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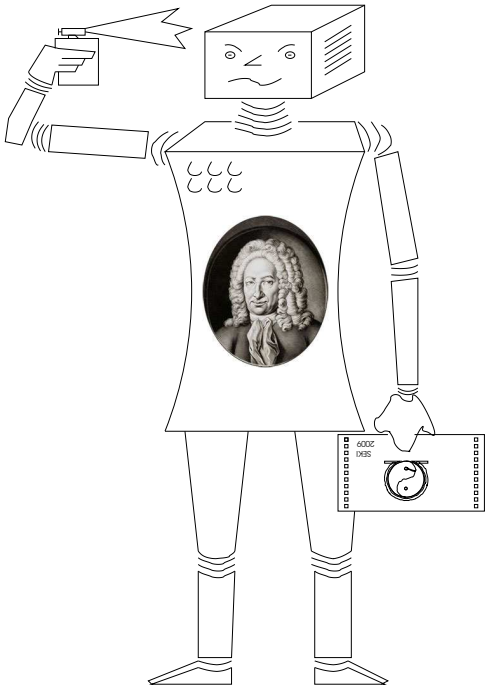

Herbrand's Fundamental Theorem: The Historical Facts and their Streamlining

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Abstract

Using Heijenoort's unpublished *generalized* rules of quantification, we discuss the proof of Herbrand's Fundamental Theorem in the form of Heijenoort's correction of Herbrand's "False Lemma" and present a didactic example. Although we are mainly concerned with the inner structure of Herbrand's Fundamental Theorem and the questions of its quality and its depth, we also discuss the outer questions of its historical context and why Bernays called it "the central theorem of predicate logic" and considered the form of its expression to be "concise and felicitous".

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1 Motivation

When working on our handbook article [Wirth &al., 2009] on Herbrand's work in logic, we were fascinated by the following three points:

Herbrand's Unification Algorithm

In addition to Heijenoort's reprint of Herbrand's PhD thesis in [Herbrand, 1968], on which also the standard translation of the 1960s by Dreben and Heijenoort in [Heijenoort, 1971], [Herbrand, 1971] is based, we accessed the original print [Herbrand, 1930]. For some unknown reason, Herbrand's unification algorithm in the reprint [Herbrand, 1968] has the confusing wording "égalités associées" instead of the original "égalités normales". Based on the correct original text, our English translation of Herbrand's unification algorithm resulted in a modern equational formulation, which is more lucid than the one of Dreben and Heijenoort and strongly supports the thesis of [Abeles, 1994] that Herbrand's unification algorithm is very close to the equational unification algorithm of [Martelli & Montanari, 1982].[1]

Heijenoort's correction of Herbrand's "False Lemma"

We were surprised that Heijenoort's lucid and modern correction of Herbrand's "False Lemma" in [Heijenoort, 1975] was still unpublished.

Herbrand's Construction of Linear First-Order Derivations

We were fascinated by that part of the proof of Herbrand's Fundamental Theorem which shows how to construct a *linear* derivation of a first-order formula A, given as input (besides A) only the natural number n such that A has the *purely sentential* Property C of order n.

Here, "linear" means that the derivation does not contain any inference step with more than one premise (such as an application of *modus ponens*), such that it has no branching when it is viewed as a tree. Moreover, all function and predicate symbols within this proof occur already in A, and all formulas in the proof are similar to A in the sense that they have the so-called *"sub-formula" property* w.r.t. A.

As a consequence of this linearity, Herbrand's *modus ponens*-free calculus and Gentzen's classical sequent calculus represent first steps toward human-orientation in the sense that a human being with a semantical understanding of an informal proof has a good chance to construct a formal proof in these calculi.[2]

[1] Note that that thesis of [Abeles, 1994] seems to have been developed without access to the uncontorted original [Herbrand, 1930]. While our translation may well be influenced by our knowledge of the development of unification algorithms in the 1970s (see [Martelli & Montanari, 1982], [Paterson & Wegman, 1978] for references), we noticed the article [Abeles, 1994] only after our translation had already been completed.

[2] Calculi that are both human- and machine-oriented are in demand for the synergetic combination of mathematicians and computing machines in interactive theorem proving systems with strong automation support [Wirth, 2012c].

> We think that HERBRAND's construction of *modus ponens*-free proofs should be part of the standard education of every logician, just as well as the famous construction of Cut-free proofs according to GENTZEN's Hauptsatz [GENTZEN, 1935], wherein GENTZEN — some years later — picked up HERBRAND's idea of formal proofs without detours in a different form.

The last two items are strongly connected because the calculus of HEIJENOORT's correction makes HERBRAND's construction of linear first-order derivations particularly elegant and noteworthy.

Therefore, we will present in this paper a novel elaboration of the last two points.

2 Introduction

Before we start, let us briefly mention the complementarity of the lives of JACQUES HERBRAND (1908–1931) and JEAN VAN HEIJENOORT (1912–1986): HERBRAND had created a significant part of the world heritage of modern logic when he died in 1931. HEIJENOORT became the leading conservator and splendid commentator and translator of the world heritage of modern logic in the 1960s, and realized it as a field for historical studies.

The scope of work on history ranges from the gathering and documenting of facts over interpretation, translation, commentation, and the speculation on the gaps in the documented facts, up to the manipulation of knowledge on history for politics or for efficient didactics, such as in the notes on history in natural-science textbooks [KUHN, 1962, § XIII].

HEIJENOORT's interest in history partly resulted from his involvement into politics and his realization of the contortion of knowledge for the manipulation of society [FEFERMAN, 1993a; 1993b]. His work on the mathematics on MARX and ENGELS [HEIJENOORT, 1986a] obviously had the intention to free himself and the society from this kind of manipulation.

Our main subject in this paper is how HEIJENOORT himself has slightly manipulated the historical facts on HERBRAND's Fundamental Theorem, mostly in his papers [HEIJENOORT, 1975; 1992], which he did not properly publish himself, but also in his published paper [HEIJENOORT, 1986b]. In these papers, HEIJENOORT changed the inference rules of HERBRAND's *modus ponens*-free calculus (which we will present in § 8), probably for didactical reasons. We think that HEIJENOORT wanted to put more emphasis on the elegance of HERBRAND's Fundamental Theorem and to correct HERBRAND's "False Lemma" in a more reasonable way than BERNAYS, GÖDEL, and DREBEN did.

Regarding HEIJENOORT's merits in publishing, translating, and commenting HERBRAND's original work,[3] we have absolutely no reason to blame HEIJENOORT for this slight manipulation. Our motivation here is a different one: We will discuss the inference system resulting from HEIJENOORT's changes to HERBRAND's *modus ponens*-free calculus in detail in §§ 3–7 (and also present an up-to-date free-variable version of it), because it helps us to put HERBRAND's Fundamental Theorem into a new light, emphasizing its elegance and practical and theoretical relevance.

[3] See [HERBRAND, 1968; 1971] and [HEIJENOORT, 1971] for HEIJENOORT's publishing, translating, and commenting of HERBRAND's work as a logician. Moreover, see [HEIJENOORT, 1982; 1986b; 1992] for HEIJENOORT's secondary discussion of HERBRAND's work as a logician.

We think that HERBRAND's work in logic still deserves to be even more widely known than it is today. We also think that our partly didactical intentions in this paper go well together with HEIJENOORT's original ones regarding his work on HERBRAND.

The still contentious assessment of HERBRAND's Fundamental Theorem is also expressed at the end of the following text, written by SOL FEFERMAN (*1928) in 1993,[4] which also serves as a further introduction into our subject:

> "HERBRAND's fundamental theorem from his 1930 dissertation gives a kind of reduction, in the limit, of quantificational logic to propositional logic. This has led to considerable further work in proof theory, among which the most noteworthy is that of GERHARD GENTZEN (beginning in 1934); it has also been used in recent years as the basis for an approach to automated deduction. However HERBRAND's own proofs were flawed; already in 1939 BERNAYS remarked that "HERBRAND's proof is hard to follow", and in 1943, GÖDEL uncovered an essential gap, though his notes on this were never published. Counter-examples to two important lemmas in HERBRAND were produced by BURTON DREBEN in collaboration with his students and colleagues PETER ANDREWS and STÅL AANDERAA in an article in 1963, in which they outlined how the arguments could be repaired. A detailed proof of the crucial lemma was later given by DREBEN and JOHN DENTON in 1966.
>
> Despite the known flaws, VAN HEIJENOORT regarded HERBRAND's theorem as one of the deepest results of logic. (In the minds of most logicians that assessment is debatable, but the importance of the theorem is undeniable.)"
>
> [FEFERMAN, 1993b, p. 383, Note 11 omitted]

We will refer to that "crucial lemma" as *"HERBRAND's 'False Lemma'"* and discuss it in § 9.

After having made all that material most obvious, we will not be able to avoid the debate on the depth and centrality of HERBRAND's Fundamental Theorem in § 10, supporting some of the well-known assessments of HEIJENOORT, BERNAYS, and FEFERMAN.

We will conclude our paper with § 11.

[4] Today, however, there is better information available on the work of GERHARD GENTZEN (1909–1945) in [MENZLER-TROTT, 2001; 2007], and GÖDEL's notes on HERBRAND's "False Lemma" have been published in [GOLDFARB, 1993].

3 HERBRAND's *Modus Ponens*-Free Calculus in the Eyes of HEIJENOORT [1975]

When HEIJENOORT refers to HERBRAND's calculus in [HEIJENOORT, 1982; 1986b; 1992], he actually describes his own calculus of [HEIJENOORT, 1975], which lacks HERBRAND's rules of passage and compensates this with generalized versions of the rules of quantification.[5]

Partly as a further introduction to our subject, we will now quote from HEIJENOORT's unpublished draft paper [HEIJENOORT, 1975], but — as a few of HEIJENOORT's technical terms are awkward and vary from paper to paper — we will replace them in our quotations consistently with modern terminology,[6] based on the classification of reductive inference rules by RAYMOND M. SMULLYAN (*1919) into α (sentential+non-branching), β (sentential+branching), γ, and δ, cf. [SMULLYAN, 1968].

> "In 1929 JACQUES HERBRAND stated a theorem that, as BERNAYS [1957] writes, is 'the central theorem of predicate logic'. The theorem has many applications, one of them being a new approach to the problem of ascertaining the validity of a formula.
>
> HERBRAND's theorem, stated in a general form, is this: Given a formula F of classical quantification theory, we can effectively generate an infinite sequence of quantifier-free formulas $F_1, F_2, \ldots$, such that F is provable in (any standard system of) quantification theory if and only if there is a k such that F_k is (sententially) valid; moreover, F can be retrieved from F_k through applications of some rules that have remarkable properties. There are several ways of generating an adequate sequence of quantifier-free formulas; and, moreover, many details can vary; here we shall consider two kinds of sequences: expansions and disjunctions.
>
> We consider a system Q adequate for classical quantification theory, without identity, in which the sentential connectives are negation, disjunction, and conjunction. A formula of Q is *rectified* if and only if it contains no vacuous quantifier, no variable has free and bound occurrences in it, and no two quantifiers bind occurrences of the same variable. A quantifier in a rectified formula is γ if and only if it is existential and in the scope of an even number of negation symbols, or universal and in the scope of an odd number of negation symbols; otherwise the quantifier is δ. (A γ-quantifier turns up as existential in a prenex form of the formula, and a δ-quantifier as universal.) A variable in a rectified formula is γ if and only if it is bound by a γ-quantifier; it is $\mid \delta$ if and only if it is free or bound by a δ-quantifier. (HERBRAND uses 'restreint' for 'γ' and 'general' for 'δ'; in work on HERBRAND's theorem in English 'restricted' and 'general' have been used.)
>
> A quantifier is *accessible* in a formula F if and only if it is in the scope of no quantifier of F. A variable is *accessible* in a formula if and only if it is either free or bound by an accessible quantifier."
>
> [HEIJENOORT, 1975, pp. 1–2. See our Note 6]

Note that both HERBRAND and HEIJENOORT consider equality of formulas only up to renaming of bound variables and often implicitly assume that a formula is rectified.

As HEIJENOORT's three-rule inference system is difficult to understand and not complete in the given form, let us present each of the rules twice in the following sub-sections: first in quotation of [HEIJENOORT, 1975], second in a hopefully more readable transcription.

Note that the three rules can be understood as operating on rectified formulas only, but this is not necessary for their soundness.

3.1 γ-Quantification

HEIJENOORT writes:

> "To pass from $F(\ Qx.\ H\ /\ H(x/t)\ )$, where $Qx.$ is an accessible γ-quantifier of F, H is the scope of $Qx.$, and t is a term of the system, to F."
> [HEIJENOORT, 1975, p. 4. See our Note 6]

Herein "$H(x/t)$" denotes a capture-avoiding replacement of all occurrences of x in H with t, where those bound variables in H are renamed whose quantifiers would otherwise capture free variables in t.

