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Generalization is Semantical

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Generalization is Semantical

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Abstract

We want to list some examples that show that it is hopeless to achieve generalization of inductive theorems by syntactical means.

1 Specification Too Poor

The following example is from Kühler (2000), p.11.

Suppose we have the definitions

(double1) $\mathsf{double}(0)=0$
(double2) $\mathsf{double}(\mathsf{s}(x^{\delta^-}))=\mathsf{s}(\mathsf{s}(\mathsf{double}(x^{\delta^-})))$

and want to prove the clause

(double-irrefl) $\mathsf{double}(x^{\delta^-})\neq x^{\delta^-},\ \ x^{\delta^-}=0$

This cannot be proved without the introduction of additional functions such as

(less1) $\mathsf{less}(0,\mathsf{s}(y^{\delta^-}))=\mathsf{true}$
(less2) $\mathsf{less}(x^{\delta^-},0)=\mathsf{false}$
(less3) $\mathsf{less}(\mathsf{s}(x^{\delta^-}),\mathsf{s}(y^{\delta^-}))=\mathsf{less}(x^{\delta^-},y^{\delta^-})$

to formulate lemmas such as

(less-s) $\mathsf{less}(x^{\delta^-},\mathsf{s}(x^{\delta^-}))=\mathsf{true}$
(less-trans) $\mathsf{less}(x^{\delta^-},z^{\delta^-})=\mathsf{true},\ \ \mathsf{less}(x^{\delta^-},y^{\delta^-})\neq\mathsf{true},\ \ \mathsf{less}(y^{\delta^-},z^{\delta^-})\neq\mathsf{true}$
(less-double) $\mathsf{less}(x^{\delta^-},\mathsf{double}(x^{\delta^-}))=\mathsf{true},\ \ x^{\delta^-}=0$
(less-$\neq$) $\mathsf{less}(x^{\delta^-},y^{\delta^-})=\mathsf{false},\ \ x^{\delta^-}\neq y^{\delta^-}$

Now (less-s) and (less-trans) are trivial by induction and prove (less-double) by induction, and this together with the inductively trivial (less-$\neq$) deductively implies (double-irrefl).

It seems obvious that the generation of all these symbols, definitions, and lemmas cannot be reasonably hoped to be achieved by syntactic manipulations in any framework.

2 Properly Generalizing does Not Find a Proof

The example of this section is a rational reconstruction and elaboration of Section 5.2 of Busch (1993) within the superior formalism of Wirth (2004). We do not agree with the conclusion of Section 5.2 of Busch (1993):

> "Although another induction proof has to be performed, the complexity of the induction compared to the original goal is reduced."

To the contrary, we do not see a reduction in complexity because a direct induction proof of the open goals is still not possible. We see this as an example, where generalization helps to find a theorem but not a proof of it.

Suppose we have the definitions

(append1) $\mathsf{append}(\mathsf{nil}, l_2^{\delta^-}) = l_2^{\delta^-}$
(append2) $\mathsf{append}(\mathsf{cons}(x_1^{\delta^-}, l_1^{\delta^-}), l_2^{\delta^-}) = \mathsf{cons}(x_1^{\delta^-}, \mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}))$
(revtail1) $\mathsf{revtail}(\mathsf{nil}, l_2^{\delta^-}) = l_2^{\delta^-}$
(revtail2) $\mathsf{revtail}(\mathsf{cons}(x_1^{\delta^-}, l_1^{\delta^-}), l_2^{\delta^-}) = \mathsf{revtail}(l_1^{\delta^-}, \mathsf{cons}(x_1^{\delta^-}, l_2^{\delta^-}))$

and want to prove

(revtail-append1) $\mathsf{revtail}(\mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}), \mathsf{nil}) = \mathsf{append}(\mathsf{revtail}(l_2^{\delta^-}, \mathsf{nil}), \mathsf{revtail}(l_1^{\delta^-}, \mathsf{nil}))$

