Category Theory

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Abstract:

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Contents

1 Basic Notions and Notations

Definition 1.1 (Category)

A *category* C (also called an *abstract category*)
consists of (left indices deliberately omitted):

- a class ${}_{\mathrm{C}}\mathrm{Obj}$ of *objects*,
- an Obj-indexed family ${}_{\mathrm{C}}\mathrm{Id} = ({}_{\mathrm{C}}\mathrm{Id}_s)_{s\in\mathrm{Obj}}$ of *identities*,
- a binary partial *composition* function ${}_{\mathrm{C}}\diamond$ with infix notation,

and either

- an (Obj×Obj)-indexed family ${}_{\mathrm{C}}\mathrm{Mor} = ({}_{\mathrm{C}}\mathrm{Mor}(s,t))_{(s,t)\in(\mathrm{Obj}\times\mathrm{Obj})}$ of sets of *morphisms*

with:

(2) $\forall s,t,u,v \in \mathrm{Obj} : \Big(\ \mathrm{Mor}(s,t) \cap \mathrm{Mor}(u,v) = \emptyset \quad \vee \quad (s,t) = (u,v) \ \Big)$

(3) $\forall s \in \mathrm{Obj} : \mathrm{Id}_s \in \mathrm{Mor}(s,s)$

(4) $\mathrm{dom}(\diamond) \ = \bigcup_{s,t,u\in\mathrm{Obj}} \mathrm{Mor}(s,t) \times \mathrm{Mor}(t,u)$

(5) $\forall s,t,u \in \mathrm{Obj} : \forall f \in \mathrm{Mor}(s,t) : \forall g \in \mathrm{Mor}(t,u) : (f \diamond g) \in \mathrm{Mor}(s,u)$

(6) $\forall s,t \in \mathrm{Obj} : \forall f \in \mathrm{Mor}(s,t) : \mathrm{Id}_s \diamond f = f = f \diamond \mathrm{Id}_t$

(7) $\forall s,t,u,v \in \mathrm{Obj} : \forall f \in \mathrm{Mor}(s,t) : \forall g \in \mathrm{Mor}(t,u) : \forall h \in \mathrm{Mor}(u,v) :$
$$(f \diamond g) \diamond h = f \diamond (g \diamond h)$$

or

- a class ${}_{\mathrm{C}}\mathrm{Arr}$ of *arrows* (or *morphisms*),
- a *domain* function ${}_{\mathrm{C}}\mathrm{Dom}\colon \ \mathrm{Arr} \to \mathrm{Obj}$,
- a *codomain* function ${}_{\mathrm{C}}\mathrm{Cod}\colon \ \mathrm{Arr} \to \mathrm{Obj}$,

with:

(1) $\forall s,t \in \mathrm{Obj} : \Big(\ \{ \ f \in \mathrm{Arr} \mid (\mathrm{Dom}(f), \mathrm{Cod}(f)) = (s,t) \ \} \ \in \ \mathcal{U} \ \Big)$

(3) $\forall s \in \mathrm{Obj} : \Big(\ \mathrm{Id}_s \in \mathrm{Arr} \quad \wedge \quad \mathrm{Dom}(\mathrm{Id}_s) = s = \mathrm{Cod}(\mathrm{Id}_s) \ \Big)$

(4) $\mathrm{dom}(\,\diamond\,) \ = \ \{ \ (f,g) \in \mathrm{Arr}{\times}\mathrm{Arr} \mid \mathrm{Cod}(f) = \mathrm{Dom}(g) \ \}$

(5) $\forall f,g \in \mathrm{Arr} : \left(\ \mathrm{Cod}(f) = \mathrm{Dom}(g) \ \Rightarrow \ \left(\begin{array}{ll} & (f \diamond g) \in \mathrm{Arr} \\ \wedge & \mathrm{Dom}(f \diamond g) = \mathrm{Dom}(f) \\ \wedge & \mathrm{Cod}(f \diamond g) = \mathrm{Cod}(g) \end{array} \right) \right)$

(6) $\forall f \in \mathrm{Arr} : \mathrm{Id}_{\mathrm{Dom}(f)} \diamond f = f = f \diamond \mathrm{Id}_{\mathrm{Cod}(f)}$

(7) $\forall f,g,h \in \mathrm{Arr} : (f \diamond g) \diamond h = f \diamond (g \diamond h)$

First consider the definition of C to be given according to the "either"-case: Then (2) allows to define Arr and Dom, Cod: $\mathrm{Arr} \to \mathrm{Obj}$ by ($f \in \mathrm{Arr}$):

$$\mathrm{Arr} := \biguplus_{s,t\in\mathrm{Obj}} \mathrm{Mor}(s,t)\ , \quad \mathrm{Dom}(f) := \bigcap_{\exists t:\, f\in\mathrm{Mor}(s,t)} s\ , \quad \text{and} \quad \mathrm{Cod}(f) := \bigcap_{\exists s:\, f\in\mathrm{Mor}(s,t)} t\ .$$

Now item (1) of the "or"-case is trivial and (3), ..., (7) follow from the respective items of the "either"-case.