Let $A[\ldots]$ be the context such that HEIJENOORT's F is $A[Qx.\ H]$. Moreover, let us write "$H\{x \mapsto t\}$" for the result of the replacement of all occurrences of x in H with t (without renaming). Then we get the following formulation.

[5]Indeed, HERBRAND did not have any *generalized* rules of quantification; so HERBRAND additionally needed his rules of passage to achieve completeness, resulting in a calculus of at least four rules. Therefore, HEIJENOORT definitely refers to his own three-rule calculus when he writes:

> "A second result of HERBRAND, connected with the first, is that the formula F can be recovered from the formula F_k by means of three rules of a very definite character. These rules are the generalized rule of γ-quantification, the generalized rule of δ^--quantification and the generalized rule of simplification." [HEIJENOORT, 1986b, p. 99. See our Note 6]

And, one page later, HEIJENOORT explicitly declares his own three-rule calculus to be HERBRAND's achievement in history:

> "The system based on HERBRAND's three rules is, historically, the first example of what we call today a cut-free system; it also has the so-called subformula property."
> [HEIJENOORT, 1986b, p. 100]

[6]Besides adding a dot after HEIJENOORT's quantifiers, this means that the following technical terms will be tacitly rewritten when quoting [HEIJENOORT, 1975; HEIJENOORT, 1986b]:

HEIJENOORT [1975]	HEIJENOORT [1986b]	HEIJENOORT [1992]	Our wording here
existentialoid	existentialoid	restricted	γ
existentialoid quantifier			γ-quantifier
Rule of existentialoid quantification	Generalized rule of existentialization	Rule of existentialization	Generalized rule of γ-quantification
Rule of simplification	Generalized rule of simplification	Rule of simplification	Generalized rule of simplification
Rule of universaloid quantification	Generalized rule of universalization	Rule of universalization	Generalized rule of δ^--quantification
universaloid	universaloid	nonrestricted	δ
universaloid quantifier			δ-quantifier

Generalized rule of γ-quantification: $\dfrac{A[H\{x \mapsto t\}]}{A[Qx.\ H]}$ where

1. $Qx.$ is an accessible[7] γ-quantifier of $A[Qx.\ H]$, and
2. the free variables of the term t must[8] not be bound by quantifiers in H.

Example 3.1 (Application of the generalized rule of γ-quantification)
If the variable z does not occur free in the term t, we get the following two inference steps with identical premises by application of the generalized rule of γ-quantification at two different positions:

- $\dfrac{(t \prec t) \quad \vee \quad \neg\forall z.\ (t \prec z)}{(t \prec t) \quad \vee \quad \exists x.\ \neg\forall z.\ (x \prec z)}$ via the meta-level substitution

  $\{ \quad A[\ldots] \ \mapsto\ (t \prec t) \vee [\ldots], \qquad H \ \mapsto\ \neg\forall z.\ (x \prec z), \qquad Q \ \mapsto\ \exists \quad \};$

- $\dfrac{(t \prec t) \quad \vee \quad \neg\forall z.\ (t \prec z)}{(t \prec t) \quad \vee \quad \neg\forall x.\ \forall z.\ (x \prec z)}$ via the meta-level substitution

  $\{ \quad A[\ldots] \ \mapsto\ (t \prec t) \vee \neg[\ldots], \qquad H \ \mapsto\ \forall z.\ (x \prec z), \qquad Q \ \mapsto\ \forall \quad \}.$ $\square$

Today's calculi for proof search have an additional kind of free variables, called "free variables" in [Fitting, 1996], [Wirth, 2013], or "(free) γ-variables" in [Wirth, 2004; 2008]. We will mark such a variable with the upper index γ, such as in "x^γ". By taking the meta-variable t of the generalized rule of γ-quantification to be such a free γ-variable, these *free-variable calculi* admit us to delay the choice of the witnessing term t in a backward application of the generalized rule of γ-quantification in reductive proof search. Later, when the state of the reductive proof attempt provides sufficient information which witness is promising, the free γ-variable can be globally instantiated in the whole proof tree.

Although free-variable calculi can still use the generalized rule of γ-quantification in the previous formulation, we will refer several times to the following sub-rule:

Generalized rule of restricted γ-quantification: $\dfrac{A[H\{x \mapsto x^\gamma\}]}{A[Qx.\ H]}$ where $Qx.$ is an accessible γ-quantifier of $A[Qx.\ H]$.

[7] The restriction of accessibility of the introduced quantifier in the generalized rule of γ-quantification is not necessary for soundness (contrary to the generalized rule of δ^--quantification). This restriction is introduced, however, because already the restricted rule suffices for completeness and for the constructions in the proof of Herbrand's Fundamental Theorem. Moreover, the restriction of accessibility guarantees the equivalence of the generalized rules of quantification with their non-generalized versions via the rules of passage; cf. [Heijenoort, 1968, p. 6].

[8] If a variable z is bound by quantifiers in H (as in Example 3.1) and occurs free in the term t (in violation of Side-Condition 2), an implicit renaming of the bound occurrences of z in H is admitted to enable a backward (i.e. *reductive*) application of the generalized rule of γ-quantification. Note that this is never required for reductive proof search if the formula $A[Qx.\ H]$ is rectified, because then, in case of violation of Side-Condition 2, the variable z cannot occur in $A[\ldots]$, and thus choosing another name for z in t can circumvent the renaming of bound occurrences of z in H — without destroying the property of being a (completed) formal proof later on, provided that this other name is chosen instead of z consistently.

3.2 δ-Quantification

HEIJENOORT writes:

> "To pass from $F(\ Qy.\ H\ /\ H\ )$, where $Qy.$ is an accessible δ-quantifier of F, H is the scope of $Qy.$, and y does not occur free in F, to F."
> [HEIJENOORT, 1975, p. 4. See our Note 6]

Let $A[\ldots]$ be the context such that HEIJENOORT's F is $A[Qy.\ H]$. Then we get the following formulation.

Generalized rule of δ^--quantification: $\dfrac{A[H]}{A[Qy.\ H]}$ where

1. $Qy.$ is an accessible δ-quantifier of $A[Qy.\ H]$, and
2. the variable y must[9] not occur free in the context $A[\ldots]$.

For soundness reasons, free-variable calculi (i.e. calculi with free γ-variables) need a different rule of δ-quantification. We take here a generalized version of the so-called δ^{+^+}-rule [BECKERT &AL., 1993]. This δ^{+^+}-rule is a *liberalized* δ-rule in the sense that, compared to the simple δ-rule (also called δ^--rule), it admits additional proofs that are more easily found by man and machine. (See § 5.4, and also [WIRTH, 2004, § 2.1.5], [WIRTH, 2006].)

Generalized rule of δ^{+^+}-quantification:

$$\frac{A[H\{y \mapsto (Qy.\ H, x_1^\gamma, \ldots, x_m^\gamma)^\delta(x_1^\gamma, \ldots, x_m^\gamma)\}]}{A[Qy.\ H]} \quad \text{where}$$

1. $Qy.$ is an accessible δ-quantifier of $A[Qy.\ H]$,
2. $x_1^\gamma, \ldots, x_m^\gamma$ are exactly the m free γ-variables occurring in $Qy.\ H$, and
3. the δ in $(Qy.\ H, x_1^\gamma, \ldots, x_m^\gamma)^\delta$ denotes the application of the function δ to $(Qy.\ H, x_1^\gamma, \ldots, x_m^\gamma)$, resulting in a function symbol of arity m that is not part of the original signature, such that, in case of $(B, x_1^\gamma, \ldots, x_m^\gamma)^\delta = (C, z_1^\gamma, \ldots, z_{m'}^\gamma)^\delta$, either $(C, z_1^\gamma, \ldots, z_{m'}^\gamma)$ is $(Bx_1^\gamma, \ldots, x_m^\gamma)$, or C results from B by renaming of bound variables and by a bijective renaming of the free γ-variables $x_1^\gamma, \ldots, x_m^\gamma$ via $\{x_1^\gamma \mapsto z_1^\gamma,\ \ldots,\ x_m^\gamma \mapsto z_{m'}^\gamma\}$.

In case of $m = 0$, i.e. in the absence of free γ-variables, there is no essential difference between the generalized rules of δ^-- and δ^{+^+}-quantification, except that the δ^{+^+}-rule may violate Side-Condition 2 of the generalized rule of δ^--quantification (when reductively applied a second time to the same formula).

[9] If y occurs in the context $A[\ldots]$, an implicit renaming of the bound occurrences of y in $Qy.\ H$ is admitted to enable backward (i.e. *reductive*) application of the inference rule. Note that this is never required for reductive proof search if the formula $A[Qy.\ H]$ is rectified, because then y cannot occur in $A[\ldots]$.

3.3 Simplification

HEIJENOORT writes:

> "To pass from $F\big(\ H\ /\ (H \vee H')\ \big)$, where H is a subformula of F and H' is a variant of H, to F." [HEIJENOORT, 1975, p. 4]

To satisfy the needs of the proof construction in HERBRAND's Fundamental Theorem, we have to correct HEIJENOORT's version a bit: On the one hand, we remove HEIJENOORT's admission of $H \vee H'$ at *negative positions* (i.e. in the scope of an odd number of negation symbols), and, on the other hand, we admit also $H \wedge H'$ at negative positions.[10]

Let $A[\ldots]$ be the context such that HEIJENOORT's F is $A[H]$. Then we get the following formulation.

Generalized rule of simplification: $\dfrac{A[H \circ H']}{A[H]}$ where

1. "$\circ$" stands for "$\vee$" if $[\ldots]$ denotes a positive position in $A[\ldots]$, and for "$\wedge$" if this position is negative,[11] and
2. H' is a variant of the sub-formula H (i.e., H' is H or can be obtained from H by the renaming of variables bound in H).

Moreover, the *generalized rule of γ-simplification* is the sub-rule for the case that H is of the form $Qy.\ C$ and $Qy.$ is a γ-quantifier of $A[Qy.\ C]$.

As the only essential function of the generalized rule of simplification is to increase γ-multiplicity, the generalized rule of γ-simplification is sufficient for completeness and for the proof of HERBRAND's Fundamental Theorem.

[10] See the formal proof in § 5, where we need the operator "$\wedge$" twice at negative positions.

[11] In the terminology of SMULLYAN's classification, this means that $\circ$ is always an α-operator.

4 Property C

4.1 Prerequisites for Property C

Definition 4.1 (Height of a Term, Champ Fini $\mathcal{T}_n(F)$)

We use $|t|$ to denote the *height* of a term t, which is given by

$$|f(t_1, \ldots, t_m)| \quad = \quad 1 + \max\{0, |t_1|, \ldots, |t_m|\}.$$

For a positive natural number n and a formula F, as a finite substitute for a typically infinite, full term universe, HERBRAND uses what he calls a *champ fini of order* n, which we will denote with $\mathcal{T}_n(F)$. The terms of $\mathcal{T}_n(F)$ are constructed from the symbols that occur free in F: the function symbols, the constant symbols (which we will tacitly subsume under the function symbols in what follows), and the free variable symbols (which can be seen as constant symbols here). Such a champ fini differs from a full term universe in containing only the terms t with $|t| \prec n$.

So we have $\mathcal{T}_1(F) = \emptyset$.

To guarantee $\mathcal{T}_n(F) \neq \emptyset$ for $n \succ 1$, in case that neither constants nor free variable symbols occur in F, we will assume that a fresh constant symbol "$\bullet$" (which does not occur elsewhere) is included in the term construction in addition to the free symbols of F. □

HERBRAND's definition of an expansion follows the traditional idea that — for a *finite* domain — universal (existential) quantification can be seen as a finite conjunction (disjunction) over the elements of the domain. To reduce the size of formal proofs according to HERBRAND's Fundamental Theorem, we will consider sub-expansions in addition.

Definition 4.2 ([Sub-] Expansion)

Let $\mathcal{T}$ be a finite set of terms. To simplify substitution, let A be a rectified formula whose bound variables do not occur in $\mathcal{T}$. A formula B is a *sub-expansion of* A *w.r.t.* $\mathcal{T}$ if A, B, $\mathcal{T}$ satisfy the recursive definition of the following table, where "s-e" abbreviates "is a sub-expansion":

form of A	form of B	required properties
quantifier free	A	—
$\neg A'$	$\neg B'$	B' s-e of A' w.r.t. $\mathcal{T}$
$A'_1 \vee A'_2$	$B'_1 \vee B'_2$	B'_i s-e of A'_i w.r.t. $\mathcal{T}$, for $i \in \{1, 2\}$
$A'_1 \wedge A'_2$	$B'_1 \wedge B'_2$	B'_i s-e of A'_i w.r.t. $\mathcal{T}$, for $i \in \{1, 2\}$
$\exists x.\ A'$	$\bigvee_{t \in \mathcal{T}'} B'\{x \mapsto t\}$	B' s-e of A' w.r.t. $\mathcal{T}$, and $\emptyset \neq \mathcal{T}' \subseteq \mathcal{T}$
$\forall x.\ A'$	$\bigwedge_{t \in \mathcal{T}'} B'\{x \mapsto t\}$	B' s-e of A' w.r.t. $\mathcal{T}$, and $\emptyset \neq \mathcal{T}' \subseteq \mathcal{T}$

If we restrict $\mathcal{T}'$ in this table to be equal to $\mathcal{T}$, then we get an *expansion* B *of* A (instead of a sub-expansion B of A) w.r.t. $\mathcal{T}$, and write $A^{\mathcal{T}}$ for B. □

Definition 4.3 (SKOLEM**ized Forms**)

Let B be a rectified formula.