Although this is inductively valid, a simple proof fails because the induction hypothesis is too weak. Both straightforward generalizations

(revtail-append2) $\mathsf{revtail}(\mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}), l_3^{\delta^-}) = \mathsf{append}(\mathsf{revtail}(l_2^{\delta^-}, l_3^{\delta^-}), \mathsf{revtail}(l_1^{\delta^-}, l_3^{\delta^-}))$
(revtail-append3) $\mathsf{revtail}(\mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}), l_3^{\delta^-}) = \mathsf{append}(\mathsf{revtail}(l_2^{\delta^-}, l_4^{\delta^-}), \mathsf{revtail}(l_1^{\delta^-}, l_5^{\delta^-}))$

can easily be refuted, however. To generalize avoiding over-generalization, we should try to prove

(revtail-append4) $\mathsf{revtail}(\mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}), l_3^{\gamma}(l_1^{\delta^-}, l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})) =$
$\mathsf{append}(\mathsf{revtail}(l_2^{\delta^-}, l_4^{\gamma}(l_1^{\delta^-}, l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})), \mathsf{revtail}(l_1^{\delta^-}, l_5^{\gamma}(l_1^{\delta^-}, l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})))$

with variable-condition $\{l_3^{\gamma}, l_4^{\gamma}, l_5^{\gamma}\} \times \{l_1^{\delta^-}, l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-}\}$. As induction over $l_1^{\delta^-}$ is unflawed, let us start it. The case $\{l_1^{\delta^-} \mapsto \mathsf{cons}(x_1^{\delta^-}, l_6^{\delta^-})\}$ adds $\{l_1^{\delta^-}\} \times \{x_1^{\delta^-}, l_6^{\delta^-}\}$ to our variable-condition and reduces to

(revtail-append4.2) $\mathsf{revtail}(\mathsf{append}(l_6^{\delta^-}, l_2^{\delta^-}), \mathsf{cons}(x_1^{\delta^-}, l_3^{\gamma}(\mathsf{cons}(x_1^{\delta^-}, l_6^{\delta^-}), l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-}))) =$
$\mathsf{append}\begin{pmatrix}\mathsf{revtail}(l_2^{\delta^-}, l_4^{\gamma}(\mathsf{cons}(x_1^{\delta^-}, l_6^{\delta^-}), l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})),\\ \mathsf{revtail}(l_6^{\delta^-}, \mathsf{cons}(x_1^{\delta^-}, l_5^{\gamma}(\mathsf{cons}(x_1^{\delta^-}, l_6^{\delta^-}), l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})))\end{pmatrix}$

Unifying the left hand side of this with the left-hand side of the induction hypothesis (revtail-append4) under the substitution $\{l_1^{\delta^-} \mapsto l_6^{\delta^-}\}$ would require

$$\mathsf{cons}(x_1^{\delta^-}, l_3^{\gamma}(\mathsf{cons}(x_1^{\delta^-}, l_6^{\delta^-}), l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})) = l_3^{\gamma}(l_6^{\delta^-}, l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-}),$$

which is hopeless. But with a substitution that replaces all unmatched free δ^--variables with free γ-variables, say $\{l_1^{\delta^-} \mapsto l_6^{\delta^-},\ l_3^{\delta^-} \mapsto l_{3'}^{\gamma},\ l_4^{\delta^-} \mapsto l_{4'}^{\gamma},\ l_5^{\delta^-} \mapsto l_{5'}^{\gamma}\}$, this requirement becomes

$$\mathsf{cons}(x_1^{\delta^-}, l_3^{\gamma}(\mathsf{cons}(x_1^{\delta^-}, l_6^{\delta^-}), l_2^{\delta^-}, l_3^{\delta^-}, l_4^{\delta^-}, l_5^{\delta^-})) = l_3^{\gamma}(l_6^{\delta^-}, l_2^{\delta^-}, l_{3'}^{\gamma}, l_{4'}^{\gamma}, l_{5'}^{\gamma}),$$

which, after substitution $\{l_3^{\gamma} \mapsto \lambda l_1, l_2, l_3, l_4, l_5.\ l_3\}$ and $\lambda\beta$-reduction, reads

$$\mathsf{cons}(x_1^{\delta^-}, l_3^{\delta^-}) = l_{3'}^{\gamma},$$

which is solved by $\{l^{\gamma}_{3'} \mapsto \mathsf{cons}(x^{\delta}_1, l^{\delta}_3)\}$. After this the original clause reads