Now consider the definition of C to be given according to the "or"-case: Then (1) allows to define an (Obj×Obj)-indexed family ${}_{\mathrm{C}}\mathrm{Mor} = ({}_{\mathrm{C}}\mathrm{Mor}(s,t))_{(s,t)\in(\mathrm{Obj}\times\mathrm{Obj})}$ of sets by ($s,t \in \mathrm{Obj}$):

$$\mathrm{Mor}(s,t) \ := \ \{\ f \in \mathrm{Arr} \mid (\mathrm{Dom}(f), \mathrm{Cod}(f)) = (s,t)\ \}\ .$$

Now item (2) of the "either"-case is trivial and (3), ..., (7) follow from the respective items of the "or"-case.

All in all, both ways of defining a category are completely equivalent. Thus, if we have to define some certain category, we can always choose the more convenient way and check the category property without translation. Moreover, by use of the translation given above, we can always use Mor together with Arr, Dom, and Cod for dealing with a single certain category, no matter which way it has been defined.

We will abbreviate "$f \diamond g$" by "fg" and "$f(gh)$" (or "$(fg)h$") by "fgh".

For many categories, item (2) must be achieved in a rather artificial way: Instead of defining $\mathrm{Mor}(s,t)$ by $\mathrm{Mor}(\text{domain-object}, \text{codomain-object}) \ := \ \{\ f \mid P(f)\ \}$ one has to define it more clumsily by

$$\mathrm{Mor}(\text{domain-object}, \text{codomain-object}) \ := \ \{\ (f, \text{domain-object}, \text{codomain-object}) \mid P(f)\ \}\ .$$

This, however, forces us to read and write lots of formulas of the form

$$\text{“ } (f, \text{domain-object}, \text{codomain-object}) \in \mathrm{Mor}(\text{domain-object}, \text{codomain-object}) \text{ ”},$$

which may be very tiresome. Instead, we prefer to write

$$\text{“ } f\text{::domain-object}\to\text{codomain-object ”}$$

and intend it to mean " $f \in \mathrm{Mor}(\text{domain-object}, \text{codomain-object})$ " unless item (2) requires it to mean

$$\text{“ } (f, \text{domain-object}, \text{codomain-object}) \in \mathrm{Mor}(\text{domain-object}, \text{codomain-object}) \text{ ” }.$$

Definition 1.2 (Functor)

Let C, D be categories.

A [*co-variant*] *functor* F *from* C *to* D (written "$\mathrm{F} : \mathrm{C} \twoheadrightarrow \mathrm{D}$") is a quadruple

$$\mathrm{F} \;=\; (\mathrm{F}_{\mathrm{Obj}}, \mathrm{F}_{\mathrm{Arr}}, \mathrm{C}, \mathrm{D})$$

with:

(1) An *object function* $\mathrm{F}_{\mathrm{Obj}}$: ${}_{\mathrm{C}}\mathrm{Obj} \to {}_{\mathrm{D}}\mathrm{Obj}$. We abbreviate "$\mathrm{F}_{\mathrm{Obj}}(s)$" by "$\mathrm{F}(s)$".

(2) An *arrow function* $\mathrm{F}_{\mathrm{Arr}}$: ${}_{\mathrm{C}}\mathrm{Arr} \to {}_{\mathrm{D}}\mathrm{Arr}$. We abbreviate "$\mathrm{F}_{\mathrm{Arr}}(f)$" by "${}_{f}\mathrm{F}$".

(3) $\forall s, t \in {}_{\mathrm{C}}\mathrm{Obj} :\; \mathrm{F}_{\mathrm{Obj}}[{}_{\mathrm{C}}\mathrm{Mor}(s,t)] \subseteq {}_{\mathrm{D}}\mathrm{Mor}(\mathrm{F}(s), \mathrm{F}(t))$.

