpoonleft}\tau) = \pi(x^{\mathbb{V}})(\tau) = \tau(y^{\mathbb{A}}).$$

This means that $(\emptyset,(\emptyset,\emptyset))$-validity of $x^{\mathbb{V}} = y^{\mathbb{A}}$ is the same as validity of

$$\forall y_0^{\mathbb{B}}.\ \exists x_0^{\mathbb{B}}.\ (x_0^{\mathbb{B}} = y_0^{\mathbb{B}}).$$

Moreover, note that $\epsilon(\pi)(\tau)$ has access to the τ-value of $y_0^{\mathbb{A}}$ just as a raising function $x_1^{\mathbb{B}}$ for $x_0^{\mathbb{B}}$ in the raised (i.e. dually Skolemized) version $\exists x_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_1^{\mathbb{B}}(y_0^{\mathbb{B}}) = y_0^{\mathbb{B}})$ of $\forall y_0^{\mathbb{B}}.\ \exists x_0^{\mathbb{B}}.\ (x_0^{\mathbb{B}} = y_0^{\mathbb{B}})$.

Contrary to this, for $R := \emptyset$ and $N := \mathbb{V}{\times}\mathbb{A}$, the same single-formula sequent $x^{\mathbb{V}} = y^{\mathbb{A}}$ is not $(\emptyset,(R,N))$-valid in general, because then the required consistency of $(R \cup S_\pi, N)$ implies $S_\pi = \emptyset$; otherwise $R{\cup}S_\pi{\cup}N$ has a cycle of length 2 with exactly one edge from N. Thus, the value of $x^{\mathbb{V}}$ cannot depend on $\tau(y^{\mathbb{A}})$ anymore:

$$\pi(x^{\mathbb{V}})({}_{S_\pi\langle\{x^{\mathbb{V}}\}\rangle}{\upharpoonleft}\tau) = \pi(x^{\mathbb{V}})({}_{\emptyset}{\upharpoonleft}\tau) = \pi(x^{\mathbb{V}})(\emptyset).$$

This means that $(\emptyset,(\emptyset,\mathbb{V}{\times}\mathbb{A}))$-validity of $x^{\mathbb{V}} = y^{\mathbb{A}}$ is the same as validity of

$$\exists x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_0^{\mathbb{B}} = y_0^{\mathbb{B}}).$$

Moreover, note that $\epsilon(\pi)(\tau)$ has no access to the τ-value of $y_0^{\mathbb{B}}$ just as a raising function $x_1^{\mathbb{B}}$ for $x_0^{\mathbb{B}}$ in the raised version $\exists x_1^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_1^{\mathbb{B}}() = y_0^{\mathbb{B}})$ of $\exists x_0^{\mathbb{B}}.\ \forall y_0^{\mathbb{B}}.\ (x_0^{\mathbb{B}} = y_0^{\mathbb{B}})$.

For a more general example let $G = \{\ A_{i,0} \ldots A_{i,n_i-1} \mid i \in I\ \}$, where, for $i \in I$ and $j \prec n_i$, the $A_{i,j}$ are formulas with variables from $\boldsymbol{v}$ and atoms from $\boldsymbol{a}$.
Then $(\emptyset,(\emptyset,\mathbb{V}{\times}\mathbb{A}))$-validity of G means validity of $\exists \boldsymbol{v}.\ \forall \boldsymbol{a}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$
whereas $(\emptyset,(\emptyset,\emptyset))$-validity of G means validity of $\forall \boldsymbol{a}.\ \exists \boldsymbol{v}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$

Ignoring the question of γ-multiplicity, also any other sequence of universal and existential quantifiers can be represented by a consistent positive/negative variable-condition (R,N), simply by starting from the consistent positive/negative variable-condition $(\emptyset,\emptyset)$ and applying the γ- and δ-rules from § 2.3. A reverse translation of a consistent positive/negative variable-condition (R,N) into a sequence of quantifiers, however, may require a strengthening of dependences, in the sense that a subsequent backward translation would result in a consistent positive/negative variable-condition (R',N') with $R \subsetneq R'$ and $N \subsetneq N'$. This means that our framework can express logical dependences more fine-grained than standard quantifiers; cf. Example 4.20.

As already noted in § 3.9, the single-formula sequent $Q_C(y^{\mathbb{V}})$ of Definition 3.10 is a formulation of axiom (ε_0) of § 3.3 in our framework.

Lemma 4.19 ($(C,(R,N))$-Validity of $Q_C(y^{\mathbb{V}})$)
Let C be an (R,N)-choice-condition. Let $y^{\mathbb{V}} \in \mathrm{dom}(C)$. Let $\mathcal{S}$ be a Σ-structure.

1. *$Q_C(y^{\mathbb{V}})$ is $(\pi,\mathcal{S})$-valid for every π that is $\mathcal{S}$-compatible with $(C,(R,N))$.*
2. *$Q_C(y^{\mathbb{V}})$ is $(C,(R,N))$-valid in $\mathcal{S}$; provided that for every return type α of C (cf. Definition 4.13), there is a generalized choice function on the power-set of the universe of $\mathcal{S}$ for the type α.*

Proof of Lemma 4.19
Let $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}.\ \ldots\lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon v_l^{\mathbb{B}}.\ B$ for a formula B. Set $\sigma := \{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}})\cdots(v_{l-1}^{\mathbb{B}})\}$. Then we have $Q_C(y^{\mathbb{V}}) = \forall v_0^{\mathbb{B}}.\ \ldots\forall v_{l-1}^{\mathbb{B}}.\ (\ \exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\sigma\ )$. Let π be $\mathcal{S}$-compatible with $(C,(R,N))$; namely, in the case of Item 1, the π mentioned in the lemma, or, in the case of Item 2, the π that exists according to Lemma 4.16. Let $\tau : \mathbb{A} \to \mathcal{S}$ be arbitrary. It now suffices to show $\mathrm{eval}(\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau)(Q_C(y^{\mathbb{V}})) = \mathsf{TRUE}$. By the backward direction of the $\forall$-Lemma, it suffices to show $\mathrm{eval}(\mathcal{S} \uplus \delta)(\exists v_l^{\mathbb{B}}.\ B \ \Rightarrow\ B\sigma) = \mathsf{TRUE}$ for an arbitrary $\chi : \{v_0^{\mathbb{B}},\ldots,v_{l-1}^{\mathbb{B}}\} \to \mathcal{S}$, setting $\delta := \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$. By the backward direction of the $\Rightarrow$-Lemma, it suffices to show $\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) = \mathsf{TRUE}$ under the assumption of $\mathrm{eval}(\mathcal{S} \uplus \delta)(\exists v_l^{\mathbb{B}}.\ B) = \mathsf{TRUE}$. From the latter, by the forward direction of the $\exists$-Lemma, there is a $\chi' : \{v_l^{\mathbb{B}}\} \to \mathcal{S}$ such that $\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi')(B) = \mathsf{TRUE}$. By Item 2 of Definition 4.15, we get $\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi')(B\sigma) = \mathsf{TRUE}$. By the Explicitness Lemma, we get $\mathrm{eval}(\mathcal{S} \uplus \delta)(B\sigma) = \mathsf{TRUE}$. **Q.e.d. (Lemma 4.19)**

In [Wirth, 2010, § 6.4.1], we showed that Henkin quantification was problematic for the variable-conditions of that paper, which had only one component, namely the positive one of our positive/negative variable-conditions here: Indeed, the only way we found to model an example of a Henkin quantification in that paper precisely, was to increase the order of some variables by raising. Let us consider the same example here again and show that now we can model its Henkin quantification directly with a *consistent* positive/*negative* variable-condition, but *without raising*.

Example 4.20 (HENKIN Quantification)

In [HINTIKKA, 1974], quantifiers in first-order logic were found insufficient to give the precise semantics of some English sentences. In [HINTIKKA, 1996], *IF logic*, i.e. Independence-Friendly logic — a first-order logic with more flexible quantifiers — was presented to overcome this weakness. In [HINTIKKA, 1974], we find the following sentence:

Some relative of each villager and
some relative of each townsman hate each other. (H0)

Let us first change to a lovelier subject:

Some loved one of each woman and
some loved one of each man love each other. (H1)

For our purposes here, we consider (H1) to be equivalent to the following sentence, which may be more meaningful and easier to understand:

Every woman could love someone and
every man could love someone, such that these loved ones could love each other.

(H1) can be represented by the following HENKIN-quantified IF-logic formula:

$$\forall x_0^{\mathbb{B}}.\left(\begin{array}{l}\mathsf{Female}(x_0^{\mathbb{B}})\\ \Rightarrow\ \exists y_1^{\mathbb{B}}.\left(\begin{array}{l}\mathsf{Loves}(x_0^{\mathbb{B}},y_1^{\mathbb{B}})\\ \wedge\ \forall y_0^{\mathbb{B}}.\left(\begin{array}{l}\mathsf{Male}(y_0^{\mathbb{B}})\\ \Rightarrow\ \exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.\left(\begin{array}{l}\mathsf{Loves}(y_0^{\mathbb{B}},x_1^{\mathbb{B}})\\ \wedge\ \mathsf{Loves}(y_1^{\mathbb{B}},x_1^{\mathbb{B}})\\ \wedge\ \mathsf{Loves}(x_1^{\mathbb{B}},y_1^{\mathbb{B}})\end{array}\right)\end{array}\right)\end{array}\right)\end{array}\right) \qquad \text{(H2)}$$

Note that Formula (H2) is already in anti-prenex form; so we cannot reduce the dependences of its quantifiers by moving them closer toward the leaves of the formula tree.

Let us refer to the standard game-theoretic semantics for quantifiers (cf. e.g. [HINTIKKA, 1996]), which is defined as follows: Witnesses have to be picked for the quantified variables outside-in. We have to pick the witnesses for the γ-quantifiers (i.e., in (H2), for the existential quantifiers), and our opponent in the game picks the witnesses for the δ-quantifiers (i.e. for the universal quantifiers in (H2)). We win iff the resulting quantifier-free formula evaluates to true. A formula is true iff we have a winning strategy.

Then the HENKIN quantifier "$\exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.$" in (H2) is a special quantifier, which is a bit different from "$\exists x_1^{\mathbb{B}}.$". Game-theoretically, it has the following semantics: It asks us to pick the loved one $x_1^{\mathbb{B}}$ independently from the choice of the woman $x_0^{\mathbb{B}}$ (by our opponent in the game), although the HENKIN quantifier occurs in the scope of the quantifier "$\forall x_0^{\mathbb{B}}.$".

An alternative way to define the semantics of HENKIN quantifiers is by describing their effect on the logically equivalent *raised* forms of the formulas in which they occur. *Raising* is a dual of SKOLEMization, cf. [MILLER, 1992]. The raised version is defined as usual, beside that a γ-quantifier, say "$\exists x_1^{\mathbb{B}}.$", followed by a slash as in "$\exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.$", are raised in such a form that $x_0^{\mathbb{B}}$ does not appear as an argument to the raising function for $x_1^{\mathbb{B}}$.