(revtail-append4) $\mathsf{revtail}(\mathsf{append}(l^{\delta}_1, l^{\delta}_2), l^{\delta}_3) =$
$\mathsf{append}(\mathsf{revtail}(l^{\delta}_2, l^{\gamma}_4(l^{\delta}_1, l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)), \mathsf{revtail}(l^{\delta}_1, l^{\gamma}_5(l^{\delta}_1, l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)))$

and the result of the induction hypothesis application is

(revtail-append4.2.1) $\mathsf{append}\begin{pmatrix}\mathsf{revtail}(l^{\delta}_2, l^{\gamma}_4(l^{\delta}_6, l^{\delta}_2, \mathsf{cons}(x^{\delta}_1, l^{\delta}_3), l^{\gamma}_{4'}, l^{\gamma}_{5'})),\\ \mathsf{revtail}(l^{\delta}_6, l^{\gamma}_5(l^{\delta}_6, l^{\delta}_2, \mathsf{cons}(x^{\delta}_1, l^{\delta}_3), l^{\gamma}_{4'}, l^{\gamma}_{5'}))\end{pmatrix} =$
$\mathsf{append}\begin{pmatrix}\mathsf{revtail}(l^{\delta}_2, l^{\gamma}_4(\mathsf{cons}(x^{\delta}_1, l^{\delta}_6), l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)),\\ \mathsf{revtail}(l^{\delta}_6, \mathsf{cons}(x^{\delta}_1, l^{\gamma}_5(\mathsf{cons}(x^{\delta}_1, l^{\delta}_6), l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)))\end{pmatrix}$

As there are many choices to close this goal by instantiation and $\lambda\beta$-reduction, let us have a look at the base case first. The case $\{l^{\delta}_1 \mapsto \mathsf{nil}\}$ reduces to

(revtail-append4.1) $\mathsf{revtail}(l^{\delta}_2, l^{\delta}_3) = \mathsf{append}(\mathsf{revtail}(l^{\delta}_2, l^{\gamma}_4(\mathsf{nil}, l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)), l^{\gamma}_5(\mathsf{nil}, l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5))$

Now induction on l^{δ}_2 is unflawed. The base case $\{l^{\delta}_2 \mapsto \mathsf{nil}\}$ reduces to

(revtail-append4.1.1) $l^{\delta}_3 = \mathsf{append}(l^{\gamma}_4(\mathsf{nil}, \mathsf{nil}, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5), l^{\gamma}_5(\mathsf{nil}, \mathsf{nil}, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5))$

Rewriting with (append1), this (unsafely) reduces to

(revtail-append4.1.1.1) $l^{\delta}_3 = l^{\gamma}_5(\mathsf{nil}, \mathsf{nil}, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)$
(revtail-append4.1.1.2) $l^{\gamma}_4(\mathsf{nil}, \mathsf{nil}, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5) = \mathsf{nil}$

As the substitution $\{l^{\gamma}_5 \mapsto \lambda l_1, l_2, l_3, l_4, l_5.\ l^{\delta}_3\}$ is blocked by our variable-condition, (revtail-append4.1.1.1) can only be closed by applying $\{l^{\gamma}_5 \mapsto \lambda l_1, l_2, l_3, l_4, l_5.\ l_3\}$ and $\lambda\beta$-reduction. With this, (revtail-append4.2.1) can be (unsafely) reduced to