The δ^--SKOLEM*ized form of a formula* B (also called: "outer SKOLEMized form") results from B by removing every δ-quantifier and replacing its bound variable y with $y^\delta(x_1, \ldots, x_m)$, where y^δ is a fresh ("SKOLEM") symbol and $x_1, \ldots, x_m$, in this order, are exactly the m variables of the m γ-quantifiers in whose scope the δ-quantifier occurs.

The δ^{+^+}-SKOLEM*ized form of* B results from B by repeating the following procedure until all δ-quantifiers have been removed: Let $Qy.$ be an outermost δ-quantifier in $A[Qy.\ H] = B$. Replace B with $A[H\{y \mapsto (Qy.\ H, x_1, \ldots, x_m)^\delta(x_1, \ldots, x_m)\}]$, where $x_1, \ldots, x_m$ are exactly the m variables bound by the m γ-quantifiers in whose scope the δ-quantifier $Qx.$ occurs and which *do actually occur in the scope* H *of the* δ*-quantifier*. Moreover, the δ in "$(Qy.\ H, x_1, \ldots, x_m)^\delta$" denotes the application of the function δ to $(Qy.\ H, x_1, \ldots, x_m)$. The result of this application must be a function symbol of arity m that is not part of the original signature, such that, in case of $(B, x_1, \ldots, x_m)^\delta = (C, x'_1, \ldots, x'_{m'})^\delta$, either $(C, x'_1, \ldots, x'_{m'})$ is $(B, x_1, \ldots, x_m)$, or C results from B by renaming of bound variables and by a bijective renaming of the free variables $x_1, \ldots, x_m$ via $\{x_1 \mapsto x'_1, \ \ldots, \ x_m \mapsto x'_{m'}\}$. □

Remark 4.4 (SKOLEM**ized Forms**)

HERBRAND has no name for the δ^--SKOLEMized form and he does not use the δ^{+^+}-SKOLEMized form, which is the current standard in two-valued first-order logic because it is closely related to the liberalized δ-rules.

We cannot get along without the δ^{+^+}-SKOLEMized form here because there is no reasonable version of the generalized rule of δ^{+^+}-quantification that is compatible with the δ^--SKOLEMized form.

Definition 4.5 (Sentential Tautology)

A first-order formula is a *sentential tautology* if it is quantifier-free and truth-functionally valid, provided its atomic subformulas are read as atomic sentential variables. □

4.2 Property C and Property C*

Definition 4.6 (Property C, Property C*)

Let A be a rectified first-order formula. Let n be a positive natural number.

A has Property C of order n if the expansion $F^{\mathcal{T}_n(F)}$ is a sentential tautology, where F is the δ^--SKOLEMized form of A.

A has Property C of order n* if the expansion $F^{\mathcal{T}_n(F)}$ is a sentential tautology, where F is the δ^{+^+}-SKOLEMized form of A. □

Note that $F^{\mathcal{T}_1(F)}$ is defined if and only if F does not contain any quantifier, because we have $\mathcal{T}_1(F) = \emptyset$, and because an empty conjunction or disjunction for the expansion of a quantifier is not admitted in Definition 4.2. Thus, we get the following corollary, which — to avoid the usage of an undefined term — may also serve as a cleaner definition for Property C and Property C* of order 1.

Corollary 4.7

The following three are logically equivalent for a rectified formula A :

1. *A has Property C of order* 1.
2. *A has Property C* of order* 1.
3. *The (no matter whether δ^-- or δ^{+^+}-)* SKOLEM*ized form of A is a sentential tautology.* □

Because all quantifiers in a SKOLEMized form F are γ-quantifiers, existential quantifiers occur only at positive positions (and are expanded with disjunctions) and universal quantifiers occur only at negative positions (and are expanded with conjunctions). Thus, we get the following corollaries:

Corollary 4.8

If any sub-expansion of a SKOLEM*ized form F w.r.t. $\mathcal{T}$ is a sentential tautology, then $F^{\mathcal{T}}$ is a sentential tautology as well.* □

Corollary 4.9

*Let A be a rectified first-order formula. Let n be a positive natural number. If any sub-expansion of the δ^--*SKOLEM*ized (or else: δ^{+^+}-*SKOLEM*ized) form F of A w.r.t. $\mathcal{T}_n(F)$ is a sentential tautology, then A has Property C (or else: Property C*) of order n.* □

By using the idea of Corollary 4.8 for the forward direction and by replacing all terms whose top function symbols are new with the same old constant or free variable for the backward direction, we get:

Corollary 4.10

Let A be a rectified first-order formula. Let n be a positive natural number.
*Let F be the δ^--SKOLEM*ized (or else: δ^{+^+}-SKOLEM*ized) form of A.*
Let $\mathcal{T}'$ be formed just as $\mathcal{T}_n(F)$, but over an extended set of function symbols.

Now the following two are logically equivalent:

1. *A has Property C (or else: Property C*) of order n.*
2. *$F^{\mathcal{T}'}$ is a sentential tautology.* □

As the replacement of δ^--SKOLEM terms with δ^{+^+}-SKOLEM terms, dropping the arguments at the missing argument positions, cannot increase the range of values of a sentential expression under all valuations (of its sentential variables named by atomic formulas), we get:

Corollary 4.11 *Let A be a rectified formula. Let n be a positive natural number.*
If A has Property C of order n, then A has Property C* of order n as well. □

Note that the converse of Corollary 4.11 does not hold in general:

Example 4.12
Let us take A to be the rectified formula $\exists a.\ \big(\ (\neg\exists b.\ p(b)) \vee p(a)\ \big)$.
Its δ^{+^+}-SKOLEMized form is $\exists a.\ \big(\ \neg p(b^\delta) \vee p(a)\ \big)$. Now, expansion w.r.t. $\{b^\delta\}$ results in a sentential tautology. Thus, A has Property C* of order 2.
Its δ^--SKOLEMized form is $\exists a.\ \big(\ \neg p(b^\delta(a)) \vee p(a)\ \big)$. Here expansion w.r.t. $\{\bullet\}$ does not result in a sentential tautology. Thus, A does not have Property C of order 2, but only of order 3. □

5 From Property C to a Linear Derivation

In this section, instead of proving our theorems, we will exemplify them by a generalizable example.

In this example, we add some sugar to our formula syntax. Let us write

$$\text{“}A \Rightarrow B\text{”}$$

instead of

$$\text{“}\neg A \vee B\text{”},$$

and let “$\Rightarrow$” have lower operator precedence than “$\wedge$” and “$\vee$”.

The following formula says that if we have an upper bound of every two elements as well as transitivity, then we also have an upper bound of every three elements.

$$\begin{array}{rl} & \forall x.\ \forall y.\ \exists m.\ (x \prec m \wedge y \prec m) \\ \wedge & \forall c.\ \forall b.\ \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \Rightarrow & \forall u.\ \forall v.\ \forall w.\ \exists z.\ (u \prec z \wedge v \prec z \wedge w \prec z) \end{array} \tag{A}$$

The (δ^-- as well as δ^{+^+}-) SKOLEMized form of A is

$$\begin{array}{rl} & \forall x.\ \forall y.\ (x \prec m^\delta(x,y) \wedge y \prec m^\delta(x,y)) \\ \wedge & \forall c.\ \forall b.\ \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \tag{F}$$

5.1 Informal Proof

Let us look for an informal proof of the last line of F, assuming the first two lines to be given as lemmas. From semantical considerations, it is obvious that a solution for z is given by the substitution

$$\sigma_3 \quad := \quad \{z \mapsto m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))\}.$$

We can prove this as follows: First we apply the first line twice instantiated by the two substitutions

$$\begin{array}{lll} \sigma_{1,1} & := & \{x \mapsto v^\delta,\ \ y \mapsto w^\delta\}, \\ \sigma_{1,2} & := & \{x \mapsto u^\delta,\ \ y \mapsto m^\delta(v^\delta, w^\delta)\} \end{array}$$

to obtain all of

$$\begin{array}{l} v^\delta \prec m^\delta(v^\delta, w^\delta),\ \ w^\delta \prec m^\delta(v^\delta, w^\delta), \\ u^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)),\ \ m^\delta(v^\delta, w^\delta) \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)). \end{array}$$

Now we can apply the second line twice instantiated by the two substitutions

$$\left.\begin{array}{lll} \sigma_{2,1} & := & \sigma_2 \cup \{a \mapsto v^\delta\}, \\ \sigma_{2,2} & := & \sigma_2 \cup \{a \mapsto w^\delta\} \end{array}\right\} \quad \text{for} \quad \sigma_2 \ := \ \{c \mapsto m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)),\ \ b \mapsto m^\delta(v^\delta, w^\delta)\}$$

to obtain also

$$v^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)),\ \ w^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)),$$

which completes our informal proof the last line of formula F.

As the maximum height of the instantiations in this informal proof is 3, it is obvious that the formula A has Property C of order 4.

5.2 Human-Oriented Formal Proof Construction in HEIJENOORT's Version of HERBRAND's *Modus Ponens*-Free Calculus

Let us use this informal proof to construct a formal proof of the formula A in HEIJENOORT's version of HERBRAND's *modus ponens*-free calculus, which we presented in § 3. More precisely, our goal is to reduce the formula A to a sub-expansion of F w.r.t. $\mathcal{T}_n(F)$ (for $n = 4$) that is a sentential tautology; our means will be the renaming of bound variables and the generalized rules of γ-simplification and δ^-- and γ-quantification.

In any of the following reductive backward steps, the parts of a formula that will be removed will be shown in red and those parts that are crucially involved (but not removed) are shown in orange. The parts changed w.r.t. the *previous* formula will be shown in green if they are crucial for triggering the previous step, and in blue otherwise.

5.2.1 Phase 1 of the Reduction: Rename Bound δ-Variables

First we use an amazing trick invented by HERBRAND: We use SKOLEM terms as names for bound δ-variables and for the free variables that result from backward application of the generalized rule of δ^--quantification.

By this trick, HERBRAND escapes the problems of giving SKOLEM functions a semantics and all involved problems, such as the need for a principle of choice. These problems are present in [LÖWENHEIM, 1915] and were overcome in [SKOLEM, 1920; 1923] only by using SKOLEM *normal form* instead of SKOLEMized form. SKOLEMized form, however, is more intuitive and more useful in automated theorem proving than SKOLEM normal form, and SKOLEM returned to SKOLEMized form in [SKOLEM, 1928].

According to HERBRAND's trick, every term that has a SKOLEM function as top symbol is seen as a variable, such that $m^\delta(v^\delta, w^\delta)$ is *not* a subterm of $m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))$, because they are both just variables. This interpretation does not affect the formula A, simply because A does not contain any SKOLEM function symbols.

Thus, we start by renaming each bound δ-variable in A to its SKOLEM term in the δ^--SKOLEMized form F of A, and obtain:

$$\begin{array}{rl} & \forall x.\ \forall y.\ \exists m^\delta(x,y).\ (x \prec m^\delta(x,y) \wedge y \prec m^\delta(x,y)) \\ \wedge & \forall c.\ \forall b.\ \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \Rightarrow & \forall u^\delta.\ \forall v^\delta.\ \forall w^\delta.\ \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (S)$$

Here, in the first line of S, we must not read the occurrences of the variables x and y as arguments of m^δ as variables bound by the outer universal quantifier, but as a part of the name "$m^\delta(x, y)$" of the variable bound by the existential quantifier.