According to this, (H2) is logically equivalent to its following raised form (H3), where $x_0^{\mathbb{B}}$ does not occur as an argument to the raising function $x_1^{\mathbb{B}}(y_0^{\mathbb{B}})$, which, however, would be the case if we had a usual γ-quantifier "$\exists x_1^{\mathbb{B}}.$" instead of "$\exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.$" in (H2).

$$\exists x_1^{\mathbb{B}}, y_1^{\mathbb{B}}.\ \forall x_0^{\mathbb{B}}, y_0^{\mathbb{B}}.\ \left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \Rightarrow & \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \\ \wedge & \left(\begin{array}{ll} & \mathsf{Male}(y_0^{\mathbb{B}}) \\ \Rightarrow & \left(\begin{array}{ll} & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}(y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}(x_0^{\mathbb{B}}), x_1^{\mathbb{B}}(y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}(y_0^{\mathbb{B}}), y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \end{array}\right) \end{array}\right) \end{array}\right) \end{array}\right) \tag{H3}$$

Beside moving-out the γ-quantifiers from (H2) to (H3), we can also move-out the range restriction $\mathsf{Male}(y_0^{\mathbb{B}})$ of $y_0^{\mathbb{B}}$, yielding the following, again logically equivalent formula, which nicely reflects the symmetry of (H1):

$$\exists x_1^{\mathbb{B}}, y_1^{\mathbb{B}}.\ \forall x_0^{\mathbb{B}}, y_0^{\mathbb{B}}.\ \left(\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}(y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}(x_0^{\mathbb{B}}), x_1^{\mathbb{B}}(y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}(y_0^{\mathbb{B}}), y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \end{array}\right)\right) \tag{H4}$$

Before we continue, let us compare Formula (H4) to the following one, which would be the result of the same raising transformation, but starting from a formula with a standard γ-quantification "$\exists x_1^{\mathbb{B}}.$" instead of the HENKIN quantification "$\exists x_1^{\mathbb{B}}/x_0^{\mathbb{B}}.$".

$$\exists x_1^{\mathbb{B}}, y_1^{\mathbb{B}}.\ \forall x_0^{\mathbb{B}}, y_0^{\mathbb{B}}.\ \left(\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{B}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{B}}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{B}}, y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{B}}, x_1^{\mathbb{B}}(x_0^{\mathbb{B}}, y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{B}}(x_0^{\mathbb{B}}), x_1^{\mathbb{B}}(x_0^{\mathbb{B}}, y_0^{\mathbb{B}})) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{B}}(x_0^{\mathbb{B}}, y_0^{\mathbb{B}}), y_1^{\mathbb{B}}(x_0^{\mathbb{B}})) \end{array}\right)\right) \tag{S}$$

Now, (H4) looks already very much like the following tentative representation of (H1) in our framework of free atoms and variables:

$$\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\mathbb{A}}) \\ \wedge & \mathsf{Male}(y_0^{\mathbb{A}}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(y_1^{\mathbb{V}}, x_1^{\mathbb{V}}) \\ \wedge & \mathsf{Loves}(x_1^{\mathbb{V}}, y_1^{\mathbb{V}}) \end{array}\right) \tag{H1$'$}$$

with choice-condition C given by

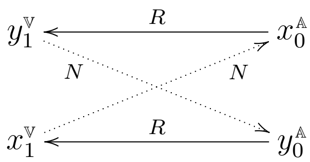
$$\begin{array}{lll} C(y_1^{\mathbb{V}}) & := & \varepsilon y_1^{\mathbb{B}}.\ (\mathsf{Female}(x_0^{\mathbb{A}}) \Rightarrow \mathsf{Loves}(x_0^{\mathbb{A}}, y_1^{\mathbb{B}})) \\ C(x_1^{\mathbb{V}}) & := & \varepsilon x_1^{\mathbb{B}}.\ (\mathsf{Male}(y_0^{\mathbb{A}}) \Rightarrow \mathsf{Loves}(y_0^{\mathbb{A}}, x_1^{\mathbb{B}})) \end{array}$$

which requires our positive/negative variable-condition (R, N) to contain $(x_0^{\mathbb{A}}, y_1^{\mathbb{V}})$ and $(y_0^{\mathbb{A}}, x_1^{\mathbb{V}})$ in the positive relation R, by Item 3 of Definition 4.13.

Here the form of our choice-condition C was chosen to reduce the difficulty of computing the semantics of Sentence (H2). Actually, however, we do not need this choice-condition here: Indeed, to find an equivalent representation in our framework, we could also work with an empty choice-condition. Crucial for our discussion, however, is that we can have $(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}), (y_0^{\mathbb{A}}, x_1^{\mathbb{V}}) \in R$; otherwise the loved ones could not depend on their lovers.

In any case, we can add $(y_1^{\mathbb{V}}, y_0^{\mathbb{A}})$ to the negative relation N here, namely to express that $y_1^{\mathbb{V}}$ must not read $y_0^{\mathbb{A}}$. This results in a logical equivalence to Formula (S).

Now we can indeed model the HENKIN quantifier by adding $(x_1^{\mathbb{V}}, x_0^{\mathbb{A}})$ to N in addition to $(y_1^{\mathbb{V}}, y_0^{\mathbb{A}})$. If we have started with the consistent positive/negative variable-condition $(\emptyset, \emptyset)$, our current positive/negative variable-condition now is given as (R, N) with $R = \{(x_0^{\mathbb{A}}, y_1^{\mathbb{V}}), (y_0^{\mathbb{A}}, x_1^{\mathbb{V}})\}$ and $N = \{(y_1^{\mathbb{V}}, y_0^{\mathbb{A}}), (x_1^{\mathbb{V}}, x_0^{\mathbb{A}})\}$. Thus, we have a single cycle in the graph, namely the following one:



But this cycle necessarily has two edges from the negative relation N. Thus, in spite of this cycle, our positive/negative variable-condition (R, N) is consistent by Corollary 4.5.

This was not possible with the variable-conditions of [WIRTH, 2002; 2004; 2008; 2010], because there was no distinction of the edges of N from the edges of R.

Thus, according to the discussion in [WIRTH, 2010, § 6.4.1], our new framework of this paper with positive/*negative* variable-conditions is the only one among all approaches suitable for describing the semantics of noun phrases in natural languages that admits us to model HENKIN quantifiers without raising.

4.11 Extended Extensions

Just like the positive/negative variable-condition (R, N), the (R, N)-choice-condition C may be extended during proofs. This kind of extension together with a simple soundness condition plays an important rôle in inference:

Definition 4.21 (Extended Extension)
$(C', (R', N'))$ is an *extended extension of* $(C, (R, N))$ if

- C is an (R, N)-choice-condition (cf. Definition 4.13),
- C' is an (R', N')-choice-condition,
- $C \subseteq C'$, and
- (R', N') is an extension of (R, N) (cf. Definition 4.6).

Lemma 4.22 (Extended Extension)
Let $(C', (R', N'))$ *be an extended extension of* $(C, (R, N))$.
If π *is* $\mathcal{S}$*-compatible with* $(C', (R', N'))$, *then* π *is* $\mathcal{S}$*-compatible with* $(C, (R, N))$ *as well.*

Proof of Lemma 4.22
Let us assume that π is $\mathcal{S}$-compatible with $(C', (R', N'))$. Then, by Item 1 of Definition 4.15, $\pi : \mathbb{V} \to (\mathbb{A} \leadsto \mathcal{S}) \leadsto \mathcal{S}$ is an $\mathcal{S}$-semantical valuation and $(R' \cup S_\pi, N')$ is consistent. As (R', N') is an extension of (R, N), we have $R \subseteq R'$ and $N \subseteq N'$. Thus, $(R' \cup S_\pi, N')$ is an extension of $(R \cup S_\pi, N)$. Thus, $(R \cup S_\pi, N)$ is consistent by Corollary 4.7. For π to be $\mathcal{S}$-compatible with $(C, (R, N))$, it now suffices to show Item 2 of Definition 4.15. As this property does not depend on the positive/negative variable-conditions anymore, it suffices to note that it actually holds because it holds for C' by assumption and we also have $C \subseteq C'$ by assumption. **Q.e.d. (Lemma 4.22)**

4.12 Extended σ-Updates of Choice-Conditions

After global application of an (R,N)-substitution σ, we now have to update both (R,N) and C:

Definition 4.23 (Extended σ-Update)
Let C be an (R,N)-choice-condition and let σ be a substitution on $\mathbb{V}$.
The *extended σ-update* $(C',(R',N'))$ *of* $(C,(R,N))$ is given as follows:

$$C' \;:=\; \{\ (x^{\mathbb{V}}, B\sigma) \mid (x^{\mathbb{V}}, B) \in C \wedge x^{\mathbb{V}} \notin \mathrm{dom}(\sigma)\ \},$$
$$(R',N') \text{ is the } \sigma\text{-update of } (R,N) \text{ (cf. Definition 4.8).}$$

Note that a σ-update (cf. Definition 4.8) is an extension (cf. Definition 4.6), whereas an extended σ-update is not an extended extension in general, because entries of the choice-condition may be modified or deleted. The remaining properties of an extended extension, however, are all satisfied:

Lemma 4.24 (Extended σ-Update)
Let C be an (R,N)-choice-condition. Let σ be an (R,N)-substitution.
Let $(C',(R',N'))$ be the extended σ-update of $(C,(R,N))$.
Then C' is an (R',N')-choice-condition.

Proof of Lemma 4.24
By assumption, $(C',(R',N'))$ is the extended σ-update of $(C,(R,N))$. Thus, (R',N') is the σ-update of (R,N). Thus, because σ is an (R,N)-substitution, (R',N') is a consistent positive/negative variable-condition by Definition 4.9. Moreover, C is an (R,N)-choice-condition. Thus, C is a partial function from $\mathbb{V}$ into the set of higher-order ε-terms, such that Items 1, 2, and 3 of Definition 4.13 hold. Thus, C' is a partial function from $\mathbb{V}$ into the set of higher-order ε-terms satisfying Items 1 and 2 of Definition 4.13 as well. For C' to satisfy also Item 3 of Definition 4.13, it now suffices to show the following Claim 1.

Claim 1: Let $y^{\mathbb{V}} \in \mathrm{dom}(C')$ and $z^{\mathbb{VA}} \in \mathbb{VA}(C'(y^{\mathbb{V}}))$. Then we have $z^{\mathbb{VA}}\,(R')^{+}\,y^{\mathbb{V}}$.

Proof of Claim 1: By the definition of C', we have $z^{\mathbb{VA}} \in \mathbb{VA}(C(y^{\mathbb{V}}))$ or else there is some $x^{\mathbb{V}} \in \mathrm{dom}(\sigma) \cap \mathbb{V}(C(y^{\mathbb{V}}))$ with $z^{\mathbb{VA}} \in \mathbb{VA}(\sigma(x^{\mathbb{V}}))$. Thus, as C is an (R,N)-choice-condition, we have either $z^{\mathbb{VA}}\,R^{+}\,y^{\mathbb{V}}$ or else $x^{\mathbb{V}}\,R^{+}\,y^{\mathbb{V}}$ and $z^{\mathbb{VA}} \in \mathbb{VA}(\sigma(x^{\mathbb{V}}))$. Then, as (R',N') is the σ-update of (R,N), by Definition 4.8, we have either $z^{\mathbb{VA}}\,(R')^{+}\,y^{\mathbb{V}}$ or else $x^{\mathbb{V}}\,(R')^{+}\,y^{\mathbb{V}}$ and $z^{\mathbb{VA}}\,R'\,x^{\mathbb{V}}$. Thus, in any case, $z^{\mathbb{VA}}\,(R')^{+}\,y^{\mathbb{V}}$. Q.e.d. (Claim 1)

Q.e.d. (Lemma 4.24)

Note that the following Lemma 4.25 gets a lot simpler when require the whole (R,N)-substitution σ to respect the (R,N)-choice-condition C by setting $O := \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$ and $O' := \emptyset$; especially all requirements on O' are trivially satisfied then. Moreover, note that its (still quite long) proof is more than a factor of 2 shorter than the proof of the analogous Lemma B.5 in [Wirth, 2004] (together with Lemma B.1, its additionally required sublemma).