(revtail-append4.2.1.1) $l^{\gamma}_4(l^{\delta}_6, l^{\delta}_2, \mathsf{cons}(x^{\delta}_1, l^{\delta}_3), l^{\gamma}_{4'}, l^{\gamma}_{5'}) = l^{\gamma}_4(\mathsf{cons}(x^{\delta}_1, l^{\delta}_6), l^{\delta}_2, l^{\delta}_3, l^{\delta}_4, l^{\delta}_5)$

The case $\{l^{\delta}_2 \mapsto \mathsf{cons}(x^{\delta}_2, l^{\delta}_7)\}$ of (revtail-append4.1) adds $\{l^{\delta}_2\} \times \{x^{\delta}_2, l^{\delta}_7\}$ to our variable-condition and reduces to

(revtail-append4.1.2) $\mathsf{revtail}(l^{\delta}_7, \mathsf{cons}(x^{\delta}_2, l^{\delta}_3)) =$
$\mathsf{append}(\mathsf{revtail}(l^{\delta}_7, \mathsf{cons}(x^{\delta}_2, l^{\gamma}_4(\mathsf{nil}, \mathsf{cons}(x^{\delta}_2, l^{\delta}_2), l^{\delta}_3, l^{\delta}_4, l^{\delta}_5))), l^{\delta}_3)$

The case $\{l^{\delta}_7 \mapsto \mathsf{nil}\}$ reduces to

(revtail-append4.1.2.1) $\mathsf{cons}(x^{\delta}_2, l^{\delta}_3) = \mathsf{append}(l^{\gamma}_4(\mathsf{nil}, \mathsf{cons}(x^{\delta}_2, l^{\delta}_2), l^{\delta}_3, l^{\delta}_4, l^{\delta}_5), \mathsf{cons}(x^{\delta}_2, l^{\delta}_3))$

By (append1) this (unsafely) reduces to

(revtail-append4.1.2.1.1) $l^{\gamma}_4(\mathsf{nil}, \mathsf{cons}(x^{\delta}_2, l^{\delta}_2), l^{\delta}_3, l^{\delta}_4, l^{\delta}_5) = \mathsf{nil}$

Now (revtail-append4.1.1.2) and (revtail-append4.1.2.1.1) can be solved by projection on the first component and the imitation of nil, i.e. $\{l^{\gamma}_4 \mapsto \lambda l_1, l_2, l_3, l_4, l_5.\ \mathsf{nil}\}$. As only the latter closes (revtail-append4.2.1.1), we apply it globally. The top open goal then is

(revtail-append4.1) $\mathsf{revtail}(l^{\delta}_2, l^{\delta}_3) = \mathsf{append}(\mathsf{revtail}(l^{\delta}_2, \mathsf{nil}), l^{\delta}_3)$

But again there is no chance to prove this by an easy induction. Indeed, even with the additional lemma

(append-assoc) $\mathsf{append}(\mathsf{append}(l^{\delta}_1, l^{\delta}_2), l^{\delta}_3) = \mathsf{append}(l^{\delta}_1, \mathsf{append}(l^{\delta}_2, l^{\delta}_3))$

Tobias Schmidt-Samoa was not able to prove (revtail-append4.1) with NQTHM, cf. Boyer & Moore (1988). The inductive theorem prover QUODLIBET, cf. Avenhaus &al. (2003), which is stronger than the famous and excellent NQTHM in the generation of induction hypotheses, proves it automatically by induction on l^{δ}_2. We have to apply (revtail-append4.1) as an induction hypothesis *twice* to