5.2.2 Phase 2 of the Reduction: Generalized Rule of γ-Simplification

We know from our informal proof that we need the first and the second line twice. Thus, we now double them at $\forall x.$ and $\forall a.$, respectively, by two backward applications of the generalized rule of γ-simplification, and obtain:

$$\begin{array}{ll} & \forall x.\ \forall y.\ \exists m^\delta(x,y).\ (x \prec m^\delta(x,y) \land y \prec m^\delta(x,y)) \\ \land & \forall x_1.\ \forall y_1.\ \exists m^\delta(x_1,y_1).\ (x_1 \prec m^\delta(x_1,y_1) \land y_1 \prec m^\delta(x_1,y_1)) \\ \land & \forall c.\ \forall b.\ \left(\begin{array}{ll} & \forall a.\ (a \prec b \land b \prec c \Rightarrow a \prec c) \\ \land & \forall a_1.\ (a_1 \prec b \land b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \forall u^\delta.\ \forall v^\delta.\ \forall w^\delta.\ \exists z.\ (u^\delta \prec z \land v^\delta \prec z \land w^\delta \prec z) \end{array} \qquad (R)$$

Note that — to keep our formula rectified — we have added the index 1 to the copies and also renamed the bound δ-variable $m^\delta(x,y)$ consistently.

5.2.3 Phase 3 of the Reduction: Generalized Rules of Quantification

Now we have to apply the generalized rules of δ^-- and γ-quantification backward until no quantifiers remain. According to the side-condition of the generalized rule of δ^--quantification, we have to guarantee that every δ-quantifier is removed before its variable has been used for the instantiation of γ-variables. According to the side-condition of the generalized rule of γ-quantification, we have to guarantee that every term substituted for a γ-variable does not contain any variables bound in the scope of its γ-quantifier. To satisfy these side-conditions, we have to remove the accessible quantifiers in the order given by the following procedure.

Procedure 5.1 (Restricting the Order of the Applications of Generalized Rules of γ- and δ^--Quantification for Reduction to a Sub-Expansion)
Do the following for every natural number i, stepping (by 1) from 1 to n:

Keep removing

- the accessible γ-quantifiers whose γ-variable is to be instantiated with a term t with $|t| \prec i$ (where variables whose names are SKOLEM terms are considered to have the height of the SKOLEM terms) as well as
- the accessible δ-quantifiers,

until no such quantifiers remain. □

Remark 5.2 (Historical Correctness)
For restricting the order of the applications, HERBRAND considered only the case of a reduction to an expansion, not to a proper sub-expansion. The order he described in the first part of the proof of his Fundamental Theorem [HERBRAND, 1930, § 5.5.1] is a complete one, which — for the case of a full expansion — is one of the orders our Procedure 5.1 admits. If we applied HERBRAND's description directly to sub-expansions, however, there would be no guarantee that the side-conditions of the applications of the generalized rule of δ^--quantification are satisfied. Nevertheless, Procedure 5.1 just captures the weakest condition to satisfy all side-conditions for the reason why HERBRAND's procedure satisfies them; cf. § 5.2.4. □

Let us see what our Procedure 5.1 means in case of our example.

For $i=1$, there cannot be any γ-quantifiers satisfying the condition, because no term can have height 0. Thus, we apply the generalized rule of δ^--quantification thrice backward to the last line, and obtain:

$$\begin{array}{ll} & \forall x.\ \forall y.\ \exists m^\delta(x,y).\ (x \prec m^\delta(x,y) \wedge y \prec m^\delta(x,y)) \\ \wedge & \forall x_1.\ \forall y_1.\ \exists m^\delta(x_1,y_1).\ (x_1 \prec m^\delta(x_1,y_1) \wedge y_1 \prec m^\delta(x_1,y_1)) \\ \wedge & \forall c.\ \forall b.\ \left(\begin{array}{ll} & \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \wedge & \forall a_1.\ (a_1 \prec b \wedge b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (Q)$$

Note that the introduced free variables all have height i.

Now, because no accessible δ-quantifiers remain, we increment i by 1.

For $i=2$, the only accessible γ-quantifiers that are to be instantiated with terms of height smaller than 2 are x and y in the first line (according to the substitution $\sigma_{1,1}$), as well as x_1 of the second line (according to the substitution $\sigma_{1,2}$).

Therefore, we apply $\sigma_{1,1}$ to all occurrences of x and y in the first line and remove their γ-quantifiers. Formally this means that we rename the bound δ-variable $m^\delta(x,y)$ to $m^\delta(v^\delta,w^\delta)$, and then we apply the generalized rule of γ-quantification twice (backward). Doing the same to x_1 according to the instantiation of the original x in $\sigma_{1,2}$, we then obtain the following formula:

$$\begin{array}{ll} & \exists m^\delta(v^\delta,w^\delta).\ (v^\delta \prec m^\delta(v^\delta,w^\delta) \wedge w^\delta \prec m^\delta(v^\delta,w^\delta)) \\ \wedge & \forall y_1.\ \exists m^\delta(u^\delta,y_1).\ (u^\delta \prec m^\delta(u^\delta,y_1) \wedge y_1 \prec m^\delta(u^\delta,y_1)) \\ \wedge & \forall c.\ \forall b.\ \left(\begin{array}{ll} & \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \wedge & \forall a_1.\ (a_1 \prec b \wedge b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (P)$$

Now, the only accessible δ-quantifier is "$\exists m^\delta(v^\delta,w^\delta)$." in the first line. Let us remove it by a backward application of the generalized rule of δ^--quantification, and obtain:

$$\begin{array}{ll} & v^\delta \prec m^\delta(v^\delta,w^\delta) \wedge w^\delta \prec m^\delta(v^\delta,w^\delta) \\ \wedge & \forall y_1.\ \exists m^\delta(u^\delta,y_1).\ (u^\delta \prec m^\delta(u^\delta,y_1) \wedge y_1 \prec m^\delta(u^\delta,y_1)) \\ \wedge & \forall c.\ \forall b.\ \left(\begin{array}{ll} & \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \wedge & \forall a_1.\ (a_1 \prec b \wedge b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (O)$$

Note that the variable of the removed δ-quantifier has height i again.

Now, because all accessible γ-quantifiers are to be replaced with terms of height not smaller than 2, we increment i by 1.

For $i=3$, according to Procedure 5.1, we now have to apply the generalized rule of γ-quantification backward to the second line according to the substitution $\sigma_{1,2}$ (for the renamed y_1 instead of the original y) (after the involved renaming of the following δ-variable), because every term in the range of $\sigma_{1,2}$ has a height smaller than 3. We obtain:

$$\begin{array}{ll} & v^\delta \prec m^\delta(v^\delta, w^\delta) \wedge w^\delta \prec m^\delta(v^\delta, w^\delta) \\ \wedge & \exists m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)). \\ & \qquad (u^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \wedge m^\delta(v^\delta, w^\delta) \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))) \\ \wedge & \forall c.\ \forall b. \left(\begin{array}{ll} & \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \wedge & \forall a_1.\ (a_1 \prec b \wedge b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (L)$$

After a subsequent application of the generalized rule of δ^--quantification, we obtain:

$$\begin{array}{ll} & v^\delta \prec m^\delta(v^\delta, w^\delta) \wedge w^\delta \prec m^\delta(v^\delta, w^\delta) \\ \wedge & u^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \wedge m^\delta(v^\delta, w^\delta) \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \\ \wedge & \forall c.\ \forall b. \left(\begin{array}{ll} & \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \wedge & \forall a_1.\ (a_1 \prec b \wedge b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (K)$$

Note that the variable of the removed δ-quantifier has height i again.

Moreover, note that the free variable $m^\delta(v^\delta, w^\delta)$ occurs twice in the first line but only once in second line, because the remaining two occurrences are just part of the name of the free variable $m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))$. Be reminded that every SKOLEM function symbol starts a name of a variable.

Now, as all accessible quantifiers in the formula K are γ-quantifiers whose variables are to be replaced with terms of height not smaller than 3, we increment i by 1 again.

5.2.4 Interlude: Why Side-Conditions are Always Satisfied

Generalized rule of δ^--quantification

Note that the side-condition of the generalized rule of δ^--quantification (i.e. that its bound δ-variable does not occur free outside its scope) is observed in the last step from the formula L back to the formula K.

Let us show that — under the reasonable conditions explained below on the terms introduced by backward applications of the generalized rule of γ-quantification — this is always the case for the order of removal of quantifiers given by Procedure 5.1:

If a bound δ-variable is turned into a free variable by the removal of its quantifier by reductive application of the generalized rule of δ^--quantification according to Procedure 5.1 during the removal of quantifiers for a given i, then its height (seen as a term) must be i.

Indeed, for $i=1$, the SKOLEM function symbol of the variable name whose δ-quantifier is removed must be a constant, simply because it cannot occur in the scope of γ-quantifiers. Moreover, for $i \succ 1$, its SKOLEM function symbol must have at least one argument of height $i-1$, simply because otherwise its δ-quantifier would have been removed already for a smaller i.

Here it is crucial that we took the δ^--SKOLEMized form instead of the δ^{++}-SKOLEMized form: All γ-variables in whose scope the δ-variable occurs must be arguments of the SKOLEM term that names the δ-variable.

Now it is immediate that the δ-variable cannot have been introduced as a free variable by a previous backward application of a generalized rule of γ-quantification, simply because all such free variables have a height strictly smaller than i.

On the other hand, if the δ-variable was introduced as a free variable by a previous backward application of a generalized rule of δ^--quantification, then — because it must have the identical SKOLEM symbol — this previous application must have occurred in a parallel branch resulting from our initial backward applications of the generalized rule of γ-simplification. If, however, we never instantiate the γ-variables of the top quantifiers of different copies introduced by backward applications of the generalized rule of γ-simplification with identical instances (and there would be no benefit of this), then we know that the SKOLEM terms that name the bound δ-variables have different arguments.

Here — for different instances of the γ-variables to result in different SKOLEM terms — it is again crucial to have the δ^-- instead of the δ^{++}-SKOLEMized form.

Therefore, the side-condition of the generalized rule of δ^--quantification (i.e. that its bound δ-variable does not occur free outside its scope) is always observed under reasonable conditions on the terms introduced by backward applications of the generalized rule of γ-quantification.

Generalized rule of γ-quantification

In a similar way, we can see that also the side-condition of the generalized rule of γ-quantification — namely that none of the free variables occurring in the term t substituted for its bound γ-variable is bound by quantifiers in the scope of the γ-quantification — is always satisfied under reasonable conditions:

If we do not admit a variable that does not start with a SKOLEM function symbol in t unless it occurs free in A (and there would be no benefit of this), then the γ-quantifications in the scope of the original γ-quantification cannot bind such a variable, simply because A is assumed to be rectified.

On the other hand, all δ-quantified variables in the scope of the original γ-quantification contain t as a subterm of their names and thus have a bigger height than all free variables occurring in t.

Here again it is crucial to have the δ^-- instead of the δ^{++}-SKOLEMized form.

After these general considerations on the side-conditions of the generalized rules of quantification, let us return to our example proof.

5.2.5 Continuation of Phase 3

For $i=4$ (i.e. for $i=n$), we remove the five γ-quantifiers remaining in the formula K by the generalized rule of γ-quantification, using $\sigma_{2,1}$ for the third line, $\sigma_{2,2}$ for the fourth line, and σ_3 for the last line, and obtain the following formula:

$$
\begin{array}{ll}
 & v^\delta \prec m^\delta(v^\delta, w^\delta) \wedge w^\delta \prec m^\delta(v^\delta, w^\delta) \\
\wedge & u^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \wedge m^\delta(v^\delta, w^\delta) \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \\
\wedge & \left(\begin{array}{ll} & v^\delta \prec m^\delta(v^\delta, w^\delta) \wedge m^\delta(v^\delta, w^\delta) \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \\ \Rightarrow & v^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \end{array} \right) \\
\wedge & \left(\begin{array}{ll} & w^\delta \prec m^\delta(v^\delta, w^\delta) \wedge m^\delta(v^\delta, w^\delta) \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \\ \Rightarrow & w^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \end{array} \right) \\
\Rightarrow & \left(\begin{array}{ll} & u^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \\ \wedge & v^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \\ \wedge & w^\delta \prec m^\delta(u^\delta, m^\delta(v^\delta, w^\delta)) \end{array} \right)
\end{array}
\qquad (J)
$$

Now, considering all atomic formulas of J as names of sentential variables, we easily see that the formula J is a sentential tautology. Moreover, J is a sub-expansion of the formula F w.r.t. $\mathcal{T}_4(F)$, the champ fini of order 4, formed over the function and free variable symbols of F, i.e. over u^δ, v^δ, w^δ, and $m^\delta(_, _)$. Thus, by Corollaries 4.8 and 4.9, the expansion $F^{\mathcal{T}_4(F)}$ is a sentential tautology as well, and so A has Property C of order 4.