(4) $\forall s \in {}_{\mathrm{C}}\mathrm{Obj} : {}_{{}_{\mathrm{C}}\mathrm{Id}_s}\mathrm{F} \;=\; {}_{\mathrm{D}}\mathrm{Id}_{\mathrm{F}(s)}$.

(5) $\forall f, g \in {}_{\mathrm{C}}\mathrm{Arr} : \Big(\; {}_{\mathrm{C}}\mathrm{Cod}(f) = {}_{\mathrm{C}}\mathrm{Dom}(g)$ [1] $\Rightarrow\; {}_{(f\;{}_{\mathrm{C}}\!\diamond\; g)}\mathrm{F} = {}_{f}\mathrm{F}\;{}_{\mathrm{D}}\!\diamond\;{}_{g}\mathrm{F} \;\Big)$.

Due to our special notation for the application of $\mathrm{F}_{\mathrm{Arr}}$ we can get the references to C and D, resp., from the level and thus may omit them. E.g., we will write "${}_{\mathrm{Id}_s}\mathrm{F} \;=\; \mathrm{Id}_{\mathrm{F}(s)}$" for "${}_{{}_{\mathrm{C}}\mathrm{Id}_s}\mathrm{F} \;=\; {}_{\mathrm{D}}\mathrm{Id}_{\mathrm{F}(s)}$" as well as "${}_{(f\diamond g)}\mathrm{F} \;=\; {}_{f}\mathrm{F} \diamond {}_{g}\mathrm{F}$" (or even "${}_{fg}\mathrm{F} \;=\; {}_{f}\mathrm{F}_{g}\mathrm{F}$") for "${}_{(f\;{}_{\mathrm{C}}\!\diamond\; g)}\mathrm{F} = {}_{f}\mathrm{F}\;{}_{\mathrm{D}}\!\diamond\;{}_{g}\mathrm{F}$".

The *identity functor* $\mathbb{I}_{\mathrm{C}}$ *for* C is the functor $\;\mathbb{I}_{\mathrm{C}} \;:=\; (\mathrm{id}|_{{}_{\mathrm{C}}\mathrm{Obj}}, \mathrm{id}|_{{}_{\mathrm{C}}\mathrm{Arr}}, \mathrm{C}, \mathrm{C})$.
The *composition of two functors* $\mathrm{F} : \mathrm{B} \twoheadrightarrow \mathrm{C}$ and $\mathrm{G} : \mathrm{C} \twoheadrightarrow \mathrm{D}$ is the functor
$\mathrm{FG} \;:=\; (\mathrm{F}_{\mathrm{Obj}}\mathrm{G}_{\mathrm{Obj}}, \mathrm{F}_{\mathrm{Arr}}\mathrm{G}_{\mathrm{Arr}}, \mathrm{B}, \mathrm{D})$.

If $\;(\mathrm{id}|_{{}_{\mathrm{C}}\mathrm{Obj}}, \mathrm{id}|_{{}_{\mathrm{C}}\mathrm{Arr}}, \mathrm{C}, \mathrm{D})\;$ is a functor, then it is called the *inclusion functor from* C *to D* and C is called a *sub-category of* D.

A *contra-variant functor* (*or contra-functor*) F *from* C *to* D (written "$\mathrm{F} : \mathrm{C} \rightleftharpoons \mathrm{D}$") differs from a co-variant functor only in the following:

(3) $\forall s, t \in {}_{\mathrm{C}}\mathrm{Obj} :\; \mathrm{F}_{\mathrm{Obj}}[{}_{\mathrm{C}}\mathrm{Mor}(s,t)] \subseteq {}_{\mathrm{D}}\mathrm{Mor}(\mathrm{F}(t), \mathrm{F}(s))$.

(5) $\forall f, g \in {}_{\mathrm{C}}\mathrm{Arr} : \Big(\; {}_{\mathrm{C}}\mathrm{Cod}(f) = {}_{\mathrm{C}}\mathrm{Dom}(g)$ [2] $\Rightarrow\; {}_{fg}\mathrm{F} = {}_{g}\mathrm{F}_{f}\mathrm{F} \;\Big)$.

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[1] The condition " ${}_{\mathrm{C}}\mathrm{Cod}(f) = {}_{\mathrm{C}}\mathrm{Dom}(g)$ " is necessary here because possibly

$${}_{\mathrm{C}}\mathrm{Cod}(f) \neq {}_{\mathrm{C}}\mathrm{Dom}(g) \;\wedge\; {}_{\mathrm{D}}\mathrm{Cod}({}_{f}\mathrm{F}) = {}_{\mathrm{D}}\mathrm{Dom}({}_{g}\mathrm{F})$$

and then ${}_{(f\diamond g)}\mathrm{F}$ is undefined while ${}_{f}\mathrm{F} \diamond {}_{g}\mathrm{F}$ is defined.