(revtail-append4.1.2) $\mathsf{revtail}(l_7^{\delta^-}, \mathsf{cons}(x_2^{\delta^-}, l_3^{\delta})) = \mathsf{append}(\mathsf{revtail}(l_7^{\delta^-}, \mathsf{cons}(x_2^{\delta^-}, \mathsf{nil})), l_3^{\delta^-})$

namely once with $\{l_2^{\delta^-} \mapsto l_7^{\delta^-},\ l_3^{\delta^-} \mapsto \mathsf{cons}(x_2^{\delta^-}, l_3^{\delta})\}$ and once with $\{l_2^{\delta^-} \mapsto l_7^{\delta^-},\ l_3^{\delta^-} \mapsto \mathsf{cons}(x_2^{\delta^-}, \mathsf{nil})\}$, both for rewriting. This results in

(revtail-append4.1.2.1) $\mathsf{append}(\mathsf{revtail}(l_7^{\delta^-}, \mathsf{nil}), \mathsf{cons}(x_2^{\delta^-}, l_3^{\delta}))$
$= \mathsf{append}(\mathsf{append}(\mathsf{revtail}(l_7^{\delta^-}, \mathsf{nil}), \mathsf{cons}(x_2^{\delta^-}, \mathsf{nil})), l_3^{\delta^-})$

the right-hand side of which can be rewritten to the left-hand side by application of (append-assoc), (append1), and (append1).

All in all, although we have been able to synthesize a sound conjecture, namely

(revtail-append4) $\mathsf{revtail}(\mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}), l_3^{\delta^-}) = \mathsf{append}(\mathsf{revtail}(l_2^{\delta^-}, \mathsf{nil}), \mathsf{revtail}(l_1^{\delta^-}, l_3^{\delta^-}))$

we did not know how to prove it. Although QUODLIBET was finally able to prove (revtail-append4.1) with some additional lemma, neither I nor NQTHM were able to do it. In our opinion, this demonstrates that generalization by syntactic means is poor. Now what does a human expert do? First he sees

(revtail-append5) $\mathsf{revtail}(\mathsf{append}(l_1^{\delta^-}, l_2^{\delta^-}), l_3^{\delta^-}) = \mathsf{revtail}(l_2^{\delta^-}, \mathsf{revtail}(l_1^{\delta^-}, l_3^{\delta^-}))$
(revtail-append6) $\mathsf{revtail}(l_1^{\delta^-}, \mathsf{append}(l_2^{\delta^-}, l_3^{\delta^-})) = \mathsf{append}(\mathsf{revtail}(l_1^{\delta^-}, l_2^{\delta^-}), l_3^{\delta^-})$

which are trivial by induction on $l_1^{\delta^-}$ and (together with (append1)) prove (revtail-append4) and—a fortiori—(revtail-append1) by deductive rewriting, where (revtail-append6) has to be applied from right to left.

Note that the proof of (revtail-append5) (whose left-hand side is identical to the one of (revtail-append4)) is what let us instantiate l_3^{γ} in the above proof attempt due to the erroneous assumption that (revtail-append4) should be proved by induction on $l_1^{\delta^-}$.

References

Jürgen Avenhaus, Ulrich Kühler, Tobias Schmidt-Samoa, Claus-Peter Wirth (2003). *How to Prove Inductive Theorems?* QUODLIBET*!*. 19th CADE 2003, LNAI 2741, pp. 328–333, Springer.

Robert S. Boyer, J S. Moore (1988). *A Computational Logic Handbook.* Academic Press.

Holger Busch (1993). *Unification-Based Induction.* In: Claesen & Gordon (1993), pp. 97–116.

Luc J. M. Claesen, Michaël J. C. Gordon (eds.) (1993). *Higher Order Logic Theorem Proving and its Applications.* North-Holland (Elsevier).

Ulrich Kühler (2000). *A Tactic-Based Inductive Theorem Prover for Data Types with Partial Operations.* Ph.D. thesis, Infix, Sankt Augustin.

Claus-Peter Wirth (2004). *Descente Infinie + Deduction.* Logic J. of the IGPL **12**, pp. 1–96, Oxford Univ. Press. `http://www.ags.uni-sb.de/~cp/p/d/welcome.html` (Sept. 12, 2003).