Lemma 4.25 ((R,N)-Substitutions and $(C,(R,N))$-Validity)

Let (R,N) be a positive/negative variable-condition.
Let C be an (R,N)-choice-condition.
Let σ be an (R,N)-substitution.
Let $(C',(R',N'))$ be the extended σ-update of $(C,(R,N))$.

Let $\mathcal{S}$ be a Σ-structure.
Let π' be an $\mathcal{S}$-semantical valuation that is $\mathcal{S}$-compatible with $(C',(R',N'))$.

Let O and O' be two disjoint sets with $O \subseteq \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$ and $O' \subseteq \mathrm{dom}(C) \setminus O$.

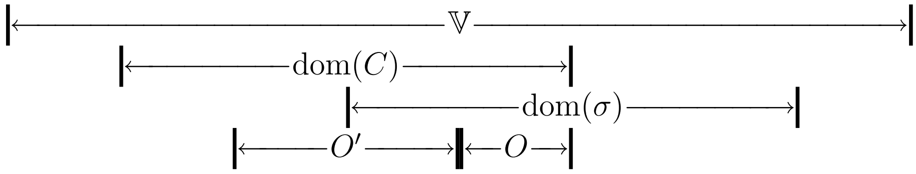
Moreover, assume that σ respects C on O in the given semantic context in the following sense (cf. Definition 3.10 for Q_C):

$$(\langle O \rangle Q_C)\sigma \text{ is } (\pi',\mathcal{S})\text{-valid.}$$

Furthermore, regarding the set O' (where σ may disrespect C), assume the following items to hold:

- *O' covers the rest of the critical variables in $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$ in the sense of*

$$\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) \subseteq O' \uplus O.$$



- *O' satisfies the closure condition $\langle O' \rangle R^+ \cap \mathrm{dom}(C) \subseteq O'$.*
- *For every $y^{\mathbb{V}} \in O'$ and for every return type α of $C(y^{\mathbb{V}})$ (cf. Definition 4.13), there is a generalized choice function on the power-set of the universe of $\mathcal{S}$ for the type α.*

Then there is an $\mathcal{S}$-semantical valuation π that is $\mathcal{S}$-compatible with $(C,(R,N))$ and that satisfies the following:

1. *For every term or formula B with $O' \cap \mathbb{V}(B) = \emptyset$ (and possibly with some unbound occurrences of bound atoms from a set $W \subseteq \mathbb{B}$), and for every $\tau : \mathbb{A} \to \mathcal{S}$ and $\chi : W \to \mathcal{S}$:*

$$\mathrm{eval}(\mathcal{S} \uplus \epsilon(\pi')(\tau) \uplus \tau \uplus \chi)(B\sigma) = \mathrm{eval}(\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau \uplus \chi)(B).$$

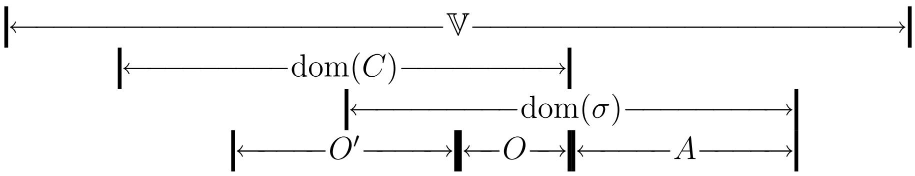
2. *For any set of sequents G with $O' \cap \mathbb{V}(G) = \emptyset$ we have:*

$$G\sigma \text{ is } (\pi',\mathcal{S})\text{-valid iff } G \text{ is } (\pi,\mathcal{S})\text{-valid.}$$

Proof of Lemma 4.25

Let us assume the situation described in the lemma.

We set $A := \mathrm{dom}(\sigma)\backslash(O'\uplus O)$. As σ is a substitution on $\mathbb{V}$, we have $\mathrm{dom}(\sigma) \subseteq O'\uplus O\uplus A \subseteq \mathbb{V}$.



Note that C' is an (R', N')-choice-condition because of Lemma 4.24.

As π' is $\mathcal{S}$-compatible with $(C', (R', N'))$, we know that $(R' \cup S_{\pi'}, N')$ is a consistent positive/negative choice-condition. Thus, $\lhd := (R' \cup S_{\pi'})^+$ is a well-founded ordering.

Let D be the dependence of σ. Set $S_\pi := {}_{\mathbb{A}}{\upharpoonright}\lhd$.

<u>Claim 1:</u> We have $R', S_{\pi'}, R, D, S_\pi \subseteq \lhd$ and $(R'\cup S_{\pi'}, N')$ is a weak extension of $(R \cup S_\pi, N)$ and of $(\lhd, N)$ (cf. Definition 4.6).

<u>Proof of Claim 1:</u> As (R', N') is the σ-update of (R, N), we have $R' = R\cup D$ and $N' = N$. Thus, $R', S_{\pi'}, R, D, S_\pi \subseteq \lhd = (R' \cup S_{\pi'})^+$. <u>Q.e.d. (Claim 1)</u>

<u>Claim 2:</u> $(R \cup S_\pi, N)$ and $(\lhd, N)$ are consistent positive/negative variable-conditions.

<u>Proof of Claim 2:</u> This follows from Claim 1 by Corollary 4.7. <u>Q.e.d. (Claim 2)</u>

<u>Claim 3:</u> ${}_{O'}{\upharpoonright}C$ is an $(\lhd, N)$-choice-condition.

<u>Proof of Claim 3:</u> By Claims 1 and 2 and the assumption that C is a (R, N)-choice-condition. <u>Q.e.d. (Claim 3)</u>

The plan for defining the $\mathcal{S}$-semantical valuation π (which we have to find) is to give $\pi(y^{\mathbb{V}})({}_{S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonright}\tau)$ a value as follows:

- For $y^{\mathbb{V}} \in \mathbb{V}\backslash(O'\uplus O\uplus A)$, we take this value to be
$$\pi'(y^{\mathbb{V}})({}_{S_{\pi'}\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonright}\tau).$$
This is indeed possible because of $S_{\pi'} \subseteq {}_{\mathbb{A}}{\upharpoonright}\lhd = S_\pi$, so ${}_{S_{\pi'}\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonright}\tau \subseteq {}_{S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonright}\tau$.

- For $y^{\mathbb{V}} \in O\uplus A$, we take this value to be
$$\mathrm{eval}(\mathcal{S} \uplus \epsilon(\pi')(\tau) \uplus \tau)(\sigma(y^{\mathbb{V}})).$$
Note that, in case of $y^{\mathbb{V}} \in O$, we know that $(Q_C(y^{\mathbb{V}}))\sigma$ is $(\pi', \mathcal{S})$-valid by assumption of the lemma. Moreover, the case of $y^{\mathbb{V}} \in A$ is unproblematic because of $y^{\mathbb{V}} \notin \mathrm{dom}(C)$. Again, π is well-defined in this case because the only part of τ that is accessed by the given value is ${}_{S_\pi\langle\!\{y^{\mathbb{V}}\}\!\rangle}{\upharpoonright}\tau$. Indeed, this can be seen as follows: By Claim 1 and the transitivity of $\lhd$, we have: ${}_{\mathbb{A}}{\upharpoonright}D \cup S_{\pi'}{\circ}D \subseteq {}_{\mathbb{A}}{\upharpoonright}\lhd = S_\pi$.

- For $y^{\mathbb{V}} \in O'$, however, we have to take care of $\mathcal{S}$-compatibility with $(C, (R, N))$ explicitly in an $\lhd$-recursive definition. This disturbance does not interfere with the semantic invariance stated in the lemma because occurrences of variables from O' in the relevant terms and formulas are explicitly excluded and, according to the statement of lemma, O' satisfies the appropriate closure condition.

Set $S_\rho := S_\pi$. Let ρ be defined by $(y^{\mathbb{V}} \in \mathbb{V},\ \tau : \mathbb{A} \to \mathcal{S})$

$$\rho(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\tau) := \begin{cases} \pi'(y^{\mathbb{V}})({}_{S_{\pi'}\langle\{y^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\tau) & \text{if } y^{\mathbb{V}} \in \mathbb{V}\backslash(O\uplus A) \\ \mathrm{eval}(\mathcal{S} \uplus \epsilon(\pi')(\tau) \uplus \tau)(\sigma(y^{\mathbb{V}})) & \text{if } y^{\mathbb{V}} \in O\uplus A \end{cases}$$

Let π be the $\mathcal{S}$-semantical valuation that exists according to Lemma 4.16 for the $\mathcal{S}$-semantical valuation ρ and the $(\lhd, N)$-choice-condition ${}_{O'}\!\upharpoonleft\! C$ (cf. Claim 3). Note that the assumptions of Lemma 4.16 are indeed satisfied here and that the resulting semantical relation S_π of Lemma 4.16 is indeed identical to our pre-defined relation of the same name, thereby justifying our abuse of notation: Indeed; by assumption of Lemma 4.25, for every return type α of ${}_{O}\!\upharpoonleft\! C$, there is a generalized choice function on the power-set of the universe of $\mathcal{S}$ for the type α; and we have

$$S_\rho \;=\; S_\pi \;=\; {}_{\mathbb{A}}\!\upharpoonleft\!\lhd \;=\; {}_{\mathbb{A}}\!\upharpoonleft\!(\lhd^+).$$

Because of $\mathrm{dom}({}_{O'}\!\upharpoonleft\! C) = O'$, according to Lemma 4.16, we then have

$${}_{\mathbb{V}\backslash O'}\!\upharpoonleft\!\pi \;=\; {}_{\mathbb{V}\backslash O'}\!\upharpoonleft\!\rho$$

and π is $\mathcal{S}$-compatible with $({}_{O'}\!\upharpoonleft\! C, (\lhd, N))$.

<u>Claim 4:</u> For all $y^{\mathbb{V}} \in O\uplus A$ and $\tau : \mathbb{A} \to \mathcal{S}$, when we set $\delta' := \epsilon(\pi')(\tau) \uplus \tau$:

$$\epsilon(\pi)(\tau)(y^{\mathbb{V}}) = \mathrm{eval}(\mathcal{S} \uplus \delta')(\sigma(y^{\mathbb{V}})).$$

<u>Proof of Claim 4:</u> We have $O\uplus A \subseteq \mathbb{V}\backslash O'$. Thus, Claim 4 follows immediately from the definition of ρ. <u>Q.e.d. (Claim 4)</u>

<u>Claim 5:</u> For all $y^{\mathbb{V}} \in \mathbb{V}\backslash(O'\uplus O\uplus A)$ and $\tau : \mathbb{A} \to \mathcal{S}$: $\epsilon(\pi)(\tau)(y^{\mathbb{V}}) = \epsilon(\pi')(\tau)(y^{\mathbb{V}})$.