5.2.6 Stepwise *Deductive* Construction of the Same Proof?

Suppose we have found a sentential tautology that is a sub-expansion of the δ^--Skolemized form F of A. Can we now construct a proof of A deductively, i.e. stepwise from such a sentential tautology *forward* to obtain formula A in the end? Of course, the reductive construction shows that there must be one such formula where this is possible, but even if we found a formula where this is actually possible (such as the formula J), it still remains much more difficult to find the steps deductively than reductively, for the following reasons:

1. To find out, which of the variables named by Skolem terms are to be turned into γ- and which in to δ-quantifiers, we have to check whether they were introduced by instantiation of F (such as the occurrences of $m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))$ in green in J), or whether they resulted from a further instantiation of a Skolem term already present in F (such as the occurrences of $m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))$ in black in the 2nd line of J). The former have to be replaced with γ-quantifiers first, so that the side-condition of the generalized rule of δ^--quantification becomes satisfied for then turning the latter into δ-quantifiers.
2. Before we can start with the introduction of γ-quantifiers for the instantiated terms with maximal depth (such as the occurrences of $m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))$ in green in the 3rd and 4th lines of J, i.e. the quantifier $\forall c.$ in formula K), we may first have to introduce the γ-quantifiers that occur more innermost (such as the γ-quantifiers for the terms printed in blue in the formula J, i.e. the quantifiers $\forall a.$, $\forall a_1.$, $\forall b.$ in formula K).

5.3 Mechanical Proof Construction in HEIJENOORT's Version of HERBRAND's *Modus Ponens*-Free Calculus

Knowing that A has Property C of order 4, we can construct a proof of A in our correction of HEIJENOORT's version of HERBRAND's *modus ponens*-free calculus also *mechanically*.

Let N denote the cardinality of $\mathcal{T}_4(F)$. Let $\mathcal{T}_4(F) = \{t_0, \ldots, t_{N-1}\}$. We have $N = 3 + 3^2 + (3 + 3^2)^2 = 156$. Therefore, the number of leaf formulas found in the expansion $F^{\mathcal{T}_4(F)}$ of the form of the three original ones of F, summed over the three original lines, is $N^2 + N^3 + N = 3820908$.

We can now reduce A mechanically to the huge[12] sentential tautology $F^{\mathcal{T}_4(F)}$ as follows: In phase 1, we rename every bound δ-variable to its SKOLEM term of the δ^--SKOLEMized form of F. In phase 2, for every γ-quantifier, we apply the generalized rule of γ-simplification $N-1$ times (backward) and associate the terms $t_0, \ldots, t_{N-1}$ to the resulting N γ-quantifiers. Finally, in phase 3, we apply Procedure 5.1 already described in § 5.2 for the reductive (i.e. backward) application of the generalized rules of δ^-- and γ-quantification, including the renaming of the bound δ-variables. When all quantifiers are removed, we have obtained $F^{\mathcal{T}_4(F)}$, and the formal proof is done.

Moreover, because the renaming of the bound δ-variables is actually determined by the path on which they occur in the formula, the final renaming of bound variables can be done right after the backward applications of the generalized rule of γ-simplification. Because our version of the generalized rule of γ-simplification admits variants of H, we can therefore restrict the renaming bound variables to the very beginning of a reductive proof of A.

This result can be used to give a constructive proof of the following lemma.

Lemma 5.3 (From Property C to a Linear Derivation)

Let A be a rectified first-order formula. Let F be the δ^--SKOLEMized form of A. Let n be a positive natural number.

If A has Property C of order n, then we can construct a derivation of A of the following form, in which we read any term starting with a SKOLEM function as an atomic variable:

Step 1: *We start with the sentential tautology $F^{\mathcal{T}_n(F)}$ (or else with some sub-expansion of F w.r.t. $\mathcal{T}_n(F)$ that is a sentential tautology).*

Step 2: *Then we may repeatedly apply the generalized rules of γ- and δ^--quantification.*

Step 3: *Then we may repeatedly apply the generalized rule of γ-simplification.*

Step 4: *Then we rename all bound δ-variables to obtain A.* □

[12] Note, however, that the HERBRAND *disjunction* used for A in [WIRTH &AL., 2009; 2014] (instead of the expansion of the δ^--SKOLEMized form) is even more than a factor of 10^7 bigger.

5.4 Formal Proof in Our Free-Variable Calculus

In this section we do the proof of § 5.2 again, but in our free-variable calculus. So we replace the generalized rule of δ^--quantification with our generalized rule of δ^{+^+}-quantification.[13] The advantage is not only the possible delay in the choice of witnessing terms,[14] but also that Procedure 5.1 of § 5.2 for restricting the order of applications of the generalized rules of quantification becomes superfluous and any order will do. Another difference is that now all SKOLEM functions are function symbols of the first-order calculus and do not start the name of a variable anymore. By application of the generalized rule of γ-simplification just as before twice, we reduce A to the following formula:

$$\begin{array}{ll} & \forall x.\ \forall y.\ \exists m.\ (x \prec m \wedge y \prec m) \\ \wedge & \forall x_1.\ \forall y_1.\ \exists m_1.\ (x_1 \prec m_1 \wedge y_1 \prec m_1) \\ \wedge & \forall c.\ \forall b.\ \left(\begin{array}{ll} & \forall a.\ (a \prec b \wedge b \prec c \Rightarrow a \prec c) \\ \wedge & \forall a_1.\ (a_1 \prec b \wedge b \prec c \Rightarrow a_1 \prec c) \end{array} \right) \\ \Rightarrow & \forall u.\ \forall v.\ \forall w.\ \exists z.\ (u \prec z \wedge v \prec z \wedge w \prec z) \end{array} \qquad (R')$$

Note that R' is similar to the formula R of § 5.2, but lacks the renaming of bound δ-variables to SKOLEM terms. As the order of applications of generalized rules of quantification does not matter anymore, let us remove all consecutively accessible quantifiers of the same kind and obtain:

$$\begin{array}{ll} & \exists m.\ (x_0^\gamma \prec m \wedge y_0^\gamma \prec m) \\ \wedge & \exists m_1.\ (x_1^\gamma \prec m_1 \wedge y_1^\gamma \prec m_1) \\ \wedge & (a_0^\gamma \prec b^\gamma \wedge b^\gamma \prec c^\gamma \Rightarrow a_0^\gamma \prec c^\gamma) \\ \wedge & (a_1^\gamma \prec b^\gamma \wedge b^\gamma \prec c^\gamma \Rightarrow a_1^\gamma \prec c^\gamma) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (Q')$$

If we denote[15] $\left(\exists m.\ (x_0^\gamma \prec m \wedge y_0^\gamma \prec m),\ x_0^\gamma,\ y_0^\gamma\right)^\delta$ with m^δ, then by application of the generalized rule of δ^{+^+}-quantification backward to the first line we obtain:

$$\begin{array}{ll} & x_0^\gamma \prec m^\delta(x_0^\gamma, y_0^\gamma) \wedge y_0^\gamma \prec m^\delta(x_0^\gamma, y_0^\gamma) \\ \wedge & \exists m_1.\ (x_1^\gamma \prec m_1 \wedge y_1^\gamma \prec m_1) \\ \wedge & (a_0^\gamma \prec b^\gamma \wedge b^\gamma \prec c^\gamma \Rightarrow a_0^\gamma \prec c^\gamma) \\ \wedge & (a_1^\gamma \prec b^\gamma \wedge b^\gamma \prec c^\gamma \Rightarrow a_1^\gamma \prec c^\gamma) \\ \Rightarrow & \exists z.\ (u^\delta \prec z \wedge v^\delta \prec z \wedge w^\delta \prec z) \end{array} \qquad (L')$$

Because $\exists m_1.\ (x_1^\gamma \prec m_1 \wedge y_1^\gamma \prec m_1)$ results from $\exists m.\ (x_0^\gamma \prec m \wedge y_0^\gamma \prec m)$ by renaming of bound variables and by a bijective renaming of free γ-variables via $\{x_0^\gamma \mapsto x_1^\gamma,\ y_0^\gamma \mapsto y_1^\gamma\}$, we can use the same function symbol m^δ again for the second line. If we also remove the γ-quantifier in the last line, we obtain:

$$\begin{array}{ll} & x_1^\gamma \prec m^\delta(x_1^\gamma, y_1^\gamma) \wedge y_1^\gamma \prec m^\delta(x_1^\gamma, y_1^\gamma) \\ \wedge & x_2^\gamma \prec m^\delta(x_2^\gamma, y_2^\gamma) \wedge y_2^\gamma \prec m^\delta(x_2^\gamma, y_2^\gamma) \\ \wedge & (a_1^\gamma \prec b_1^\gamma \wedge b_1^\gamma \prec c_1^\gamma \Rightarrow a_1^\gamma \prec c_1^\gamma) \\ \wedge & (a_2^\gamma \prec b_2^\gamma \wedge b_2^\gamma \prec c_2^\gamma \Rightarrow a_2^\gamma \prec c_2^\gamma) \\ \Rightarrow & u^\delta \prec z^\gamma \wedge v^\delta \prec z^\gamma \wedge w^\delta \prec z^\gamma \end{array} \qquad (J')$$

Finally, the sentential tautology J of § 5.2.5 results from J' by an appropriate instantiation of free γ-variables.

This example proof admits generalization to the following lemma.

Lemma 5.4 (From Property C* to a Linear Derivation)

*Let A be a rectified first-order formula. Let F be the δ^{+^+}-*SKOLEM*ized form of A. Let n be a positive natural number.*

If A has Property C^* of order n,
then we can construct a derivation of A of the following form:

Step 1: *We start with a quantifier-free formula B with free γ-variables, such that — by replacing each free γ-variable x^γ occurring in B with a term t_{x^γ} (over the free symbols in F and possibly $\bullet$) with $|t_{x^\gamma}| \prec n$ — we obtain the sentential tautology $F^{\mathcal{T}_n(F)}$ (or else some sub-expansion of F w.r.t. $\mathcal{T}_n(F)$ that is a sentential tautology).*

Step 2: *Then we may repeatedly apply the generalized rules of δ^{+^+}-quantification and restricted γ-quantification.*

Step 3: *Then we may repeatedly apply the generalized rule of γ-simplification to obtain A.* □

[13]Cf. § 3.2.

[14]Cf. § 3.1.

[15]For the function δ, see the generalized rule of δ^{+^+}-quantification in § 3.2, and also Definition 4.3 in § 4.1.

6 HERBRAND's Fundamental Theorem

If we now have a look on what our inference rules do with a formula that has Property C of order n, then we see that the results of applications of the generalized rules of simplification and δ^--quantification (and renaming of bound variables) are formulas that have Property C of order n again.[16] Moreover, an application of the generalized rule of γ-quantification where a term t is replaced with a bound γ-variable in a formula that has Property C of order n results in a formula that has Property C of order $n + |t|$.[17] Using Corollary 4.7, we can summarize this as follows:

Lemma 6.1 (From a Linear Derivation to Property C)
If there is a derivation of the rectified first-order formula A from a sentential tautology by applications of the generalized rules of simplification, and of γ- and δ^--quantification (and renaming of bound variables),

then A has Property C of order $1 + \sum_{i=1}^{m} |t_i|$,

where $t_1, \ldots, t_m$ are the instances for the meta-variable t of the generalized rule of γ-quantification in its m applications in the derivation of A. □

[16] Indeed, regarding the generalized rule of δ^--quantification, the SKOLEMization just removes the δ-quantifier and replaces its bound variable with a constant that occurs at exactly the same places where the variable occurred free before application of the generalized rule of δ^--quantification. Moreover, regarding the generalized rule of simplification, the expansion of the δ^--SKOLEMized form of $A[H \circ H']$ can be transformed into $S \vee S'$ where S and S' are the expansions of the δ^--SKOLEMized forms of $A[H]$ and $A[H']$, respectively, over the terms that include the SKOLEM functions of $A[H \circ H']$. Clearly, all or none of S, S', $S \vee S'$ are sentential tautologies, and the rest follows from Corollary 4.10.

[17] Note that the replacement seemingly inverting the effect of the application of the generalized rule of γ-quantification, namely the removal of the new γ-quantifier and the replacement of its bound variable with the term t would result only in an order of $\max\{n, \ |t| + 1\}$ (because the free variables of t remain free also after the substitution). This is not a complete inversion, however, because t may occur in the expansion also as a newly added first argument of SKOLEM functions. Indeed, the arity of all SKOLEM functions for the δ-quantifiers in the scope of the γ-quantifier may be increased; this requires the order $n + |t|$ in general. Finally we have $\max\{n, \ |t| + 1, \ n + |t|\} = n + |t|$.

For examples of this, consider the formulas of § 5.2: The formula L has Property C of order 2,[18] but the formula O — derived from L by applications the generalized rule of γ-quantification with[18] $|t| = 1$ — has Property C of order 3. Similarly, the formula P has Property C of order 3 (it has to have the same order as formula O because it results from it by application of the generalized rule of δ^--quantification), but the formula Q — derived from P by applications the generalized rule of γ-quantification with $|t| = 1$ — has Property C of order 4. In both examples, the order is not increased by inverse substitution of the term t, because then all formulas would have Property C invariantly of order $\max\{n, \ |t| + 1\} = \max\{n, \ 2\} = n$. The order is increased because the depth of the SKOLEM term resulting from the previously introduced δ-quantifier is increased by a newly added first argument.