[2] The condition " ${}_{\mathrm{C}}\mathrm{Cod}(f) = {}_{\mathrm{C}}\mathrm{Dom}(g)$ " is necessary here because possibly

$${}_{\mathrm{C}}\mathrm{Cod}(f) \neq {}_{\mathrm{C}}\mathrm{Dom}(g) \;\wedge\; {}_{\mathrm{D}}\mathrm{Cod}({}_{g}\mathrm{F}) = {}_{\mathrm{D}}\mathrm{Dom}({}_{f}\mathrm{F})$$

and then ${}_{fg}\mathrm{F}$ is undefined while ${}_{g}\mathrm{F}_{f}\mathrm{F}$ is defined.

Definition 1.3 (Small and Large Categories)
A category C is *small* if ${}_{\mathrm{C}}\mathrm{Obj} \in \mathcal{U}$ (i.e. iff ${}_{\mathrm{C}}\mathrm{Obj}$ is a set);
and *large* otherwise (i.e. iff ${}_{\mathrm{C}}\mathrm{Obj}$ is a proper class).

Corollary 1.4
Let C *and* D *be categories.*
If C *and* D *are a small, then* ${}_{\mathrm{C}}\mathrm{Obj}$, ${}_{\mathrm{C}}\mathrm{Id}$, ${}_{\mathrm{C}}\diamond$, ${}_{\mathrm{C}}\mathrm{Mor}$, ${}_{\mathrm{C}}\mathrm{Arr}$, ${}_{\mathrm{C}}\mathrm{Dom}$, ${}_{\mathrm{C}}\mathrm{Cod}$, C, *and all functors* $\mathrm{F} : \mathrm{C} \twoheadrightarrow \mathrm{D}$ *are sets.*
If C *is large, then* ${}_{\mathrm{C}}\mathrm{Obj}$, ${}_{\mathrm{C}}\mathrm{Id}$, ${}_{\mathrm{C}}\diamond$, ${}_{\mathrm{C}}\mathrm{Mor}$, ${}_{\mathrm{C}}\mathrm{Arr}$, ${}_{\mathrm{C}}\mathrm{Dom}$, ${}_{\mathrm{C}}\mathrm{Cod}$, C, *and all functors* $\mathrm{F} : \mathrm{C} \twoheadrightarrow \mathrm{D}$ *are proper classes.*

2 Basic Examples

Example 2.1 (Categories of Sets)

The *relation category of sets* is the large category REL given as follows:

$_{\text{REL}}\mathrm{Obj} := \mathcal{U}$

$_{\text{REL}}\mathrm{Mor}(A,B) \ := \ \{\ (R,A,B) \mid R \subseteq A{\times}B\ \}$ for $A,B \in \mathcal{U}$.

$(R,A,B)\ {}_{\text{REL}}\diamond\ (S,B,C) \ := \ (RS,A,C)$ for $(R,A,B),(S,B,C) \in {}_{\text{REL}}\mathrm{Arr}$.

$_{\text{REL}}\mathrm{Id}_A \ := \ (\mathrm{id}|_A, A, A)$ for $A \in \mathcal{U}$.

The *partial function category of sets* is the large sub-category FUN of the category REL given as follows:

$_{\text{FUN}}\mathrm{Obj} := \mathcal{U}$

$_{\text{FUN}}\mathrm{Mor}(A,B) \ := \ \{\ (f,A,B) \mid f\colon\ A \rightsquigarrow B\ \}$ for $A,B \in \mathcal{U}$.

The (*total function*) *category of sets* is the large sub-category SET of the category FUN given as follows:

$_{\text{SET}}\mathrm{Obj} := \mathcal{U}$

$_{\text{SET}}\mathrm{Mor}(A,B) \ := \ \{\ (f,A,B) \mid f\colon\ A \to B\ \}$ for $A,B \in \mathcal{U}$.

3 Duality

The *duality principle* is a very important tool for doing universal category theory because it halves the proof work. The principle is based on the cognition that things do not change when all arrows and their composition are reversed.