<u>Proof of Claim 5:</u> For $y^{\mathbb{V}} \in \mathbb{V}\backslash(O'\uplus O\uplus A)$, we have $y^{\mathbb{V}} \in \mathbb{V}\backslash O'$ and $y^{\mathbb{V}} \in \mathbb{V}\backslash(O\uplus A)$. Thus, $\epsilon(\pi)(\tau)(y^{\mathbb{V}}) = \pi(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\tau) = \rho(y^{\mathbb{V}})({}_{S_\pi\langle\{y^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\tau) = \pi'(y^{\mathbb{V}})({}_{S_{\pi'}\langle\{y^{\mathbb{V}}\}\rangle}\!\upharpoonleft\!\tau) = \epsilon(\pi')(\tau)(y^{\mathbb{V}})$. <u>Q.e.d. (Claim 5)</u>

<u>Claim 6:</u> For any term or formula B (possibly with some unbound occurrences of bound atoms from the set $W \subseteq \mathbb{B}$) with $O' \cap \mathbb{V}(B) = \emptyset$, and for every $\tau : \mathbb{A} \to \mathcal{S}$ and every $\chi : W \to \mathcal{S}$, when we set $\delta := \epsilon(\pi)(\tau) \uplus \tau$ and $\delta' := \epsilon(\pi')(\tau) \uplus \tau$, we have

$$\mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi)(B\sigma) = \mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi)(B).$$

<u>Proof of Claim 6:</u> $\mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi)(B\sigma) =$ (by the SUBSTITUTION [VALUE] LEMMA)
$\mathrm{eval}(\mathcal{S} \uplus (\sigma \uplus {}_{\mathbb{VAB}\backslash\mathrm{dom}(\sigma)}\!\upharpoonleft\!\mathrm{id}) \circ \mathrm{eval}(\mathcal{S} \uplus \delta' \uplus \chi))(B) =$
(by the EXPLICITNESS LEMMA and the VALUATION LEMMA (for the case of $l = 0$))
$\mathrm{eval}(\mathcal{S} \uplus (\sigma \circ \mathrm{eval}(\mathcal{S} \uplus \delta')) \uplus {}_{\mathbb{VA}\backslash\mathrm{dom}(\sigma)}\!\upharpoonleft\!\delta' \uplus \chi)(B) =$
(by $O\uplus A \subseteq \mathrm{dom}(\sigma) \subseteq O'\uplus O\uplus A$, $O'\cap\mathbb{V}(B) = \emptyset$, and the EXPLICITNESS LEMMA)
$\mathrm{eval}(\mathcal{S} \uplus {}_{O\uplus A}\!\upharpoonleft\!\sigma \circ \mathrm{eval}(\mathcal{S} \uplus \delta') \uplus {}_{\mathbb{VA}\backslash(O'\uplus O\uplus A)}\!\upharpoonleft\!\delta' \uplus \chi)(B) =$ (by Claim 4 and Claim 5)
$\mathrm{eval}(\mathcal{S} \uplus {}_{O\uplus A}\!\upharpoonleft\!\delta \uplus {}_{\mathbb{VA}\backslash(O'\uplus O\uplus A)}\!\upharpoonleft\!\delta \uplus \chi)(B) =$
(by $O'\cap\mathbb{V}(B) = \emptyset$ and the EXPLICITNESS LEMMA)
$\mathrm{eval}(\mathcal{S} \uplus \delta \uplus \chi)(B)$. <u>Q.e.d. (Claim 6)</u>

<u>Claim 7:</u> For every set of sequents G' (possibly with some unbound occurrences of bound atoms from the set $W \subseteq \mathbb{B}$) with $O' \cap \mathbb{V}(G') = \emptyset$, and for every $\tau : \mathbb{A} \to \mathcal{S}$ and for every $\chi : W \to \mathcal{S}$: Truth of G' in $\mathcal{S} \uplus \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$ is logically equivalent to truth of $G'\sigma$ in $\mathcal{S} \uplus \epsilon(\pi')(\tau) \uplus \tau \uplus \chi$.

Proof of Claim 7: This is a trivial consequence of Claim 6. Q.e.d. (Claim 7)

Claim 8: For $y^{\mathbb{V}} \in \mathrm{dom}(C) \setminus O'$, we have $O' \cap \mathbb{V}(C(y^{\mathbb{V}})) = \emptyset$.

Proof of Claim 8: Otherwise there is some $y^{\mathbb{V}} \in \mathrm{dom}(C) \setminus O'$ and some $z^{\mathbb{V}} \in O' \cap \mathbb{V}(C(y^{\mathbb{V}}))$. Then $z^{\mathbb{V}} R^{+} y^{\mathbb{V}}$ because C is an (R, N)-choice-condition, and then, as $\langle O' \rangle R^{+} \cap \mathrm{dom}(C) \subseteq O'$ by assumption of the lemma, we have the contradicting $y^{\mathbb{V}} \in O'$. Q.e.d. (Claim 8)

Claim 9: Let $y^{\mathbb{V}} \in \mathrm{dom}(C)$ and $C(y^{\mathbb{V}}) = \lambda v_0^{\mathbb{B}}. \ \ldots \lambda v_{l-1}^{\mathbb{B}}. \ \varepsilon v_l^{\mathbb{B}}. \ B.$ Let $\tau : \mathbb{A} \to \mathcal{S}$ and $\chi : \{v_0^{\mathbb{B}}, \ldots, v_l^{\mathbb{B}}\} \to \mathcal{S}$. Set $\delta := \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$. Set $\mu := \{v_l^{\mathbb{B}} \mapsto y^{\mathbb{V}}(v_0^{\mathbb{B}}) \cdots (v_{l-1}^{\mathbb{B}})\}$. If B is true in $\mathcal{S} \uplus \delta$, then $B\mu$ is true in $\mathcal{S} \uplus \delta$ as well.

Proof of Claim 9: Set $\delta' := \epsilon(\pi')(\tau) \uplus \tau \uplus \chi$.

$y^{\mathbb{V}} \notin O' \uplus O$: In this case, because of $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) \subseteq O' \uplus O$, we have $y^{\mathbb{V}} \notin \mathrm{dom}(\sigma)$. Thus, as $(C', (R', N'))$ is the extended σ-update of $(C, (R, N))$, we have $C'(y^{\mathbb{V}}) = (C(y^{\mathbb{V}}))\sigma$. By Claim 8, we have $O' \cap \mathbb{V}(B) = \emptyset$.
And then, by our case assumption, also $O' \cap \mathbb{V}(B\mu) = \emptyset$.
By assumption of Claim 9, B is true in $\mathcal{S} \uplus \delta$. Thus, by Claim 7, $B\sigma$ is true in $\mathcal{S} \uplus \delta'$. Thus, as π' is $\mathcal{S}$-compatible with $(C', (R', N'))$, we know that $(B\sigma)\mu$ is true in $\mathcal{S} \uplus \delta'$. Because of $y^{\mathbb{V}} \notin \mathrm{dom}(\sigma)$, this means that $(B\mu)\sigma$ is true in $\mathcal{S} \uplus \delta'$. Thus, by Claim 7, $B\mu$ is true in $\mathcal{S} \uplus \delta$.

$y^{\mathbb{V}} \in O$: By Claim 8, we have $O' \cap \mathbb{V}(B) = \emptyset$.
And then, by our case assumption, also $O' \cap \mathbb{V}(B\mu) = \emptyset$.
Moreover, $(Q_C(y^{\mathbb{V}}))\sigma$ is equal to $\forall v_0^{\mathbb{B}}. \ \ldots \forall v_{l-1}^{\mathbb{B}}. \ \big(\ \exists v_l^{\mathbb{B}}. \ B \ \Rightarrow \ B\mu \ \big)\sigma$ and $(\pi', \mathcal{S})$-valid by assumption of the lemma. Thus, by the forward direction of the $\forall$-Lemma, $\big(\ \exists v_l^{\mathbb{B}}. \ B \ \Rightarrow \ B\mu \ \big)\sigma$ is true in $\mathcal{S} \uplus \delta'$. Thus, by Claim 7, $\exists v_l^{\mathbb{B}}. \ B \ \Rightarrow \ B\mu$ is true in $\mathcal{S} \uplus \delta$. As, by assumption of Claim 9, B is true in $\mathcal{S} \uplus \delta$, by the backward direction of the $\exists$-Lemma, $\exists v_l^{\mathbb{B}}. \ B$ is true in $\mathcal{S} \uplus \delta$ as well. Thus, by the forward direction of the $\Rightarrow$-Lemma, $B\mu$ is true in $\mathcal{S} \uplus \delta$ as well.

$y^{\mathbb{V}} \in O'$: π is $\mathcal{S}$-compatible with $({}_{O'}{\upharpoonleft} C, (\lhd, N))$ by definition, as explicitly stated before Claim 4. Q.e.d. (Claim 9)

By Claims 2 and 9, π is $\mathcal{S}$-compatible with $(C, (R, N))$. And then Items 1 and 2 of the lemma are trivial consequences of Claims 6 and 7, respectively.

Q.e.d. (Lemma 4.25)

4.13 Reduction

Reduction is the reverse of consequence. It is the backbone of logical reasoning, especially of abduction and goal-directed deduction. Our version of reduction does not only reduce a set of problems to another set of problems, but also guarantees that the solutions of the latter also solve the former; here "solutions" means the $\mathcal{S}$-semantical valuations of the (rigid) (free) variables from $\mathbb{V}$.

Definition 4.26 (Reduction)
Let C be an (R,N)-choice-condition. Let G_0 and G_1 be sets of sequents.
Let $\mathcal{S}$ be a Σ-structure.
G_0 *$(C,(R,N))$-reduces to G_1 in $\mathcal{S}$* if for every π that is $\mathcal{S}$-compatible with $(C,(R,N))$:

If G_1 is $(\pi,\mathcal{S})$-valid, then G_0 is $(\pi,\mathcal{S})$-valid as well.

Theorem 4.27 (Reduction)
Let C be an (R,N)-choice-condition. Let G_0, G_1, G_2, and G_3 be sets of sequents.
Let $\mathcal{S}$ be a Σ-structure.

1. (Validity) *If G_0 $(C,(R,N))$-reduces to G_1 in $\mathcal{S}$ and G_1 is $(C,(R,N))$-valid in $\mathcal{S}$, then G_0 is $(C,(R,N))$-valid in $\mathcal{S}$, too.*

2. (Reflexivity) *In case of $G_0 \subseteq G_1$: G_0 $(C,(R,N))$-reduces to G_1 in $\mathcal{S}$.*

3. (Transitivity) *If G_0 $(C,(R,N))$-reduces to G_1 in $\mathcal{S}$ and G_1 $(C,(R,N))$-reduces to G_2 in $\mathcal{S}$, then G_0 $(C,(R,N))$-reduces to G_2 in $\mathcal{S}$.*

4. (Additivity) *If G_0 $(C,(R,N))$-reduces to G_2 in $\mathcal{S}$ and G_1 $(C,(R,N))$-reduces to G_3 in $\mathcal{S}$, then $G_0 \cup G_1$ $(C,(R,N))$-reduces to $G_2 \cup G_3$ in $\mathcal{S}$.*

5. (Monotonicity) *For $(C',(R',N'))$ being an extended extension of $(C,(R,N))$:*

(a) If G_0 is $(C',(R',N'))$-valid in $\mathcal{S}$, then G_0 is also $(C,(R,N))$-valid in $\mathcal{S}$.

(b) If G_0 $(C,(R,N))$-reduces to G_1 in $\mathcal{S}$, then G_0 also $(C',(R',N'))$-reduces to G_1 in $\mathcal{S}$.