[18] In the peculiar notation of § 5.2, where variables are denoted by terms with a SKOLEM function as top symbol, this is a bit difficult to see: To show that formula L has Property C of order 2, according to the substitution σ_3, we have to replace the bound δ-variable $m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))$ with the SKOLEM constant $\overline{m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))}$, where the whole overlined term is to be interpreted as the name of the constant. To show that formula O has Property C, we instantiate the bound δ-variable with the same term in principle, but now we have to denote this term in the form of $\overline{m^\delta(u^\delta, m^\delta(v^\delta, w^\delta))}$, where $\overline{m^\delta(u^\delta, _)}$ is the singulary SKOLEM function symbol introduced by the expansion, that was a nullary one before, now with the new argument $m^\delta(v^\delta, w^\delta)$, which is the name of a variable introduced by the generalized rule of γ-quantification. This term has height 2; so formula O has Property C of order 3.

By the similarity of the expansion of the δ^{++}-SKOLEMized form with the instantiation of the result of backward applications of the generalized rules of δ^{++}- and restricted γ-quantification, and by Lemma 4.10, we get:

Lemma 6.2 (From a Linear Derivation to Property C*)
Let n be a positive natural number. Let A be a rectified first-order formula.
If there is a derivation of A

- *starting from a quantifier-free formula B with free γ-variables, such that we get a sentential tautology by replacing each free γ-variable x^γ occurring in B with a term t_{x^γ} with $|t_{x^\gamma}| \prec n$,*
- *proceeding by applications of the generalized rules of simplification and of δ^{++}- and restricted γ-quantification (and renaming of bound variables),*

then A has Property C^* of order n. □

As a corollary of Lemmas 5.3 and 6.1, we can now state HERBRAND's Fundamental Theorem in the form presented by HEIJENOORT in [HEIJENOORT, 1975]:

Theorem 6.3 (HERBRAND's Fundamental Theorem à la HEIJENOORT)

Let A be a rectified first-order formula.

The following two statements are logically equivalent. Moreover, we can construct a witness for each statement from a witness for the other one.

1. *There is a positive natural number n such that A has Property C of order n.*
2. *There is a sentential tautology B, and*
   there is a derivation of A from B that consists in applications of the generalized rules of simplification, γ-quantification, and δ^--quantification
   (and in the renaming of bound variables). □

As a corollary of Lemmas 5.4 and 6.2, we can now state HERBRAND's Fundamental Theorem in an up-to-date form as follows.

Theorem 6.4 (HERBRAND's Fundamental Theorem à la WIRTH)

Let A be a rectified first-order formula. Let n be a positive natural number.

The following two statements are logically equivalent. Moreover, we can always construct witnesses for the second statement if the first one holds.

1. *A has Property C^* of order n.*
2. *There is a quantifier-free formula B with free γ-variables, and*
   there is a substitution σ from these γ-variables to terms t with $|t| \prec n$
   such that $B\sigma$ is a sentential tautology, and
   there is derivation of A from B that consists in applications of the generalized rules of simplification, δ^{++}-quantification, and restricted γ-quantification. □

7 Generalized Rules of Quantification in the Literature

Regarding HERBRAND's *modus ponens*-free calculus, HEIJENOORT's invention is restricted to the *generalized* rules of γ- and δ^--quantification. Let us briefly discuss their chronology.

- In 1960, KURT SCHÜTTE (1909–1998) published the generalized rules of γ- and δ^--quantification under the names "S 3" (with a built-in application of the generalized rule of γ-simplification) and "S 2" in his first monograph on proof theory (in German) [SCHÜTTE, 1960, p. 78]. SCHÜTTE's versions, however, come with an additional restriction regarding the positions where quantifiers may be introduced. SCHÜTTE's top-down definition of positive and negative positions ("Positivteile" and "Negativteile" [SCHÜTTE, 1960, p.11]) covers only α-operators (according to SMULLYAN's classification).[19] SCHÜTTE's rules restrict the introduction of quantifiers to positions that are either positive or negative in this sense.

- In 1968, the analogs of the generalized rules of γ- and δ^--quantification without restrictions for a tableau calculus are introduced in HEIJENOORT's unpublished paper [HEIJENOORT, 1968], under the names "(V′)" and "(IV′)".

- In 1970, in BERNAYS' second[20] edition [HILBERT & BERNAYS, 1970, p.166] of Vol. II of the fundamental work on proof theory, *Foundations of Mathematics* [HILBERT & BERNAYS, 2013a; 2013b] by DAVID HILBERT (1862–1943) and PAUL BERNAYS (1888–1977), the generalized rules of γ- and δ^--quantification are introduced under the names "(μ^*)" and "(ν^*)", but again in a restricted form. Let positive positions now be defined recursively over α- and β-operators, but without the mutually recursive definition of negative positions.[21] BERNAYS' rules restrict the introduction of quantifiers to positive positions in this sense.

- In 1975, HEIJENOORT introduced the generalized rules of γ- and δ^--quantification (without restrictions and for a HILBERT calculus) in the unpublished paper [HEIJENOORT, 1975], which we quoted extensively in § 3.

- In 1977, in his second monograph on proof theory [SCHÜTTE, 1977, p. 20], SCHÜTTE published versions of the generalized rules of quantification whose names and restrictions are in principle the same as the before-mentioned ones of [SCHÜTTE, 1960], but over a different set of sentential operators: falsehood and implication instead of conjunction, disjunction, and negation.

- In the 1980s, HEIJENOORT refers to the generalized rules of γ- and δ^--quantification in [HEIJENOORT, 1982; 1986b], but does not define them. See our Note 5.

- In 2009, to the best of our knowledge, the first correct definition of the generalized rules of γ- and δ^--quantification and (γ-) simplification without restrictions was published in our handbook article on HERBRAND [WIRTH &AL., 2009; 2014].

[19] E.g., in the formula $A \wedge B$, the top position is positive, but the positions of the sub-formulas A and B are neither positive nor negative because a positive $\wedge$ is a β-operator.

[20] In the first edition [HILBERT & BERNAYS, 1939], however, these rules did not occur yet.

[21] E.g., in the formula $A \wedge \neg\neg B$, the top position and the position of the sub-formulas A and $\neg\neg B$ are positive, but the position of the sub-formulas $\neg B$ and B are neither positive nor negative.

8 HERBRAND's Original Rules

To see that HEIJENOORT's inference rules are not just a slight variation of HERBRAND's original rules, let us have a look at the latter ones.

HERBRAND had only the non-generalized versions of the rules of γ- and δ^{-}-quantification, which (just as the non-generalized version of the rule of simplification) result from our formalization of the generalized rules in § 3 by restricting $A[\ldots]$ to the empty context (i.e. $A[Qx.\ H]$, e.g., is just $Qx.\ H$). HERBRAND introduced them in § 2.2 of his PhD thesis [HERBRAND, 1930]; he named the rules of γ- and δ^{-}-quantification "second" and "first rule of generalization" [HERBRAND, 1971, p. 74f.], respectively ("deuxième" and "première règle de généralisation" [HERBRAND, 1968, p. 68f.]).

At the same places, we also find the (non-generalized) rule of simplification ("règle de simplification"), *modus ponens* ("règle d'implication"), and the "rules of passage" ("règles de passage"). While HERBRAND does not need *modus ponens* for completeness, the deep inference rules of passage are needed (in anti-prenex direction) because the shallow rules of quantification — contrary to the generalized ones — cannot introduce quantifiers at non-top positions.

Rules of Passage: The following six logical equivalences may be used for rewriting from left to right (*prenex direction*) and from right to left (*anti-prenex direction*), resulting in twelve deep inference rules:

$$
\begin{array}{lrcl}
(1) & \neg\forall x.\ A & \Leftrightarrow & \exists x.\ \neg A \\
(2) & \neg\exists x.\ A & \Leftrightarrow & \forall x.\ \neg A \\
(3) & (\forall x.\ A)\ \vee\ B & \Leftrightarrow & \forall x.\ (A \vee B) \\
(4) & B\ \vee\ \forall x.\ A & \Leftrightarrow & \forall x.\ (B \vee A) \\
(5) & (\exists x.\ A)\ \vee\ B & \Leftrightarrow & \exists x.\ (A \vee B) \\
(6) & B\ \vee\ \exists x.\ A & \Leftrightarrow & \exists x.\ (B \vee A)
\end{array}
$$

Here, B is a formula in which the variable x does not occur free.

As explained in § 3, if x occurs free in B, an implicit renaming of the bound occurrences of x in A is admitted to enable rewriting in prenex direction.

Note that HERBRAND did not need rules of passage for conjunction (besides the rules of passage for negation $(1, 2)$ and for disjunction $(3, 4, 5, 6)$), because he considered conjunction $A \wedge B$ a meta-level notion defined as $\neg(\neg A \vee \neg B)$.

Finally, in § 5.6.A of his PhD Thesis [HERBRAND, 1930], HERBRAND also introduces a "generalized rule of simplification" [HERBRAND, 1971, p. 175] ("règle de simplification généralisée" [HERBRAND, 1968, p. 143]). It is a deep inference rule for rewriting $H \vee H$ to H. Assuming that $\neg$ and $\vee$ are the only sentential operators in our formula language, HERBRAND's rule it is tantamount to HEIJENOORT's generalized rule of simplification presented in § 3.3, because HERBRAND implicitly equates the variants H and H'.

9 HERBRAND's "False Lemma" and its Corrections

For a given positive natural number n, HERBRAND *'s "False Lemma"* says that Property C of order n is invariant under the application of the rules of passage.

HERBRAND's "False Lemma" is wrong because the rules of passage may change the δ^--SKOLEMized form. This happens whenever a γ-quantifier binding a variable x is moved over a binary operator whose unchanged operand B contains a δ-quantifier.[22]

Counterexample 9.1
Let us again consider the rectified formula A of Example 4.12 of § 4, i.e. the formula $\exists a.\ \big(\ (\neg\exists b.\ p(b)) \vee p(a)\ \big)$, which has the δ^--SKOLEMized form $\exists a.\ \big(\ \neg p(b^\delta(a)) \vee p(a)\ \big)$, and thus Property C of order 3 (by expansion w.r.t. $\{\bullet, b^\delta(\bullet)\}$), but not Property C of order 2. Let us move the γ-quantifier "$\exists a.$" inward by applying the last equivalence of the rules of passage in anti-prenex direction. Then we obtain $(\neg\exists b.\ p(b)) \vee \exists a.\ p(a)$. The ($\delta^-$- as well as δ^{+^+}-) SKOLEMized form of this formula is $\neg p(b^\delta) \vee \exists a.\ p(a)$. Now, expansion w.r.t. $\{b^\delta\}$ results in a sentential tautology. Thus, A has Property C of order 2. □

This, however, is not really a counterexample for HERBRAND's "False Lemma" because HERBRAND treated our fresh constant "$\bullet$" from Definition 4.1 as a variable and defined the height of a SKOLEM constant to be 1 (just as we did in Definition 4.1), but the height of a variable to be 0 (instead of 1), such that $|b^\delta| = 1 = |b^\delta(\bullet)|$. As free variables and SKOLEM constants play exactly the same rôle in the given context, HERBRAND's definition of height is a bit counterintuitive and was possibly introduced to avoid this counterexample. To find a counterexample for HERBRAND's "False Lemma" also for HERBRAND's definition of height, we just have to find a way to replace "$\bullet$" with a SKOLEM constant, as shown in the following example.

Counterexample 9.2
Let us take B to be the rectified formula

$$\big(\ (\neg\exists b.\ p(b)) \vee \exists a.\ q(a)\ \big)\ \vee\ \big(\ (\exists x.\ p(x)) \wedge \neg\exists y.\ q(y)\ \big).$$

The (δ^-- as well as δ^{+^+}-) SKOLEMized form of B is

$$\big(\ \neg p(b^\delta) \vee \exists a.\ q(a)\ \big)\ \vee\ \big(\ (\exists x.\ p(x)) \wedge \neg q(y^\delta)\ \big).$$

Now, expansion w.r.t. $\{b^\delta, y^\delta\}$ results in a sentential tautology. Thus, B has Property C of order 2. Let us move the γ-quantifier "$\exists a.$" in formula B outward by applying the last equivalence of the rules of passage in prenex direction. Then we obtain

$$\big(\ \exists a.\ \big(\ (\neg\exists b.\ p(b)) \vee q(a)\ \big)\ \big)\ \vee\ \big(\ (\exists x.\ p(x)) \wedge \neg\exists y.\ q(y)\ \big).$$

Let us call this formula B'. The δ^--SKOLEMized (but not the δ^{+^+}-SKOLEMized)[23] form of B' is

$$\big(\ \exists a.\ \big(\ \neg p(b^\delta(a)) \vee q(a)\ \big)\ \big)\ \vee\ \big(\ (\exists x.\ p(x)) \wedge \neg q(y^\delta)\ \big).$$

Now the smallest expansion that results in a sentential tautology is the one w.r.t. $\{y^\delta, b^\delta(y^\delta)\}$. Thus, the formula B' has Property C of order 3, but not of order 2, no matter in which of the two ways we define the height of variables. □

[22] Here we use the same meta-variables as in our description of the rules of passage in § 8 and assume that x does not occur free in B.