Definition 3.1 (Dual Category and Functor)
Let C be a category. The *dual*[3] *category of* C is the category Dual(C) given by
${}_{\mathrm{Dual(C)}}\mathrm{Obj} := {}_{\mathrm{C}}\mathrm{Obj}$; ${}_{\mathrm{Dual(C)}}\mathrm{Arr} := {}_{\mathrm{C}}\mathrm{Arr}$; ${}_{\mathrm{Dual(C)}}\mathrm{Id} := {}_{\mathrm{C}}\mathrm{Id}$; (attention!:)
${}_{\mathrm{Dual(C)}}\mathrm{Dom} := {}_{\mathrm{C}}\mathrm{Cod}$; ${}_{\mathrm{Dual(C)}}\mathrm{Cod} := {}_{\mathrm{C}}\mathrm{Dom}$; (attention!:)
$f \;{}_{\mathrm{Dual(C)}}\!\diamond\; g := g \;{}_{\mathrm{C}}\!\diamond\; f$; ${}_{\mathrm{Dual(C)}}\mathrm{Mor}(s,t) := {}_{\mathrm{C}}\mathrm{Mor}(t,s)$.

Let $\mathrm{F} : \mathrm{C} \twoheadrightarrow \mathrm{D}$. The *dual functor* $\mathrm{Dual(F)} : \mathrm{Dual(C)} \twoheadrightarrow \mathrm{Dual(D)}$ *of* F is given by $\mathrm{Dual(F)}_{\mathrm{Obj}} := \mathrm{F}_{\mathrm{Obj}}$; $\mathrm{Dual(F)}_{\mathrm{Arr}} := \mathrm{F}_{\mathrm{Arr}}$. Do not confuse this co-variant functor with the contra-variant one having the same object and arrow functions: The *dual contra-functor* $\mathrm{Contradual(F)} : \mathrm{C} \rightleftharpoons \mathrm{Dual(D)}$ *of* F is also given by $\mathrm{Contradual(F)}_{\mathrm{Obj}} := \mathrm{F}_{\mathrm{Obj}}$; $\mathrm{Contradual(F)}_{\mathrm{Arr}} := \mathrm{F}_{\mathrm{Arr}}$. Of special interest is $\mathrm{Contradual}(\mathbb{I}_{\mathrm{C}}) : \mathrm{C} \rightleftharpoons \mathrm{Dual(C)}$, which is no co-variant functor: Note that C is in general not isomorphic to Dual(C).

The *dual expression* of some expression ϕ is obtained by replacing each occurrence of a category or functor symbol X by $\mathrm{Dual}(X)$ and reversing the arguments of all concatenations.

Simply speaking of the *dual* we mean what is obtained by substituting everything by its dual.

Note that the dual expression has the same truth value in the dual as the original expression in the original. Since $\mathrm{Dual(Dual(C))} = \mathrm{C}$ and $\mathrm{Dual(Dual(F))} = \mathrm{F}$ for $\mathrm{F} : \mathrm{C} \twoheadrightarrow \mathrm{D}$, the dual of the dual is the original. Therefore, the dual of all categories and functors are all categories and functors. Moreover, the dual of the expression "C is a category" is logically equivalent to the expression "C is a category" itself. The same holds for the expression "F is a functor". Since the dual of a logic inference is a logic inference, the dual of a consequence of a set of axioms A which is logically equivalent to its dual is a consequence of A. Thus, the universal theory of categories and functors does not differ from its dual. Stated differently: If a general theorem about categories and functors is valid, then the dual theorem is valid, too. This "*duality principle*" is a very important tool for doing universal category theory because it halves the proof work.

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[3] called *opposite* by some authors

4 On the Inside of Categories

In the whole section we assume C to be some category and omit the left indices to C.

Definition 4.1

Let $f{::}s{\to}t$.

Now (w.r.t. C) f is called ...

... *monic* if $\forall g,h \in \mathrm{Arr} : \Big(\ (\mathrm{Cod}(g) = \mathrm{Cod}(h) = s \ \wedge \ gf = hf \ ) \ \Rightarrow \ g = h \ \Big)$.

... *epi* if $\forall g,h \in \mathrm{Arr} : \Big(\ (t = \mathrm{Dom}(g) = \mathrm{Dom}(h) \ \wedge \ fg = fh \ ) \ \Rightarrow \ g = h \ \Big)$.

Thus, “epi” is the dual of “monic”.

... *split-monic* if $\exists r \in \mathrm{Arr} : fr = \mathrm{Id}_s$. Such an r is called a *retraction of* f.

... *split-epi* if $\exists l \in \mathrm{Arr} : lf = \mathrm{Id}_t$. Such an l is called a *section of* f.

Thus, “split-epi” and “section” are the dual of “split-monic” and “retraction”.