6. (Instantiation) *Let σ be an (R,N)-substitution.*
Let $(C',(R',N'))$ be the extended σ-update of $(C,(R,N))$.
Set $M := \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$.
Set $O := M \cap R^{*}\langle\mathbb{V}(G_0,G_1)\rangle$.
Set $O' := \mathrm{dom}(C) \cap \langle M \backslash O \rangle R^{*}$.
Assume that for every $y^{\mathbb{V}} \in O'$ and for every return type α of $C(y^{\mathbb{V}})$, there is a generalized choice function on the power-set of the universe of $\mathcal{S}$ for the type α.

(a) If $G_0\sigma \cup (\langle O \rangle Q_C)\sigma$ is $(C',(R',N'))$-valid in $\mathcal{S}$, then G_0 is $(C,(R,N))$-valid in $\mathcal{S}$.

(b) If G_0 $(C,(R,N))$-reduces to G_1 in $\mathcal{S}$, then $G_0\sigma$ $(C',(R',N'))$-reduces to $G_1\sigma \cup (\langle O \rangle Q_C)\sigma$ in $\mathcal{S}$.

Proof of Theorem 4.27

The first four items are trivial (Validity, Reflexivity, Transitivity, Additivity).

(5a): If G_0 is $(C',(R',N'))$-valid in $\mathcal{S}$, then there is some π that is $\mathcal{S}$-compatible with $(C',(R',N'))$ such that G_0 is $(\pi,\mathcal{S})$-valid. By Lemma 4.22, π is also $\mathcal{S}$-compatible with $(C,(R,N))$. Thus, G_0 is $(C,(R,N))$-valid, in $\mathcal{S}$.

(5b): Suppose that π is $\mathcal{S}$-compatible with $(C',(R',N'))$, and that G_1 is $(\pi,\mathcal{S})$-valid. By Lemma 4.22, π is also $\mathcal{S}$-compatible with $(C,(R,N))$. Thus, since G_0 $(C,(R,N))$-reduces to G_1, also G_0 is $(\pi,\mathcal{S})$-valid as was to be shown.

(6): Assume the situation described in the lemma.

Claim 1: $O' \subseteq \mathrm{dom}(C) \setminus O$.

Proof of Claim 1: By definition, $O' \subseteq \mathrm{dom}(C)$. It remains to show $O' \cap O = \emptyset$. To the contrary, suppose that there is some $y^{\mathrm{V}} \in O' \cap O$. Then, by the definition of O', there is some $z^{\mathrm{V}} \in M \backslash O$ with $z^{\mathrm{V}}\, R^*\, y^{\mathrm{V}}$. By definition of O, however, we have $y^{\mathrm{V}} \in R^*\langle \mathbb{V}(G_0,G_1)\rangle$. Thus, $z^{\mathrm{V}} \in R^*\langle \mathbb{V}(G_0,G_1)\rangle$. Thus, $z^{\mathrm{V}} \in O$, a contradiction. Q.e.d. (Claim 1)

Claim 2: $\langle O'\rangle R^+ \cap \mathrm{dom}(C) \subseteq O'$.

Proof of Claim 2: Assume $y^{\mathrm{V}} \in O'$ and $z^{\mathrm{V}} \in \mathrm{dom}(C)$ with $y^{\mathrm{V}}\, R^+\, z^{\mathrm{V}}$. It now suffices to show $z^{\mathrm{V}} \in O'$. Because of $y^{\mathrm{V}} \in O'$, there is some $x^{\mathrm{V}} \in M \backslash O$ with $x^{\mathrm{V}}\, R^*\, y^{\mathrm{V}}$. Thus, $x^{\mathrm{V}}\, R^*\, z^{\mathrm{V}}$. Thus, $z^{\mathrm{V}} \in O'$. Q.e.d. (Claim 2)

Claim 3: $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) \subseteq O' \cup O$.

Proof of Claim 3: $\mathrm{dom}(\sigma) \cap \mathrm{dom}(C) = \mathrm{dom}(C) \cap M \subseteq O \cup (\mathrm{dom}(C) \cap (M \backslash O)) \subseteq O \cup (\mathrm{dom}(C) \cap \langle M \backslash O\rangle R^*) = O \cup O'$. Q.e.d. (Claim 3)

Claim 4: $O' \cap \mathbb{V}(G_0,G_1) = \emptyset$.

Proof of Claim 4: To the contrary, suppose that there is some $y^{\mathrm{V}} \in O' \cap \mathbb{V}(G_0,G_1)$. Then, by the definition of O', there is some $z^{\mathrm{V}} \in M \backslash O$ with $z^{\mathrm{V}}\, R^*\, y^{\mathrm{V}}$. By definition of O, however, we have $z^{\mathrm{V}} \in O$, a contradiction. Q.e.d. (Claim 4)

(6a): In case that $G_0\sigma \cup (\langle O\rangle Q_C)\sigma$ is $(C',(R',N'))$-valid in $\mathcal{S}$, there is some π' that is $\mathcal{S}$-compatible with $(C',(R',N'))$ such that $G_0\sigma \cup (\langle O\rangle Q_C)\sigma$ is $(\pi',\mathcal{S})$-valid. Then both $G_0\sigma$ and $(\langle O\rangle Q_C)\sigma$ are $(\pi',\mathcal{S})$-valid. By Claims 1, 2, 3, and 4, let π be given as in Lemma 4.25. Then G_0 is $(\pi,\mathcal{S})$-valid. Moreover, as π is $\mathcal{S}$-compatible with $(C,(R,N))$, G_0 is $(C,(R,N))$-valid in $\mathcal{S}$.

(6b): Let π' be $\mathcal{S}$-compatible with $(C',(R',N'))$, and suppose that $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$ is $(\pi',\mathcal{S})$-valid. Then both $G_1\sigma$ and $(\langle O\rangle Q_C)\sigma$ are $(\pi',\mathcal{S})$-valid. By Claims 1, 2, 3, and 4, let π be given as in Lemma 4.25. Then π is $\mathcal{S}$-compatible with $(C,(R,N))$, and G_1 is $(\pi,\mathcal{S})$-valid. By assumption, G_0 $(C,(R,N))$-reduces to G_1. Thus, G_0 is $(\pi,\mathcal{S})$-valid, too. Thus, by Lemma 4.25, $G_0\sigma$ is $(\pi',\mathcal{S})$-valid as was to be shown.

Q.e.d. (Theorem 4.27)

4.14 Soundness, Safeness, and Solution-Preservation

Soundness of inference rules has the global effect that if we reduce a set of sequents to an empty set, then we know that the original set is valid. *Safeness* of inference rules has the global effect that if we reduce a set of sequents to an invalid set, then we know that already the original set was invalid. Soundness is an essential property of inference rules. Safeness is helpful in rejecting false assumptions and in patching failed proof attempts. As explained before, soundness is not sufficient for us in our framework, because we want *solution-preservation* in the sense that our $\mathcal{S}$-semantical valuations π that turn the current proof state $(\pi, \mathcal{S})$-valid is guaranteed do the same for the original input proposition.

All our inference rules of § 2.3 have all of these properties. For the inference rules that are critical in the sense that this is not obvious, we state these properties in the following theorem.

Theorem 4.28

Let (R, N) be a positive/negative variable-condition. Let C be an (R, N)-choice-condition. Let $\mathcal{S}$ be a Σ-structure.
Let us consider any of the γ-, δ^--, and δ^+-rules of § 2.3.
Let G_0 and G_1 be the sets of the sequent above and of the sequents below the bar of that rule, respectively.
Let C'' be the set of the pair indicated to the upper right of the bar if there is any (applies only to the δ^+-rules) or the empty set otherwise.
Let V be the relation indicated to the lower right of the bar if there is any (applies only to the δ^-- and δ^+-rules) or the empty set otherwise.
Let us weaken the informal requirement "Let $x^{\mathbb{A}}$ be a new free atom" of the δ^--rules to its technical essence "$x^{\mathbb{A}} \in \mathbb{A} \setminus (\ \mathrm{dom}(R) \cup \mathbb{A}(\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi)\ )$".
Let us weaken the informal statement "Let $x^{\mathbb{V}}$ be a new free variable" of the δ^+-rules to its technical essence "$x^{\mathbb{V}} \in \mathbb{V} \setminus (\ \mathrm{dom}(C \cup R \cup N) \cup \mathbb{V}(\forall x^{\mathbb{B}}.\ A)\ )$".
Let us set $C' := C \cup C''$, $R' := R \cup V{\upharpoonright_{\mathbb{V}}}$, $N' := N \cup V{\upharpoonright_{\mathbb{A}}}$.

Then $(C', (R', N'))$ is an extended extension of $(C, (R, N))$ (cf. Definition 4.21); moreover, the considered inference rule is sound and safe in the sense that G_0 and G_1 mutually $(C', (R', N'))$-reduce to each other in $\mathcal{S}$.

Proof of Theorem 4.28

To illustrate our techniques, we only treat the first rule of each kind; the other rules can be treated most similar. In the situation described in the theorem, it suffices to show that C' is an (R', N')-choice-condition (because the other properties of an extended extension are trivial), and that, for every $\mathcal{S}$-semantical valuation π that is $\mathcal{S}$-compatible with $(C', (R', N'))$, the sets G_0 and G_1 of the upper and lower sequents of the inference rule are equivalent w.r.t. their $(\pi, \mathcal{S})$-validity.

γ-rule: In this case we have $(C', (R', N')) = (C, (R, N))$. Thus, C' is an (R', N')-choice-condition by assumption of the theorem. Moreover, for every $\mathcal{S}$-valuation $\tau : \mathbb{A} \to \mathcal{S}$, and for $\delta := \epsilon(\pi)(\tau) \uplus \tau$, the truths of

$$\{\Gamma\ \ \exists y^{\mathbb{B}}.\ A\ \ \Pi\} \quad \text{and} \quad \{A\{y^{\mathbb{B}}{\mapsto}t\}\ \ \Gamma\ \ \exists y^{\mathbb{B}}.\ A\ \ \Pi\}$$

in $\mathcal{S}{\uplus}\delta$ are indeed equivalent. The implication from left to right is trivial because the former sequent is a sub-sequent of the latter. For the other direction, assume that $A\{y^{\mathbb{B}}{\mapsto}t\}$

is true in $\mathcal{S}\uplus\delta$. Thus, by the SUBSTITUTION [VALUE] LEMMA (second equation) and the VALUATION LEMMA for $l=0$ (third equation):

$$\begin{aligned}\mathsf{TRUE} &= \mathrm{eval}(\mathcal{S}\uplus\delta)(A\{y^{\mathbb{B}}\mapsto t\})\\ &= \mathrm{eval}(\mathcal{S}\uplus((\{y^{\mathbb{B}}\mapsto t\}\uplus{}_{\mathbb{VAB}\setminus\{y^{\mathbb{B}}\}}\upharpoonright\mathrm{id})\circ\mathrm{eval}(\mathcal{S}\uplus\delta)))(A)\\ &= \mathrm{eval}(\mathcal{S}\uplus\{y^{\mathbb{B}}\mapsto\mathrm{eval}(\mathcal{S}\uplus\delta)(t)\}\uplus\delta)(A)\end{aligned}$$

Thus, by the backward direction of the $\exists$-LEMMA, $\exists y^{\mathbb{B}}.\ A$ is true in $\mathcal{S}\uplus\delta$. Thus, the upper sequent is true $\mathcal{S}\uplus\delta$.