The basic function of HERBRAND's "False Lemma" in the proof of HERBRAND's Fundamental Theorem is to establish the logical equivalence of Property C of a formula A with Property C of the prenex and anti-prenex forms of A.

Let us see how this flaw of HERBRAND's "False Lemma" has been corrected.

9.1 BERNAYS' Correction

In 1939, BERNAYS remarked that HERBRAND's proof is hard to follow[24] and — for the first time — published a sound proof of a version of HERBRAND's Fundamental Theorem. This version is restricted to prenex form, but more efficient in the number of terms that have to be considered in a sub-expansion than HERBRAND's quite global limitation to an expansion w.r.t. *all terms t with* $|t| \prec n$, related to Property C of order n.[25]

9.2 GÖDEL's and DREBEN's Correction

According to a conversation with HEIJENOORT in autumn 1963,[26] KURT GÖDEL (1906–1978) noticed the flaw in the proof of HERBRAND's "False Lemma" in 1943 and wrote a private note, but did not publish it. While GÖDEL's documented attempts to construct a counterexample to HERBRAND's "False Lemma" failed, he had actually worked out a *correction* of HERBRAND's "False Lemma", which is sufficient for the proof of HERBRAND's Fundamental Theorem.[27]

In 1962, when GÖDEL's correction was still unknown, a young student, PETER B. ANDREWS (*1937), had the audacity to tell his advisor ALONZO CHURCH (1903–1995) that there seemed to be a gap in the proof of HERBRAND's "False Lemma". CHURCH sent ANDREWS to BURTON DREBEN (1927–1999), who finally came up with a counterexample. And then ANDREWS constructed a simpler counterexample (similar to the one we presented in Counterexample 9.2) and joint work found a correction similar to GÖDEL's.[28]

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[23] From this example, one might get the idea that the flaw in HERBRAND's "False Lemma" would be a peculiarity of the δ^--SKOLEMized form. Indeed, for the δ^{+^+}-SKOLEMized form (cf. Definition 4.3), moving γ-quantifiers with the rules of passage cannot change the number of arguments of the SKOLEM functions. This does not help, however, to overcome the flaw in HERBRAND's "False Lemma", because, for the δ^{+^+}-SKOLEMized form, moving δ-quantifiers may change the number of arguments of the SKOLEM functions if the rule of passage is applied within the scope of a γ-quantifier whose bound variable occurs in B but not in A (cf. Note 22). The δ^{+^+}-SKOLEMized form of $\exists y_1.\, \forall z_1.\, p(y_1, z_1) \vee \exists y_2.\, \forall z_2.\, q(y_2, z_2)$ is $\exists y_1.\, p(y_1, z_1^\delta(y_1)) \vee \exists y_2.\, q(y_2, z_2^\delta(y_2))$, but the δ^{+^+}-SKOLEMized form of any prenex form has a *binary* SKOLEM function, unless we use HENKIN quantifiers as found in HINTIKKA's first-order logic, cf. [HINTIKKA, 1996], [WIRTH, 2013].

[24] In both the 1939 edition [HILBERT & BERNAYS, 1939, Note 1, p.158] and in the 1970 edition [HILBERT & BERNAYS, 1970, Note 1, p.161], we read: "Die HERBRANDsche Beweisführung ist schwer zu verfolgen"

[25] Cf. § 3.3 of the 1939 edition [HILBERT & BERNAYS, 1939]. In the 1970 edition [HILBERT & BERNAYS, 1970], BERNAYS also indicates how to remove the restriction to prenex formulas.

[26] Cf. [HERBRAND, 1968, p. 8, Note j].

[27] Cf. [GOLDFARB, 1993].

[28] Cf. [ANDREWS, 2003], [DREBEN &AL., 1963], [DREBEN & DENTON, 1963].

Roughly speaking, the corrected lemma says that — to keep Property C of A invariant under (a single application of) a rule of passage — we may have to step from order n to order

$$n \cdot (N^r + 1)^n.$$

Here r is the number of γ-quantifiers in whose scope the rule of passage is applied and N is the cardinality of $\mathcal{T}_n(F)$ for the δ^--SKOLEMized form F of A.[29]

GÖDEL's and DREBEN's correction is not particularly elegant because — iterated several times until a prenex form is reached — it can lead to pretty high orders. Thus, although this correction serves well for a finitistic proof of HERBRAND's Fundamental Theorem, it results in a complexity that is completely unacceptable in practice (e.g. intractable in automated reasoning), and this already for small non-prenex formulas.

9.3 HEIJENOORT's Correction

The lack of generalized versions of the rules of quantification force HERBRAND's formal derivations in his *modus ponens*-free calculus to take a detour over prenex form, which was standard at HERBRAND's time. For example, LÖWENHEIM and SKOLEM had always reduced their problems to prenex forms of various kinds. The reduction of a proof task to prenex form has several disadvantages, however, such as serious negative effects on proof complexity.[30]

The surprising — but, as a matter of fact, correct — thesis of IRVING H. ANELLIS (1946–2013) in [ANELLIS, 1991] is that, building on the LÖWENHEIM–SKOLEM Theorem, it was HERBRAND's work in elaborating HILBERT's concept of "being a proof" that gave rise to the development of the variety of first-order calculi in the 1930s, such as the ones of the HILBERT school [HILBERT & BERNAYS, 2013b], and such as natural deduction and sequent calculi in [GENTZEN, 1935]. So instead of fiddling around with HILBERT's ε and other existing calculi, HERBRAND showed that the design of tailor-made calculi suiting the given situation and its requirements can be most fertile.[31]

As a consequence of this attitude of HERBRAND, if he had become aware of the flaw in his "False Lemma", he would probably have avoided the whole detour over prenex forms. And as there is not much of a choice for this avoidance, he would probably have proceeded in the way of HEIJENOORT [1975], which we have presented §§ 3–7. Indeed, not only HEIJENOORT's generalized rules of quantification are *deep* inference rules, already HERBRAND's rules of passage and of generalized simplification were such rules which manipulate formulas at an arbitrary depth.

[29] Cf. [DREBEN & DENTON, 1963, p. 393].

[30] Cf. e.g. [BAAZ & FERMÜLLER, 1995], [BAAZ & LEITSCH, 1995].

[31] This is a finding that still today is not sufficiently respected in many areas, especially in those where logic is applied as a tool. Cf. e.g. [WIRTH, 2012d] for a discussion of this topic in the representation and computation of the semantics of natural language dialogs.

In HERBRAND's proof of the equivalence of Property C and derivability in HERBRAND's *modus ponens*-free calculus (stated in HERBRAND's Fundamental Theorem), there is only one application of HERBRAND's "False Lemma", namely for dealing with applications of the rules of passage. As the rules of passage are not part of HEIJENOORT's version of HERBRAND's *modus ponens*-free calculus, there is no need for HERBRAND's "False Lemma" in "HEIJENOORT*'s correction*" anymore.

Contrary to BERNAYS' correction, HEIJENOORT's correction avoids the detour over the Extended First ε-Theorem of the proof of BERNAYS mentioned before; cf. Note 25.

Contrary to the GÖDEL's and DREBEN's correction, the corrections of HEIJENOORT and BERNAYS have the minor disadvantage[32] that they do not provide a corrected version of HERBRAND's "False Lemma".

We see a unique quality of HEIJENOORT's correction in the following: It corrects exactly what went wrong in the execution HERBRAND's proof plan. As we have shown in §§ 5 and 6, HERBRAND's original proof of his Fundamental Theorem can easily be adapted to a proof of its version à la HEIJENOORT (i.e. our Theorem 6.3) (omitting the detour over prenex form). The idea to such a correction of HERBRAND's proof seems to have been published first in [HEIJENOORT, 1971, Note 77, p. 555] and [HERBRAND, 1971, Note 60, p.171], where, however, no generalized versions of the rules of quantification are mentioned.

Moreover, the version of HERBRAND's Fundamental Theorem à la HEIJENOORT resulting from HEIJENOORT's correction is more elegant and concise than HERBRAND's original one: It is not only more easily memorized by humans. Also the intractable and unintuitive[33] rise in complexity introduced by GÖDEL's and DREBEN's correction is avoided. And because the relation to Property C is more straightforward after HEIJENOORT's correction, the Fundamental Theorem à la HEIJENOORT gives human beings a better chance to construct a proof of a manageable size from a sub-expansion of the SKOLEMized form as we have done in § 5.

Finally, our new free-variable version of HERBRAND's Fundamental Theorem (i.e. Theorem 6.4) adapts HEIJENOORT's correction to our novel free-variable calculus, and makes the relation to modern reductive free-variable calculi[34] for computer-assisted theorem proving with strong automation support[35] most obvious and clear.

[32]Note that a corrected version of HERBRAND's "False Lemma" (in the sense of GÖDEL's and DREBEN's correction) is still needed if we want to prove that a derivation in HERBRAND's calculus that includes *modus ponens* implies Property C in a finitistic sense. As the example on top of page 201 in [HERBRAND, 1971] shows, however, an intractable increase of the order of Property C cannot be avoided in general for inference steps by *modus ponens*.

[33]Unintuitive e.g. in the sense of [TAIT, 2006].

[34]Such as free-variable tableau, sequent, and matrix calculi [FITTING, 1996], [WIRTH, 2004; 2013].

[35]Cf. [AUTEXIER, 2003], [AUTEXIER &AL., 2006], [DIETRICH, 2011].

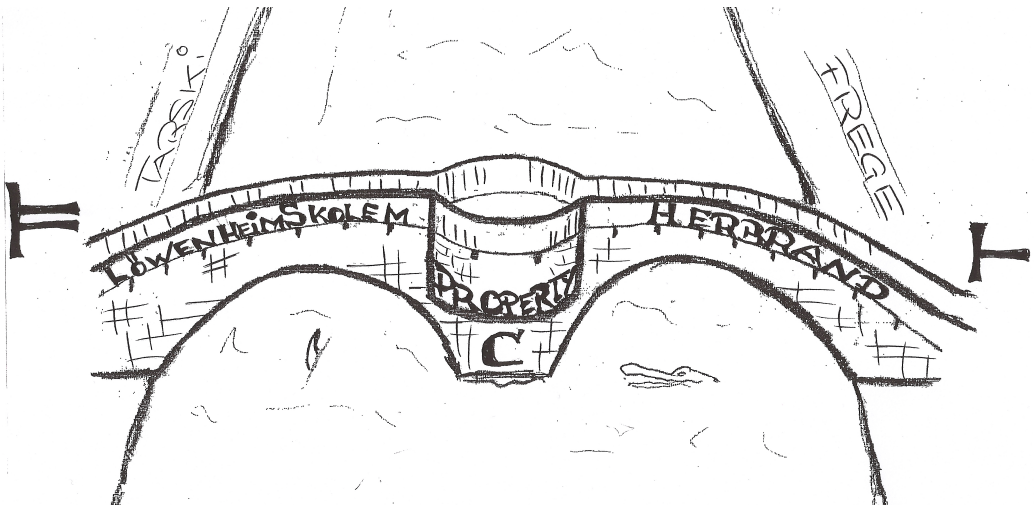

Figure 1: The bridge of the LÖWENHEIM–SKOLEM Theorem and HERBRAND's Fundamental Theorem, based on the sentential Property C standing firm in the river that divides the banks of valid and derivable formulas in the land of first-order predicate logic.

10 Why HERBRAND's Fundamental Theorem is Central, Though Not Deep

Right at the beginning of § 3, we have quoted from [HEIJENOORT, 1975], where HEIJENOORT again quotes a statement of BERNAYS [1957], saying that HERBRAND's Fundamental Theorem would be the central theorem of predicate logic. HEIJENOORT has quoted this statement also in [HERBRAND, 1968, p.1] and in [HEIJENOORT, 1976].

Referring to the place of appearance of HERBRAND's Fundamental Theorem in [HERBRAND, 1930, Chapter 5], HEIJENOORT also ends his article [HEIJENOORT, 1967] as follows:

> "Let me say simply, in conclusion, that *Begriffsschrift* [FREGE, 1879], LÖWENHEIM's paper [LÖWENHEIM, 1915], and Chapter 5 of HERBRAND's thesis [HERBRAND, 1930] are the three cornerstones of modern logic."