... an *equalizer for* $p,q{::}t{\to}u$ if $fp = fq$ and
$\forall r \in \mathrm{Obj} : \forall f' \in \mathrm{Mor}(r,t) : (f'p = f'q \Rightarrow \exists! h \in \mathrm{Mor}(r,s) : f' = hf)$.

... a *co-equalizer for* $p,q{::}r{\to}s$ if $pf = qf$ and
$\forall u \in \mathrm{Obj} : \forall f' \in \mathrm{Mor}(s,u) : (pf' = qf' \Rightarrow \exists! h \in \mathrm{Mor}(t,u) : f' = fh)$.

Thus, “co-equalizer” is the dual of “equalizer”.

Lemma 4.2

Let $f{::}s{\to}t$.

(1) *If f is split-monic, then f is monic.*

(2) *If f is an equalizer for $p,q{::}t{\to}u$, then f is monic.*

Dual(1) *If f is split-epi, then f is epi.*

Dual(2) *If f is a co-equalizer for $p,q{::}r{\to}s$, then f is epi.*

Proof of Lemma 4.2

(1): For $gf = hf$ we get $g = g\,\mathrm{Id}_s = gfr = hfr = h\,\mathrm{Id}_s = h$.
(2): By the first property of an equalizer we get $(gf)p = (gf)q$ for each $g \in \mathrm{Arr}$ with $\mathrm{Cod}(g) = s$. Thus, (by the uniqueness part of the second property of an equalizer) $gf = hf$ implies $g = h$.

Lemma 4.3
In the relation category[4] REL *of sets the following items are logically equivalent for* $R{::}A{\to}B$ *and* $C \;:=\; \{\; y \mid \quad |R^{-1}[\{y\}]| = 1 \;\}$ *:*

1. $R{::}A{\to}B$ *is split-monic.*
2. $R{::}A{\to}B$ *is monic.*
3. $\forall X, Y \subseteq A : \; (\; R[X] = R[Y] \;\Rightarrow\; X{=}Y \;).$
4. $\forall x \in A : \;\; R[A] \neq R[A \setminus \{x\}].$
5. $\mathrm{dom}(R\; \mathrm{id}|_C) = A.$
6. $R\; \mathrm{id}|_C \, R^{-1} \;=\; \mathrm{id}|_A.$

Proof of Lemma 4.3
(1) $\Rightarrow$ (2): By Lemma 4.2.
(2) $\Rightarrow$ (3): Suppose $R[X] = R[Y]$. Define $S := \{\emptyset\}{\times}X$ and $T := \{\emptyset\}{\times}Y$. Then $SR = \{\emptyset\}{\times}R[X] = \{\emptyset\}{\times}R[Y] = TR$. Since $R{::}A{\to}B$ is monic, we get $S = T$, i.e. $X = Y$.
(3) $\Rightarrow$ (4): If (4) was not true, (3) would imply $A = A \setminus \{x\}$.
(4) $\Rightarrow$ (5): "$\subseteq$" is trivial. For showing "$\supseteq$" assume $x \in A$ but $y \notin C$ for all y with $(x, y) \in \mathrm{R}$. Then we get $R[A] = R[A \setminus \{x\}]$, which contradicts (4).
(5) $\Rightarrow$ (6): "$\subseteq$" is true by definition of C. "$\supseteq$" follows from (5).
(6) $\Rightarrow$ (1): We get $\mathrm{id}|_C \, R^{-1}{::}B{\to}A$ and $(R, A, B)(\mathrm{id}|_C \, R^{-1}, B, A) = (\mathrm{id}|_A, A, A) = {}_{\mathrm{REL}}\mathrm{Id}_A$.

Lemma 4.4
In the relation category[4] REL *of sets the following items are logically equivalent for* $R{::}A{\to}B$ *and* $C \;:=\; \{\; y \mid \quad |R[\{y\}]| = 1 \;\}$ *:*

1. $R{::}A{\to}B$ *is split-epi.*
2. $R{::}A{\to}B$ *is epi.*
3. $\forall X, Y \subseteq B : \; (\; R^{-1}[X] = R^{-1}[Y] \;\Rightarrow\; X{=}Y \;).$
4. $\forall x \in B : \;\; R^{-1}[B] \neq R^{-1}[B \setminus \{x\}].$
5. $\mathrm{ran}(\mathrm{id}|_C \, R) = B.$
6. $R^{-1}\; \mathrm{id}|_C \, R \;=\; \mathrm{id}|_B.$

Since Dual(REL) is isomorphic to REL via F : Dual(REL) $\twoheadrightarrow$ REL given by $\mathrm{F}(A) := A$ and ${}_{(R,A,B)}\mathrm{F} := (R^{-1}, B, A)$ and since Lemma 4.4 is the F-isomorphic of the dual of Lemma 4.3, we do not give a proof of Lemma 4.4 — even if the duality principle[5] is actually only sufficient to infer the logical equivalence of (1) and (2).