<u>δ^--rule:</u> In this case, we have $x^{\mathbb{A}}\in\mathbb{A}\setminus(\mathrm{dom}(R)\cup\mathbb{A}(\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi))$, $C''=\emptyset$, and $V=\mathbb{V}(\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi)\times\{x^{\mathbb{A}}\}$. Thus, $C'=C$, $R'=R$, and $N'=N\cup V$.

<u>Claim 1:</u> C' is an (R',N')-choice-condition.

<u>Proof of Claim 1:</u> By assumption of the theorem, C is an (R,N)-choice-condition. Thus, (R,N) is a consistent positive/negative variable-condition. By Definition 4.4, R is well-founded and $R^+\circ N$ is irreflexive. Since $x^{\mathbb{A}}\notin\mathrm{dom}(R)$, we have $x^{\mathbb{A}}\notin\mathrm{dom}(R^+)$. Thus, because of $\mathrm{ran}(V)=\{x^{\mathbb{A}}\}$, also $R^+\circ N'$ is irreflexive. Thus, (R',N') is a consistent positive/negative variable-condition, and C' is a (R',N')-choice-condition. <u>Q.e.d. (Claim 1)</u>

Now, for the soundness direction, it suffices to show the contrapositive, namely to assume that there is an $\mathcal{S}$-valuation $\tau:\mathbb{A}\to\mathcal{S}$ such that $\{\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi\}$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau)\uplus\tau$, and to show that there is an $\mathcal{S}$-valuation $\tau':\mathbb{A}\to\mathcal{S}$ such that $\{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}\ \ \Gamma\ \ \Pi\}$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau')\uplus\tau'$. Under this assumption, $\Gamma\Pi$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau)\uplus\tau$.

<u>Claim 2:</u> $\Gamma\Pi$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau')\uplus\tau'$ for all $\tau':\mathbb{A}\to\mathcal{S}$ with ${}_{\mathbb{A}\setminus\{x^{\mathbb{A}}\}}\upharpoonright\tau'={}_{\mathbb{A}\setminus\{x^{\mathbb{A}}\}}\upharpoonright\tau$.

<u>Proof of Claim 2:</u> Because of $x^{\mathbb{A}}\notin\mathbb{A}(\Gamma\Pi)$, by the EXPLICITNESS LEMMA, if Claim 2 did not hold, there would have to be some $u^{\mathbb{V}}\in\mathbb{V}(\Gamma\Pi)$ with $x^{\mathbb{A}}\ S_\pi\ u^{\mathbb{V}}$. Then we have $u^{\mathbb{V}}\ N'\ x^{\mathbb{A}}$. Thus, we know that $(R'\cup S_\pi)^+\circ N'$ is not irreflexive, which contradicts π being $\mathcal{S}$-compatible with $(C',(R',N'))$. <u>Q.e.d. (Claim 2)</u>

Moreover, under the above assumption, also $\forall x^{\mathbb{B}}.\ A$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau)\uplus\tau$. By the backwards direction of the $\forall$-LEMMA, this means that there is some object o such that A is false in $\mathcal{S}\uplus\{x^{\mathbb{B}}\mapsto o\}\uplus\epsilon(\pi)(\tau)\uplus\tau$. Set $\tau':={}_{\mathbb{A}\setminus\{x^{\mathbb{A}}\}}\upharpoonright\tau\uplus\{x^{\mathbb{A}}\mapsto o\}$. Then, by the SUBSTITUTION [VALUE] LEMMA (first equation), by the VALUATION LEMMA for $l=0$ (second equation), and by the EXPLICITNESS LEMMA and $x^{\mathbb{A}}\notin\mathbb{A}(A)$ (third equation), we have:

$$\begin{aligned}\mathrm{eval}(\mathcal{S}\uplus\epsilon(\pi)(\tau)\uplus\tau')(A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}) &=\\ \mathrm{eval}(\mathcal{S}\uplus((\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}\uplus{}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}\upharpoonright\mathrm{id})\circ\mathrm{eval}(\mathcal{S}\uplus\epsilon(\pi)(\tau)\uplus\tau')))(A) &=\\ \mathrm{eval}(\mathcal{S}\uplus\{x^{\mathbb{B}}\mapsto o\}\uplus\epsilon(\pi)(\tau)\uplus\tau')(A) &=\\ \mathrm{eval}(\mathcal{S}\uplus\{x^{\mathbb{B}}\mapsto o\}\uplus\epsilon(\pi)(\tau)\uplus\tau)(A) &= \mathsf{FALSE}.\end{aligned}$$

<u>Claim 4:</u> $A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau')\uplus\tau'$.

<u>Proof of Claim 4:</u> Otherwise, there must be some $u^{\mathbb{V}}\in\mathbb{V}(A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\})$ with $x^{\mathbb{A}}\ S_\pi\ u^{\mathbb{V}}$. Then we have $u^{\mathbb{V}}\ N'\ x^{\mathbb{A}}$. Thus, we know that $(R'\cup S_\pi)^+\circ N'$ is not irreflexive, which contradicts π being $\mathcal{S}$-compatible with $(C',(R',N'))$. <u>Q.e.d. (Claim 4)</u>

By the Claims 4 and 2, $\{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}\ \ \Gamma\ \ \Pi\}$ is false in $\mathcal{S}\uplus\epsilon(\pi)(\tau')\uplus\tau'$, as was to be show for the soundness direction of the proof.

Finally, for the safeness direction of the proof, assume that the upper sequent $\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi$ is $(\pi,\mathcal{S})$-valid. For arbitrary $\tau:\mathbb{A}\to\mathcal{S}$, we have to show that the lower sequent $A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}\ \Gamma\ \Pi$ is true in $\mathcal{S}\uplus\delta$ for $\delta:=\epsilon(\pi)(\tau)\uplus\tau$. If some formula in $\Gamma\Pi$ is true in $\mathcal{S}\uplus\delta$, then the lower sequent is true in $\mathcal{S}\uplus\delta$ as well. Otherwise, $\forall x^{\mathbb{B}}.\ A$ is true in $\mathcal{S}\uplus\delta$.

Then, by the forward direction of the ∀-LEMMA, this means that A is true in $\mathcal{S}\uplus\chi\uplus\delta$ for all $\mathcal{S}$-valuations $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$. Then, by the SUBSTITUTION [VALUE] LEMMA (first equation), and by the VALUATION LEMMA for $l = 0$ (second equation), we have:

$$\begin{aligned} \mathrm{eval}(\mathcal{S} \uplus \delta)(A\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\}) &= \\ \mathrm{eval}(\mathcal{S} \uplus ((\{x^{\mathbb{B}}\mapsto x^{\mathbb{A}}\} \uplus {}_{\mathbb{VAB}\setminus\{x^{\mathbb{B}}\}}\rceil\mathrm{id}) \circ \mathrm{eval}(\mathcal{S} \uplus \delta)))(A) &= \\ \mathrm{eval}(\mathcal{S} \uplus \{x^{\mathbb{B}}\mapsto\delta(x^{\mathbb{A}})\} \uplus \delta)(A) &= \mathsf{TRUE}. \end{aligned}$$

δ^{+}-rule: In this case, we have $x^{\mathbb{V}} \in \mathbb{V} \setminus (\mathrm{dom}(C \cup R \cup N) \cup \mathbb{V}(\forall x^{\mathbb{B}}.\ A))$,
$C'' = \{(x^{\mathbb{V}}, \varepsilon x^{\mathbb{B}}.\ \neg A)\}$, and $V = \mathbb{VA}(\forall x^{\mathbb{B}}.\ A) \times \{x^{\mathbb{V}}\}$.
Thus, $C' = C \cup \{(x^{\mathbb{V}}, \varepsilon x^{\mathbb{B}}.\ \neg A)\}$, $R' = R \cup V$, and $N' = N$.
By assumption of the theorem, C is an (R, N)-choice-condition. Thus, (R, N) is a consistent positive/negative variable-condition. Thus, by Definition 4.4, R is well-founded and $R^{+}\circ N$ is irreflexive.

Claim 5: R' is well-founded.
Proof of Claim 5: Let B be a non-empty class. We have to show that there is an R'-minimal element in B. Because R is well-founded, there is some R-minimal element in B. If this element is V-minimal in B, then it is an R'-minimal element in B. Otherwise, this element is $x^{\mathbb{V}}$ and there is an element $n^{\mathbb{VA}} \in B \cap \mathbb{VA}(\forall x^{\mathbb{B}}.\ A)$. Set $B' := \{\ b^{\mathbb{VA}} \in B \mid b^{\mathbb{VA}}\ R^{*}\ n^{\mathbb{VA}}\ \}$. Because of $n^{\mathbb{VA}} \in B'$, we know that B' is a non-empty subset of B. Because R is well-founded, there is some R-minimal element $m^{\mathbb{VA}}$ in B'. Then $m^{\mathbb{VA}}$ is also an R-minimal element of B. Because of $x^{\mathbb{V}} \notin \mathbb{VA}(\forall x^{\mathbb{B}}.\ A) \cup \mathrm{dom}(R)$, we know that $x^{\mathbb{V}} \notin B'$. Thus, $m^{\mathbb{VA}} \neq x^{\mathbb{V}}$. Thus, $m^{\mathbb{VA}}$ is also a V-minimal element of B. Thus, $m^{\mathbb{VA}}$ is also an R'-minimal element of B.
Q.e.d. (Claim 5)

Claim 6: $(R')^{+} \circ N'$ is irreflexive.
Proof of Claim 6: Suppose the contrary. Because $R^{+} \circ N$ is irreflexive, $R^{*} \circ (V \circ R^{*})^{+} \circ N$ must be reflexive. Because of $\mathrm{ran}(V) = \{x^{\mathbb{V}}\}$ and $\{x^{\mathbb{V}}\} \cap \mathrm{dom}(R \cup N) = \emptyset$, we have $V \circ R = \emptyset$ and $V \circ N = \emptyset$. Thus, $R^{*} \circ (V \circ R^{*})^{+} \circ N = R^{*} \circ V^{+} \circ N = \emptyset$. Q.e.d. (Claim 6)

Claim 7: C' is a (R', N')-choice-condition.
Proof of Claim 7: By Claims 5 and 6, (R', N') is a consistent positive/negative variable-condition. As $x^{\mathbb{V}} \in \mathbb{V}\setminus\mathrm{dom}(C)$, we know that C' is a partial function on $\mathbb{V}$ just as C. Moreover, for $y^{\mathbb{V}} \in \mathrm{dom}(C')$, we either have $y^{\mathbb{V}} \in \mathrm{dom}(C)$ and then
$\mathbb{VA}(C'(y^{\mathbb{V}})) \times \{y^{\mathbb{V}}\} = \mathbb{VA}(C(y^{\mathbb{V}})) \times \{y^{\mathbb{V}}\} \subseteq R^{+} \subseteq (R')^{+}$, or $y^{\mathbb{V}} = x^{\mathbb{V}}$ and then
$\mathbb{VA}(C'(y^{\mathbb{V}})) \times \{y^{\mathbb{V}}\} = \mathbb{VA}(\varepsilon x^{\mathbb{B}}.\ \neg A) \times \{x^{\mathbb{V}}\} = V \subseteq R' \subseteq (R')^{+}$. Q.e.d. (Claim 7)

Now it suffices to show that, for each $\tau : \mathbb{A} \to \mathcal{S}$, and for $\delta := \epsilon(\pi)(\tau) \uplus \tau$, the truth of $\{\mathit{\Gamma}\quad \forall x^{\mathbb{B}}.\ A \quad \mathit{\Pi}\}$ in $\mathcal{S} \uplus \delta$ is logically equivalent that of $\{A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\}\quad \mathit{\Gamma}\quad \mathit{\Pi}\}$.