We did not find any place, however, where HEIJENOORT called HERBRAND's Fundamental Theorem "deep" (or "one of the deepest results of logic" as reported by FEFERMAN in our quotation at the end of § 2).[36] Among the statements of HEIJENOORT we found on HERBRAND's Fundamental Theorem, closest to "deep" (but not really close) comes the following sentence in [HEIJENOORT, 1992, p. 247]:

[36] We also did not find a French publication of HEIJENOORT where he calls HERBRAND's Fundamental Theorem "profond" ("deep"). This does not mean, of course, that such a publication does not exist, although we have carefully searched through all English and French texts of HEIJENOORT listed in our references (some of them even by machine).

"In his doctoral dissertation [Herbrand, 1930], Jacques Herbrand (1908–1931) presented a theorem that revealed a profound feature of first-order logic."

On the one hand, it is obvious that HERBRAND's Fundamental Theorem reveals a profound feature of first-order logic and that it is one of the cornerstones of modern logic.

On the other hand, however, we cannot really justify why it should be called "deep", and agree with FEFERMAN's statement in our quotation at the end of § 2 that such an assessment would be "debatable". In the vernacular of working mathematicians, a "deep theorem" or "deep result" is a theorem whose proof is very hard to find, especially because it requires ingenuity,[37] such as the proof of FERMAT's Last Theorem (cf. [TAYLOR & WILES, 1995], [WILES, 1995]). Although the proof of HERBRAND's Fundamental Theorem in 1929 was more "avantgarde" than most other proofs in the history of modern logic at their respective times, and although the theorem itself is ingenious, the proof of HERBRAND's Fundamental Theorem is *not* hard to find. As we have seen in § 5, the proof of HERBRAND's Fundamental Theorem basically verifies the equivalence of Property C with HERBRAND's *modus ponens*-free calculus, which was up to his own design.[38] HERBRAND was a creative genius in designing all the novel features he needed for that proof at his time, but we do not think that all this classifies that proof as being deep in the sense we described here.

Compared to the great achievements of HERBRAND in his PhD thesis, the gap in his original proof is negligible, at least w.r.t. HEIJENOORT's correction as discussed in § 9.3. Moreover, there can be no doubt that he would have been able to patch the gap if he had only become aware of it. Therefore, we think that this gap does not justify to say that HERBRAND failed to prove his Fundamental Theorem, nor that the theorem is a deep one.

All in all, HEIJENOORT was definitely right in calling HERBRAND a *génie créateur* [HERBRAND, 1968, p.1], whereas it remains an open question here whether he really called HERBRAND's Fundamental Theorem "deep".

To find out, on the other hand, why HERBRAND's Fundamental Theorem should be called "central", let us now have a look at a translation of BERNAYS' briefly quoted statement in its context:

> "In its proof-theoretic form, HERBRAND's Theorem can be seen as the central theorem of predicate logic. It expresses the relation of predicate logic to propositional logic in a concise and felicitous form."[39]

So it is quite obvious that BERNAYS did not want to say that HERBRAND's Fundamental Theorem is situated in the land of first-order predicate logic like Paris in France;[40] he just wanted to say that it establishes an (in the limit complete) reduction of the problem of derivability in one calculus to the problem of derivability in a simpler one (a HERBRAND *reduction* as it is called today), and that this reduction is "concise and felicitous".[41]

[37] A "deep theorem", however, does not necessarily require computational power, such as in the case of the very hard proof of the Four-Color Theorem [GONTHIER, 2008].

[38] In terms of Figure 1, we could say that the place where HERBRAND's part of the bridge entered the bank of derivable formulas in the land of first-order predicate logic was up to HERBRAND's choice.

[39] The German original text is: "Der HERBRAND'sche Satz in seiner beweistheoretischen Fassung kann als das zentrale Theorem der Prädikatenlogik angesehen werden. In ihm wird die Beziehung der Prädikatenlogik zur Aussagenlogik auf eine prägnante Form gebracht."

[40] You can hardly avoid Paris when traveling in France.

[41] The original German word "prägnant" (which we translate as "concise and felicitous") has the same etymological root as the English word "pregnant", but it has mostly lost that direct meaning.

Let us elaborate a bit on the questions why HERBRAND's Fundamental Theorem is central and why the form of its expression is concise and felicitous.

As an explicit notion, HERBRAND's Property C was first formulated in HERBRAND's thesis [HERBRAND, 1930]. Implicitly it was used already before, namely by LEOPOLD LÖWENHEIM (1878–1957) in [LÖWENHEIM, 1915] and by THORALF SKOLEM (1887–1963) in [SKOLEM, 1928]. It is the main property of HERBRAND's work and may well be called the central property of first-order logic, for reasons to be explained in the following.

Contrary to the the unjustified[42] criticism of SKOLEM, HERBRAND, and HEIJENOORT, there are no essential gaps in the proof of LÖWENHEIM in [LÖWENHEIM, 1915] of the later so-called LÖWENHEIM–SKOLEM Theorem, which says that Property C is equivalent to model-theoretic first-order validity. In his PhD thesis, HERBRAND also showed the equivalence of his calculi with those of the HILBERT school [HILBERT & BERNAYS, 2013b] and the Principia Mathematica [WHITEHEAD & RUSSELL, 1910–1913]. Therefore, as a consequence of the LÖWENHEIM–SKOLEM Theorem, the completeness of all these calculi is an immediate corollary of HERBRAND's Fundamental Theorem. But HERBRAND did not trust the left arc of the bridge in our illustration of Figure 1. HERBRAND's standpoint was so radically finitistic that in the area of logic he did not accept model theory or set theory at all. And so GÖDEL proved the completeness of first-order logic first when he submitted his thesis [GÖDEL, 1930] in 1929, in the same year as HERBRAND, and the theorem is now called GÖDEL*'s Completeness Theorem* in all textbooks on logic.

So, although Property C is purely sentential and does not really belong to the land of first-order predicate logic with its semantical and syntactical banks (cf. Figure 1), it can be seen as the central property of first-order predicate logic and HERBRAND's Fundamental Theorem connects this property with derivability in the standard calculi. Moreover, the way this connection is established, is concise and felicitous because the formal proof in HERBRAND's *modus ponens*-free calculus is linear, has the "sub-formula" property, and does not take any detours because it does not have rules that eliminate intermediate results, such as *modus ponens* and GENTZEN's cut rule do. Furthermore, the proof construction of HERBRAND's Fundamental Theorem out of Property C gets even more concise and felicitous in HEIJENOORT's correction; cf. § 9.3.

Moreover, Property C has strongly influenced the early history of automated deduction in the 2nd half of the 20th century [WIRTH &AL., 2009; 2014]. The different treatment of δ-quantifiers and γ-quantifiers in Property C, namely by SKOLEMization and expansion, respectively, as found in [HERBRAND, 1930], [SKOLEM, 1928], rendered the reduction to sentential logic by hand for small examples (and later with a computer) practically executable for the first time.[43] This different treatment of the two kinds of quantification is inherited from the PEIRCE–SCHRÖDER tradition[44] which came on HERBRAND via LÖWEN-

[42]Cf. [WIRTH &AL., 2009, § 3.12, esp. Note 98] or [WIRTH &AL., 2014, § 14, esp. Note 114]

[43]For instance, the elimination of both γ- and δ-quantifiers with the help of HILBERT's ε-operator suffers from an exponential complexity in formula size. As a result, already small formulas grow so large that the mere size makes them inaccessible to human inspection; and this is still the case for the term-sharing representation of ε-terms of [WIRTH, 2008, Example 8], which is further elaborated in [WIRTH, 2013, Example 3.7].

[44]For the heritage of PEIRCE see [BRADY, 2000], [PEIRCE, 1885]; for that of SCHRÖDER see [PECKHAUS, 2004], [SCHRÖDER, 1895].

HEIM and SKOLEM. RUSSELL and HILBERT had already merged that tradition with the one of FREGE, sometimes emphasizing their FREGE heritage over the one of PEIRCE and SCHRÖDER.[45] It was HERBRAND who completed the bridge between these two traditions with his Fundamental Theorem, as depicted in Figure 1.

So one could well say with some justification that, as a consequence of Property C being the central property of first-order predicate logic, the LÖWENHEIM–SKOLEM Theorem and HERBRAND's Fundamental Theorem are the central theorems of first-order predicate logic.

Moreover, HERBRAND's *modus ponens*-free calculus — at least after HEIJENOORT's correction and our adaption to free-variable calculi — is most close to modern approaches in automated theorem proving that aim at a synergetic combination of the semantical strength of human mathematicians with the syntactical and computational strength of computing machines, as we have indicated already at the end of § 9.3. The manner in which automated theorem-proving systems based on modern sequent, tableau, and matrix calculi organize proof search[46] does not follow the HILBERT school and their ε-elimination theorems, but GENTZEN's and HERBRAND's calculi. Moreover, regarding their SKOLEM-ization, their deep inference,[47] and their focus on γ-quantifiers and their multiplicity, these modern proof-search calculi are even more in HERBRAND's tradition than in GENTZEN's.

[45] While this emphasis on FREGE will be understood by everybody who ever had the fascinating experience of reading FREGE, it put some unjustified bias to the historiography of modern logic, still present in the selection of HEIJENOORT's famous source book [HEIJENOORT, 1971]; cf. e.g. [ANELLIS, 1992, Chapter 3].

[46] Cf. e.g. [AUTEXIER, 2005], [WALLEN, 1990], [WIRTH, 2004].

[47] Note that although the deep inference rules of *generalized* quantification are an extension of HERBRAND's calculi by HEIJENOORT, the deep inference rules of passage and of generalized simplification are HERBRAND's original contributions.

11 Conclusion and Future Work

We have most clearly described and discussed HEIJENOORT's version of HERBRAND's *modus ponens*-free calculus and its relation to HERBRAND's Fundamental Theorem in §§ 3–6, where our presentation had a didactical focus because (as stated already in § 1) we hope that our improvements of this calculus will become part of the standard education of logicians, just as well as the famous construction of Cut-free proofs according to GENTZEN's Hauptsatz [GENTZEN, 1935].

Already from our brief discussion in § 7 on the publication history of the *generalized* rules of quantification alone, it becomes clear that HEIJENOORT's unpublished handout [HEIJENOORT, 1975] would have deserved publication in an international journal, provided that HEIJENOORT had worked out the proof properly, involving our correction of HEIJENOORT's generalized rule of simplification.[48]

After clarifying HERBRAND's original rules in § 8, it became possible to discuss HERBRAND's "False Lemma" in § 9: After briefly describing BERNAYS', GÖDEL's, and DREBEN's corrections, we explained why we think that "HEIJENOORT's correction" (consisting in the material we presented in §§ 3–6) is actually the one that fits HERBRAND's style in his work on logic best and offers the most tractable solution, which also has a close relation to automated and human-oriented theorem proving.

Finally, in § 10, we have checked in which context BERNAYS called HERBRAND's Fundamental Theorem "central" and — to defend HEIJENOORT's and BERNAYS' assessment of HERBRAND's Fundamental Theorem against FEFERMAN's critique quoted at the end of § 2 — elaborated on the question why its form of expression is "concise and felicitous", though not deep.

The strong dependencies

1. between the δ^--SKOLEMized form and the proof of HERBRAND's Fundamental Theorem (actually Lemma 5.3) described in § 5.2, and
2. between the δ^{+^+}-SKOLEMized form and the liberalized δ-rule in the form of our generalized rule of δ^{+^+}-quantification (which forced us to introduce Property C* in Definition 4.6),

became clear only in this article and may deserve further consideration.

A careful bilingual edition of HERBRAND's complete works on the basis of the previous editorial achievements [HERBRAND, 1936; 1971], which was on the agenda until HEIJENOORT died [FEFERMAN, 1993b, p. 383], is still in high demand.

Last but not least, we would like to suggest our handbook article [WIRTH &AL., 2009; 2014] for further reading on HERBRAND's work in logic, also because a comparison with the material we have further elaborated in this article will make some additional aspects of HERBRAND's Fundamental Theorem clear.[49]

[48] That this proof would have been within HEIJENOORT's reach is out of question; although he had not worked it out, as we can see from the bug in his generalized rule of simplification.

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[49]For example, as indicated already in our quotation from Heijenoort's [Heijenoort, 1975] at the beginning of § 2, besides the expansions used in this article (nowadays called "Herbrand expansions"), there is the alternative of Herbrand disjunctions used in [Wirth &al., 2009; 2014]. Considering the intractable size of a Herbrand disjunction compared to a Herbrand expansion (cf. Note 12), we have come to the conclusion that expansions should be preferred in general, not only because they are Herbrand's own choice, but also because they are more intuitive and more tractable, especially in combination with the "sub-expansions" we have introduced in Definition 4.2. This means, for instance, that presentations of Herbrand's Fundamental Theorem via Herbrand disjunctions, such as in [Heijenoort, 1992, pp. 247–249], should be avoided because they are none of the following: historically adequate, tractable, intuitive.

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