Lemma 4.5
In the partial function category[6] FUN *of sets the following items are logically equivalent:*

[4] cf. 2.1
[5] cf. section 3
[6] cf. 2.1

(1) $f{::}A{\to}B$ *is split-monic.*

(2) $f{::}A{\to}B$ *is monic.*

(3) $f\colon A \rightsquigarrow B$ *is injective and* $\mathrm{dom}(f) = A$ *.*

Proof of Lemma 4.5
(1) $\Rightarrow$ (2): By Lemma 4.2.
(2) $\Rightarrow$ (3): If f was not injective, there would be $a, b \in \mathrm{dom}(f)$ with $a \neq b$ and $f(a) = f(b)$. Define $g, h{::}\{\emptyset\}{\to}A$ by $g(\emptyset) := a$ and $h(\emptyset) := b$. Thus, $gf = hf$ and $g \neq h$, which contradicts (2). If f was not total on A, there would be an $a \in A\backslash\mathrm{dom}(f)$. Thus, $\mathrm{id}|_{\{a\}}f = \emptyset f$ and $\mathrm{id}|_{\{a\}} \neq \emptyset$, which contradicts (2).
(3) $\Rightarrow$ (1): For injective $f\colon A \to B$ we get $f^{-1}{::}B{\to}A$ and $(f, A, B)(f^{-1}, B, A) = (\mathrm{id}|_A, A, A)$ $= {}_{\mathrm{FUN}}\mathrm{Id}_A$.

Lemma 4.6
In the partial function category[6] FUN *of sets the following items are logically equivalent:*

(1) $f{::}A{\to}B$ *is epi.*

(2) $f\colon A \rightsquigarrow B$ *is surjective (i.e.* $\mathrm{ran}(f) = B$*).*

Furthermore, (1) and (2) are implied by

(3) $f{::}A{\to}B$ *is split-epi.*

"(1) $\Rightarrow$ (3)*", however, is equivalent to the set-version of (the function choice function version of) the Axiom of Choice.*

Proof of Lemma 4.6
(1) $\Rightarrow$ (2): If $f\colon A \rightsquigarrow B$ was not surjective, there would be an $b \in B\backslash\mathrm{ran}(f)$. Define $g, h{::}B{\to}\{0, 1\}$ by $g(b) := 1$, $g(x) := 0$ for $x \in B\backslash\{b\}$, and $h(x) := 0$ for $x \in B$. Thus, $fg = fh$ and $g \neq h$, which contradicts (1).
(2) $\Rightarrow$ (1): Assume $g, h\colon B \rightsquigarrow C$ with $fg = fh$. Since f is a function and then by (2) we get $f^{-1}f = \mathrm{id}|_{\mathrm{ran}(f)} = \mathrm{id}|_B$. Thus $g = \mathrm{id}|_B\, g = f^{-1}fg = f^{-1}fh = \mathrm{id}|_B\, h = h$.
(3) $\Rightarrow$ (1): By Lemma 4.2.
Axiom of Choice, (2) $\Rightarrow$ (3): Since $f\colon A \rightsquigarrow B$ is surjective, let $f''\colon B \to A$ be a function choice function of $f'\colon B \to \mathfrak{P}(A)$ given by $f'(x) := f^{-1}[\{x\}]$. Since f is a function and then by (2) we get $f''f = \mathrm{id}|_{\mathrm{ran}(f)} = \mathrm{id}|_B$. Thus $f''{::}B{\to}A$ is a section of $f{::}A{\to}B$.
((2) $\Rightarrow$ (3)) $\Rightarrow$ Axiom of Choice: Assume $g\colon B \to \mathfrak{P}(\mathcal{U})$ with $\forall x \in B : \emptyset \notin g(x)$. Define $A := \bigcup_{x \in B}(\{x\}{\times}g(x))$ and the surjective $f\colon A \to B$ by $f(x, y) := x$. By "(2) $\Rightarrow$ (3)" there must be a section f' of f. Define $h\colon A \to \mathcal{U}$ by $h(x, y) := y$. Now $f'h$ is a function choice function of g. Indeed, for $x \in B$ we get $f(f'(x)) = x$, $f'(x) \in \{x\}{\times}\, g(x)$, and $h(f'(x)) \in g(x)$.