Then, for the soundness direction, it suffices to show that the former sequent is true in $\mathcal{S}\uplus\delta$ under the assumption that the latter is. If some formula in $\mathit{\Gamma\Pi}$ is true in $\mathcal{S}\uplus\delta$, then the former sequent is true in $\mathcal{S}\uplus\delta$ as well. Otherwise, this means that $A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\}$ is true in $\mathcal{S}\uplus\delta$. Then $\neg A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\}$ is false in $\mathcal{S}\uplus\delta$. By the EXPLICITNESS LEMMA, $\neg A\{x^{\mathbb{B}}\mapsto x^{\mathbb{V}}\}$ is false in $\mathcal{S}\uplus\delta\uplus\chi$ for all $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$. Because π is $\mathcal{S}$-compatible with $(C', (R', N'))$ and because of $C'(x^{\mathbb{V}}) = \varepsilon x^{\mathbb{B}}.\ \neg A$, by Item 2 of Definition 4.15, $\neg A$ is false in $\mathcal{S}\uplus\delta\uplus\chi$ for all $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$. Then A is true in $\mathcal{S}\uplus\delta\uplus\chi$ for all $\chi : \{x^{\mathbb{B}}\} \to \mathcal{S}$. Then, by the backwards direction of the ∀-LEMMA, $\forall x^{\mathbb{B}}.\ A$ is true in $\mathcal{S}\uplus\delta$.

The safeness direction is perfectly analogous to the case of the δ^{-}-rule.

Q.e.d. (Theorem 4.28)

5 Summary and Discussion of our Free-Variable Framework

5.1 Positive/Negative Variable-Conditions

Where is the syntax of formulas defined?

We take a *sequent* to be a list of formulas which denotes the disjunction of these formulas. In addition to the standard frameworks of two-valued logics, our formulas may contain free atoms and variables with a context-independent semantics: Although we admit explicit quantification to bind only *bound atoms*, our *free atoms* (written $x^{\mathbb{A}}$) are implicitly universally quantified. Moreover, free variables (written $x^{\mathbb{V}}$) are implicitly existentially quantified. The structure of this implicit form of quantification without quantifiers is represented globally in a *positive/negative variable-condition* (R, N), which is a directed graph on free atoms and variables whose edges are either elements of R or of N. Roughly speaking, on the one hand, a *free variable* $y^{\mathbb{V}}$ is put into the scope of another free variable or atom $x^{\mathbb{VA}}$ by an edge $(x^{\mathbb{VA}}, y^{\mathbb{V}})$ in R; and, on the other hand, a *free atom* $y^{\mathbb{A}}$ is put into the scope of another free variable or atom $x^{\mathbb{VA}}$ by an edge $(x^{\mathbb{VA}}, y^{\mathbb{A}})$ in N. More precisely, on the one hand, an edge $(x^{\mathbb{VA}}, y^{\mathbb{V}})$ must be put into R

- if $y^{\mathbb{V}}$ is introduced in a δ^{+}-step where $x^{\mathbb{VA}}$ occurs in the principal[2] formula, and also
- if $y^{\mathbb{V}}$ is globally replaced with a term in which $x^{\mathbb{VA}}$ occurs;

and, on the other hand, an edge $(x^{\mathbb{VA}}, y^{\mathbb{A}})$ must be put into N

- if $x^{\mathbb{VA}}$ is actually a free *variable*, and $y^{\mathbb{V}}$ is introduced in a δ^{-}-step where $x^{\mathbb{VA}}$ occurs anywhere in the sequent.

Such edges *may* always be added to the positive/negative variable-condition. This might be appropriate especially in the formulation of a new proposition: partly, because we may need this for modeling the intended semantics by representing the intended quantificational structure for the free variables and atoms of the new proposition; partly, because we may need this for enabling induction in the form of FERMAT's *descente infinie* on the free atoms of the proposition; cf. [WIRTH, 2004, §§ 2.5.2 and 3.3].

A positive/negative variable-condition (R, N) is *consistent* if each cycle in of the directed graph has more than one edge from N.

5.2 Semantics of Positive/Negative Variable-Conditions

The value assigned to a free variable $y^{\mathbb{V}}$ by an $\mathcal{S}$*-semantical valuation* π may depend on the value assigned to an atom $x^{\mathbb{A}}$ by an $\mathcal{S}$*-valuation* of the atoms. In that case, the *semantical relation* S_{π}, contains an edge $(x^{\mathbb{A}}, y^{\mathbb{V}})$. Moreover, π is enforced to obey the quantificational structure by the requirement that $(R \cup S_{\pi}, N)$ must be consistent; cf. Definitions 4.10 and 4.15.

5.3 Replacing ε-terms with free variables

Suppose that an ε-term $\varepsilon z^{\mathbb{B}}.\ B$ has free occurrences of exactly the bound atoms $v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}$ which are not free atoms of our framework, but are actually bound in the syntactic context in which this ε-term occurs. Then we replace it in its context with the application term $z^{\mathbb{V}}(v_0^{\mathbb{B}})\cdots(v_{l-1}^{\mathbb{B}})$ for a fresh free variable $z^{\mathbb{V}}$ and set the value of a global function C (called the *choice-condition*) at $z^{\mathbb{V}}$ according to

$$C(z^{\mathbb{V}}) \quad := \quad \lambda v_0^{\mathbb{B}}.\ \ldots \lambda v_{l-1}^{\mathbb{B}}.\ \varepsilon z^{\mathbb{B}}.\ B,$$

and augment R with an edge $(y^{\mathbb{VA}}, z^{\mathbb{V}})$ for each free variable or atom $y^{\mathbb{VA}}$ occurring in B.

5.4 Semantics of Choice-Conditions

A free variable $z^{\mathbb{V}}$ in the domain of the global choice-condition C must take a value that makes $C(z^{\mathbb{V}})$ true — if such a choice is possible. This can be formalized as follows.

Let "eval" be the standard evaluation function. Let $\mathcal{S}$ be any of the semantical structures (or models) under consideration. Let δ be a valuation of the free variables and atoms (resulting from an $\mathcal{S}$-semantical valuation of the variables and an $\mathcal{S}$-valuation of the atoms). Let χ be an arbitrary $\mathcal{S}$-valuation of the bound atoms $v_0^{\mathbb{B}}, \ldots, v_{l-1}^{\mathbb{B}}, z^{\mathbb{B}}$. Then $\delta(z^{\mathbb{V}})$ must be a function which chooses a value that makes B true whenever possible, in the sense that $\mathrm{eval}(\mathcal{S}\uplus\delta\uplus\chi)(B) = \mathsf{TRUE}$ implies $\mathrm{eval}(\mathcal{S}\uplus\delta\uplus\chi)(B\mu) = \mathsf{TRUE}$ for

$$\mu := \{z^{\mathbb{B}} \mapsto z^{\mathbb{V}}(v_0^{\mathbb{B}})\cdots(v_{l-1}^{\mathbb{B}})\}.$$

5.5 Substitution of Free Variables ("ε-Substitution")

The kind of logical inference we essentially need is (problem-) *reduction*, the backbone of abduction and goal-directed deduction; cf. § 4.13. In reduction steps our free atoms and variables show the following behavior with respect to their instantiation:

Atoms behave as constant parameters. A free variable $y^{\mathbb{V}}$, however, may be globally instantiated with any term by application of a substitution σ; unless, of course, in case it is in the domain of the global choice-condition C, in which case σ must additionally satisfy $C(y^{\mathbb{V}})$, in a sense to be explained below. Precise ref.

In addition, the applied substitution σ must always be an (R, N)-*substitution*. This means that the current positive/negative variable-condition (R, N) remains consistent when we extend it to its so-called σ-*update*, which augments R with the edges from the free variables and atoms in $\sigma(z^{\mathbb{V}})$ to $z^{\mathbb{V}}$, for each free variable $z^{\mathbb{V}}$ in the domain $\mathrm{dom}(\sigma)$.

Moreover, the global choice-condition C must be updated by removing $z^{\mathbb{V}}$ from its domain $\mathrm{dom}(C)$ and by applying σ to the C-values of the free variables remaining in $\mathrm{dom}(C)$.

Now, in case of a free variable $z^{\mathbb{V}} \in \mathrm{dom}(\sigma) \cap \mathrm{dom}(C)$, σ satisfies the current choice-condition C if $(Q_C(z^{\mathbb{V}}))\sigma$ is valid in the context of the updated variable-condition and choice-condition. Here, for a choice-condition $C(z^{\mathbb{V}})$ given as above, $Q_C(z^{\mathbb{V}})$ denotes the formula

$$\forall v_0^{\mathbb{B}}.\ \ldots \forall v_{l-1}^{\mathbb{B}}.\ \big(\ \exists z^{\mathbb{B}}.\ B \Rightarrow B\mu\ \big),$$

which is nothing but our version of HILBERT's axiom (ε_0); cf. Definition 3.10. Under these conditions, the invariance of reduction under substitution is stated in Theorem 4.27(6b).

Finally, note that $Q_C(z^{\mathbb{V}})$ itself is always valid in our framework; cf. Lemma 4.19.

5.6 Where have all the ε-terms gone?

The ε-symbol does not occur anymore in our terms, and our formulas are much more readable than in the standard approach of in-line presentation of ε-terms, which was always just a theoretical presentation because in practical proofs the ε-terms would have grown so large that the mere size of them made them inaccessible to human inspection. To see this, compare our presentation in Example 3.9 to the one in Example 3.7, which is still hard to read although we have invested some efforts in finding a readable form of presentation. From a mathematical point of view, however, the original ε-terms are still present in our approach; up to isomorphism and with the exception of some irrelevant term sharing. To make these ε-terms explicit in a formula A for a given (R, N)-choice-condition C, we just have to do the following:

1. Let us consider the relation C not as a function, but as a ground term rewriting system: This means that we read $(z^{\mathbb{V}}, \lambda v_0^{\mathbb{B}}.\ \ldots v_{l-1}^{\mathbb{B}}.\ \varepsilon z^{\mathbb{B}}_{.}B) \in C$ as a rewrite rule saying that we may replace the free variable $z^{\mathbb{V}}$ (the left-hand side of the rule, which is not a variable but a constant w.r.t. the rewriting system) with the right-hand side $\lambda v_0^{\mathbb{B}}.\ \ldots v_{l-1}^{\mathbb{B}}.\ \varepsilon z^{\mathbb{B}}.\ B$ in any given context as long as we want. By Definition 4.13(3), we know that all variables in B are smaller than $z^{\mathbb{V}}$ in R^+. By the consistency of our positive/negative variable-condition R (according to Definition 4.13), we know that R^+ is well-founded, and so is its multi-set extension. Thus, the multiset of the free variable of the left-hand side is bigger than the multi-set of the free-variable occurrences in the right-hand side in the well-founded multiset extension of R^+. Thus, if we rewrite a formula, the multi-set of the free-variable occurrences in the rewritten formula is smaller than the multi-set of the free-variable occurrences in the original formula. Therefore, normalization of any formula A with these rewrite rules terminates with a formula A'.

2. As typed $\lambda\alpha\beta$-reduction is also terminating, we can apply it to remove the λ-terms introduced to A' by the rewriting of Step 1. This results in a formula A'' without free variables in the domain of C. Moreover, if the free variables in A resulted from the elimination of ε-terms as described in §§ 3.8 and 5.3, then all λ-terms that were not already present in A are provided with arguments and can be removed by this rewriting.

For example, if we normalize $\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}})$ with respect to the rewriting system given by the (R, N)-choice-condition C of of Example 3.9, and then by $\lambda\alpha\beta$-reduction, we end up in a normal form which is the first-order ε-formula (3.7.1) of Example 3.7, with the exception of the renaming of some bound atoms bound by ε. Note that the normal form even preserves our information on committed choice when we consider any ε-term binding an occurrence of a bound atom of the same name to be committed to the same choice. In this sense, the representation given by the normal form is isomorphic to our original one given by $\mathsf{P}(w_a^{\mathbb{V}}, x_b^{\mathbb{V}}, y_d^{\mathbb{V}}, z_h^{\mathbb{V}})$ and C.

5.7 Are we breaking with the traditional treatment of HILBERT's ε?

Our new semantical free-variable framework was actually developed to meet the requirements analysis for the combination of mathematical induction in the liberal style of FERMAT's *descente infinie* with state-of-the-art logical deduction. The framework provides a formal system in which a working mathematician can straightforwardly develop his proofs supported by powerful automation; cf. [WIRTH, 2004].

If traditionality meant restriction to the expressional means of the first half of the 20th century — with its foundational crisis and special emphasis on constructivism, intuitionism, and proof transformation — then our approach would not classify as traditional. (In the meanwhile the fear of inconsistency should have been cured by [WITTGENSTEIN, 1939].) But with its equivalents for the traditional ε-terms (cf. § 5.6) and for the ε-substitution methods (cf. §§ 3.9 and 4.13), our framework is deeply rooted in this tradition.

The main disadvantage of a constructive syntactical framework for the ε as compared to a semantical one is the following: Constructive proofs of practically relevant theorems become too huge and too tedious, whereas semantical proofs are of a better manageable size. More important is the possibility to invent *new and more suitable logics for new applications* with semantical means, whereas proof transformations can only refer to already existing logics, cf. § 3.4.

We intend to pass the heritage of HILBERT's ε on to new generations interested in computational linguistics, automated theorem proving, and mathematics assistance systems; fields in which — with very few exceptions — the overall common opinion still is (the wrong one) that the ε hardly can be of any practical benefit.

5.8 Comparison to the original ε

The differences between our free-variable framework for the ε and HILBERT's original underspecified ε-operator, in the order of increasing importance, are the following:

1. The term-sharing of ε-terms with the help of free variables improves the readability of our formulas considerably.

2. We do not have the requirement of globally committed choice for any ε-term: Different free variables with the same choice-condition may take different values. Nevertheless, *ε-substitution* works at least as well as in the original framework of ACKERMANN, BERNAYS, and HILBERT.

3. Opposed to all other classical semantics for the ε (including the ones of [ASSER, 1957], [HERMES, 1965], and [LEISENRING, 1969]), the implicit quantification of our free variables is existential instead of universal. This change simplifies formal reasoning in all relevant contexts; namely because we have to consider only an arbitrary single solution (or substitution) instead of checking all of them.

6 Conclusion

Our novel indefinite semantics for HILBERT's ε and our novel free-variable framework presented in this paper were developed to solve the difficult soundness problems arising during the combination of mathematical induction in the liberal style of FERMAT's *descente infinie* with state-of-the-art deduction.[5] Thereby, they had passed an evaluation of their usefulness even before they were recognized as a candidate for the semantics that DAVID HILBERT probably had in mind for his ε. While the speculation on this question will go on, the semantical framework for HILBERT's ε proposed in this paper definitely has the following advantages:

Syntax: The requirement of a commitment to a choice is expressed syntactically and most clearly by the sharing of a free variable; cf. § 3.8.

Semantics: The semantics of the ε is simple and straightforward in the sense that the ε-operator becomes similar to the referential use of the indefinite article in some natural languages.

Our semantics for the ε is based on an abstract formal approach that extends a semantics for closed formulas (satisfying only very weak requirements, cf. § 4.7) to a semantics with existentially quantified "free variables" and universally quantified "free atoms", replacing the three kinds of free variables of [WIRTH, 2004; 2008; 2010] (existential (free γ-variables), universal (free δ^--variables), and ε-constrained (free δ^+-variables)). This simplification reduces the complexity of our framework considerably and will make its adaptation to applications much easier.

In spite of this simplification, we have enhanced the expressiveness of our framework by replacing the variable-conditions of [WIRTH, 2004; 2008; 2010] with our *positive/negative* variable-conditions here, such that our framework now admits us to represent HENKIN quantification directly; cf. Example 4.20. From a philosophical point of view, this clearer differentiation also provides a deep insight into the true nature and the relation of the δ^-- and the δ^+-rules.

Reasoning: In a reductive proof step, our representation of an ε-term $\varepsilon x^{\mathbb{B}}.\ A$ can be replaced with *any* term t that satisfies the formula $\exists x^{\mathbb{B}}.\ A \ \Rightarrow\ A\{x^{\mathbb{B}}{\mapsto}t\}$, cf. § 3.9. Thus, the soundness of such a replacement is likely to be expressible and verifiable in the original calculus. Our free-variable framework for the ε is especially convenient for developing proofs in the style of a working mathematician, cf. [WIRTH, 2004; 2006]. Indeed, our approach makes proof work most simple because we do not have to consider all proper choices t for x (as in all other semantical approaches) but only a single arbitrary one, which is fixed in a proof step, just as choices are settled in program steps, cf. § 3.7.

Finally, we hope that new semantical framework will help to solve further practical and theoretical problems with the ε and improve the applicability of the ε as a logical tool for description and reasoning. And already without the ε (i.e. for the case that the choice-condition is empty), our free-variable framework should find a multitude of applications in all areas of computer-supported reasoning.

Acknowledgments

Murdoch J. Gabbay

Notes

Note 1 "Bound" atoms (or variables) should actually be called "bindable" instead of "bound", because we will always have to have unbound occurrences of bound variables. When the name of the notion was coined, however, neither "bindable" nor the German "bindbar" were considered proper words of their languages.

Note 2 The notions of a *principal formula* and a *side formula* were introduced in [Gentzen, 1935] and refined in [Schmidt-Samoa, 2006]. Very roughly speaking, the principal formula of a reductive inference rule is the formula that is taken to pieces by that rule, and the side formulas are the resulting pieces. In our inference rules here, the principal formulas are the formulas above the lines except the ones in Γ, Π, and the side formulas are the formulas below the lines except the ones in Γ, Π.

Note 3 Regarding the classification of one of the δ-rules as "liberalized", we could try to object with the following two points:

- $\mathbb{VA}(\forall x^{\mathbb{B}}.\ A)$ is not necessarily a subset of $\mathbb{V}(\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi)$, because it may include some additional atoms.

  First note that δ^{-}-rules and their free atoms do not occur in inference systems with δ^{+}-rules before [Wirth, 2004], so that in the earlier systems $\mathbb{VA}(\forall x^{\mathbb{B}}.\ A)$ is indeed a subset of $\mathbb{V}(\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi)$.

  Moreover, the additional atoms blocked by the δ^{+}-rules (as compared to the δ^{-}-rules) do not block proofs in practice. This has following reason: With a reasonably minimal positive/negative variable-condition (R, N), the only additional cycles that could occur as a consequence of these additional atoms are of the form $y^{\mathbb{V}}\ N\ z^{\mathbb{A}}\ R\ x^{\mathbb{V}}\ R^{+}\ y^{\mathbb{V}}$ with $z^{\mathbb{A}} \in \mathbb{A}(\forall x^{\mathbb{B}}.\ A)$ and $y^{\mathbb{V}} \in \mathbb{V}(\Gamma\ \ \forall x^{\mathbb{B}}.\ A\ \ \Pi)$; unless we globally replace $x^{\mathbb{V}}$ during the proof attempt by an (R, N)-substitution (which, however, would not be possible for an atom $x^{\mathbb{A}}$ introduced by a δ^{-}-rule anyway). And, in this case, the corresponding δ^{-}-rule would result in the cycle $y^{\mathbb{V}}\ N\ x^{\mathbb{A}}\ R^{+}\ y^{\mathbb{V}}$ anyway.

- The δ^{+}-rule may contribute an R-edge to a cycle with exactly one edge from N, whereas the analogous δ^{-}-rule would contribute an N-edge instead, so the analogous cycle would then not count as counterexample to the consistency of the positive/negative variable-condition because it has two edges from N.

  Also in this case we conjecture that δ^{-}-rules do not admit any successful proofs that are not possible with the analogous δ^{+}-rules. A proof of this conjecture, however, is not easy: First, it is a global property which requires us to consider the whole inference system. Second, δ^{-}-rules indeed admit some extra (R, N)-substitutions: If we want to prove $\forall y^{\mathbb{B}}.\ \mathsf{Q}(a^{\mathbb{V}}, y^{\mathbb{B}}) \wedge \forall x^{\mathbb{B}}.\ \mathsf{Q}(x^{\mathbb{B}}, b^{\mathbb{V}})$, which is true for a reflexive ordering Q with a minimal and a maximal element, β- and δ^{-}-rules reduce this to the two goals $\mathsf{Q}(a^{\mathbb{V}}, y^{\mathbb{A}})$ and $\mathsf{Q}(x^{\mathbb{A}}, b^{\mathbb{V}})$ with positive/negative variable-condition (R, N) with $R = \emptyset$ and $N = \{(a^{\mathbb{V}}, y^{\mathbb{A}}),\ (b^{\mathbb{V}}, x^{\mathbb{A}})\}$. Then $\sigma_{\mathbb{A}} := \{a^{\mathbb{V}}{\mapsto}x^{\mathbb{A}},\ b^{\mathbb{V}}{\mapsto}y^{\mathbb{A}}\}$ is an (R, N)-substitution. The analogous δ^{+}-rules would have resulted in the positive/negative variable-condition (R', N') with $R' = \{(a^{\mathbb{V}}, y^{\mathbb{V}}),\ (b^{\mathbb{V}}, x^{\mathbb{V}})\}$ and $N' = \emptyset$. But $\sigma_{\mathbb{V}} := \{a^{\mathbb{V}}{\mapsto}x^{\mathbb{V}},\ b^{\mathbb{V}}{\mapsto}y^{\mathbb{V}}\}$ is not an (R', N')-substitution!

Note 4 More precisely: Lemma 4.19 depends on the backward directions of the ∀-Lemma and the ⇒-Lemma, and on the forward direction of the ∃-Lemma. Lemma 4.25 and Theorem 4.27(6) depend on the forward directions of the ∀-Lemma and the ⇒-Lemma, and on the backward direction of the ∃-Lemma. Theorem 4.28 depends on the backward direction of the ∃-Lemma and on both directions of the ∀-Lemma.

Note 5 The well-foundedness required for the soundness of *descente infinie* gave rise to a notion of reduction which preserves solutions, cf. Definition 4.26. The liberalized δ-rules as found in [Fitting, 1996] do not satisfy this notion. The addition of our choice-conditions finally turned out to be the only way to repair this defect of the liberalized δ-rules. Cf. [Wirth, 2004] for more details.

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Do we need dependence at level 2 or can we deduce it from level 1?