Lemma 4.7
In the (total function) category[7] SET *of sets the following items are logically equivalent:*

(1) $f{::}A{\to}B$ *is an equalizer*
[for each $p,q{::}B{\to}C$ *with* $\forall b\in B:\ (\ p(b)=q(b)\ \Leftrightarrow\ b\in\mathrm{ran}(f)\ )$ *].*

(2) $f{::}A{\to}B$ *is monic.*

(3) $f\colon\ A\to B$ *is injective.*

Furthermore, (1) - (3) are implied by

(4) $f{::}A{\to}B$ *is split-monic.*

In general, however, (4) is not implied by (1) - (3), but it is equivalent to

(5) $f\colon\ A\to B$ *is injective and* $(\ A=\emptyset\ \Rightarrow\ \mathrm{B}=\emptyset\ )$.

Proof of Lemma 4.7
(1) $\Rightarrow$ (2): By Lemma 4.2.
(2) $\Rightarrow$ (3): By Lemma 4.5, since SET is a sub-category of FUN.
(3) $\Rightarrow$ (1): For p, q as indicated, we get $fp = fq$. Furthermore, for f': $D\to B$ with $f'p = f'q$ we get $\mathrm{ran}(f')\subseteq\mathrm{ran}(f)$. Thus, since f is injective, we then can define h: $D\to A$ by $h := f'f^{-1}$, and get $f' = f'\,\mathrm{id}|_{\mathrm{ran}(f)} = f'f^{-1}f = hf$. Finally, for any other h': $D\to A$ with $f' = h'f$ we get $h' = h'\,\mathrm{id}|_A = h'ff^{-1} = f'f^{-1} = h$.
(4) $\Rightarrow$ (2): By Lemma 4.2.
(4) $\Rightarrow$ (5): If f has a retraction $r{::}B{\to}\emptyset$, we must have $B=\emptyset$.
(5) $\Rightarrow$ (4): If $A=\emptyset=B$, f has the retraction $\emptyset{::}\emptyset{\to}\emptyset$. If there is some $a\in A$, f has the retraction $r{::}B{\to}A$ given by $r(x) := f^{-1}(x)$ for $x\in\mathrm{ran}(f)$ and $r(x) := a$ for $x\in B\backslash\mathrm{ran}(f)$.
In general not "(3) $\Rightarrow$ (4)": Take $f := A := \emptyset$ and $B := \{\emptyset\}$. Now f is injective but cannot have a retraction since ${}_{\mathrm{SET}}\mathrm{Mor}(B,\emptyset)=\emptyset$.

Lemma 4.8
In the (total function) category[7] SET *of sets the following items are logically equivalent:*

(1) $f{::}A{\to}B$ *is a co-equalizer*
[for $p,q{::}C{\to}A$*;* $C := \ker(f) := ff^{-1}$ *; and*
p,q *given by* $((x,y)\in C)$ $p(x,y) := x$ *and* $q(x,y) := y$ *]*

(2) $f{::}A{\to}B$ *is epi.*

(3) $f\colon\ A\to B$ *is surjective (i.e.* $\mathrm{ran}(f)=B$*).*

Furthermore, (1) - (3) are implied by

(4) $f{::}A{\to}B$ *is split-epi.*

"(2) $\Rightarrow$ (4)*", however, is equivalent to the set-version of the Axiom of Choice.*

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[7] cf. 2.1

Proof of Lemma 4.8
(1) $\Rightarrow$ (2): By Lemma 4.2.
(2) $\Rightarrow$ (3): By Lemma 4.5, since SET is a sub-category of FUN.
(3) $\Rightarrow$ (1): For p, q as indicated, we get $pf = qf$. Furthermore, for f': $A \to D$ with $pf' = qf'$ we get $\ker(f) \subseteq \ker(f')$. Thus, since f is surjective, we then can define h: $B \to D$ by $h := f^{-1}f'$ and get $f' = ff^{-1}f' = fh$. Finally, for any other h': $B \to D$ with $f' = fh'$, since f is surjective we get $h' = \mathrm{id}|_B \, h' = \mathrm{id}|_{\mathrm{ran}(f)} \, h' = f^{-1}fh' = f^{-1}f' = h$.
(4) $\Rightarrow$ (2): By Lemma 4.2.
Logical equivalence with the Axiom of Choice: The proof for this does not differ from the respective part of the proof of Lemma 4.6.